Extremes of Random Volatility Models

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1 Introduction

Extreme value theory for financial models mostly concerns the martingale part of the logarithm of a price process, since random volatility determines the extreme risk in price fluctuations. The increments $(Y_n)_{n\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{R}}$ of length 1 of this martingale part often have the structure

$$Y_n = \sigma_n \varepsilon_n\,, \quad n \in \mathbb{Z}\,, \quad \text{or} \quad Y_t = \int_{(t-1,t]} \sigma_{s-} dL_s\,, \quad t \in \mathbb{R}\,,$$

for a discrete time or continuous time model, respectively. Here, the volatility is modelled by σ, and $(\varepsilon_n)_{n\in\mathbb{Z}}$ or $(L_t)_{t\in\mathbb{R}}$ are typically i.i.d. sequences or a Lévy process, respectively. The usual prerequisite of extreme value theory for a stochastic process is its strict stationarity. Note that in most cases strict stationarity of the log price increment process Y is inherited from stationarity of the volatility process σ. Consequently, we will present conditions for strict stationarity of the models below, followed by the extremal analysis of a stationary version.

The importance of extreme value theory for such pricing models is two-fold. Firstly, the *tail behavior* for large absolute arguments describes the fluctuations of the prices. We distinguish between light- and heavy-tailed models. Whereas light-tailed means normal or exponential models, heavy-tailed models are defined via regular variation. We say a random variable X has regularly varying tail, if there are $\alpha > 0$ and slowly varying $\ell : (0,\infty) \to (0,\infty)$, i.e., $\lim_{x\to\infty} \ell(tx)/\ell(x) = 1$ for all $t > 0$, such that

$$P(X > x) = \ell(x) x^{-\alpha}\,, \quad x > 0\,.$$

We write $X \in \mathcal{R}(-\alpha)$. If the volatility σ has regularly varying tail and the noise ε or L is light tailed and independent of the volatility, then the log price increments are again regularly varying by Breiman's classical result. The same holds for a light-tailed volatility and independent regularly varying noise. If both, volatility and noise, are light-tailed, the distribution of the product depends even asymptotically on both factors and may be more difficult to estimate. Symmetry or tail balance of the right and left tail of the noise often simplifies the calculations.

Secondly, *volatility clusters on high levels* induce extreme price clusters, which can cause a particularly risky situation. For discrete time models such clusters are described by a limit of point processes of exceedances over high thresholds. For certain models this limit is simply a Poisson process, indicating that exceedances happen as single points and at completely random times. However, more realistic models capture the fact that not a Poisson process, but a compound Poisson turns up in the limit, which describes the cluster size distribution. A crude, but simple measure of the cluster size of extremes is the *extremal index* $\theta \in (0,1]$, where $1/\theta$ can be interpreted as the mean size of a cluster. It also appears in the distributional limit of the running maxima

$$M_n^Z = \max\{Z_1, \ldots, Z_n\}, \quad n \in \mathbb{N},$$

of a stationary sequence $(Z_n)_{n\in\mathbb{Z}}$. More precisely, under weak conditions on Z_1, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that the *Poisson condition*

$$nP(Z_1 > a_n x + b_n) = -\log G(x), \quad x \in \mathbb{R}, \tag{1}$$

holds, and if the process satisfies further mixing conditions, then the extremal index of $(Z_n)_{n\in\mathbb{Z}}$ is the unique number $\theta \in (0,1]$ such that

$$\lim_{n\to\infty} P(a_n^{-1}(M_n^Z - b_n) \le x) = G^\theta(x), \quad x \in \mathbb{R}. \tag{2}$$

Here, G is an extreme value distribution, i.e. it is of the same type as either a Fréchet distribution $\Phi_\alpha(x) = \mathbf{1}_{(0,\infty)}(x)\exp(-x^{-\alpha})$ for some $\alpha > 0$ (heavy-tailed case), a Gumbel distribution $\Lambda(x) = \exp(-e^{-x})$ (light-tailed case), or a Weibull distribution Ψ_α for some $\alpha > 0$. We refer to the books [11, 24] for this and further information about extreme value theory.

For $\theta = 1$ we interpret the stochastic process as a process without clusters in the extremes, whereas if $\theta < 1$ we speak of a process with cluster possibilities in the extremes. Using a point process approach the whole cluster size distribution can be derived giving much more insight into the extremal behaviour of time series models. This, however, would go beyond this article and we refer to the references given throughout for more details.

As for discrete time models, the extremal behaviour of continuous time models can be described by the limit behaviour of point processes. A simple measure is again given by an extension of the extremal index: if $(Z_t)_{t\in\mathbb{R}}$ is a continuous time process, we define for fixed $h > 0$ the discrete time process

$$M_k^Z(h) := \sup_{(k-1)h \le t \le kh} Z_t, \quad k \in \mathbb{Z}.$$

Denoting $\theta(h)$ the extremal index of the sequence $(M_k^Z(h))_{k\in\mathbb{Z}}$, we follow [13] and call $\theta(h)$ for $h \in (0,\infty)$ the *extremal index function* of Z. The function θ is increasing, and we shall say that the continuous time process Z has extremal clusters if $\lim_{h\to 0}\theta(h) < 1$, i.e. if there is some $h > 0$ such that the discrete skeleton $(M_k^Z(h))_{k\in\mathbb{Z}}$ has extremal index less than 1, i.e. clusters.

There exist various publications on extreme value theory for time dependent data; we mention e.g. [7, 8, 11, 17, 18, 19, 20, 24] and references therein.

2 Discrete-time models

2.1 Stochastic volatility models

The simple stochastic volatility model is given by

$$Y_n = \sigma_n \varepsilon_n\,, \quad \log \sigma_n^2 = \alpha_0 + \sum_{j=0}^{\infty} c_j \eta_{n-j}\,, \quad n \in \mathbb{Z}\,, \tag{3}$$

where $(\eta_n)_{n\in\mathbb{Z}}$ is iid $N(0,s^2)$ with $\sum_{j=0}^{\infty} c_j^2 < \infty$ and $(\varepsilon_n)_{n\in\mathbb{Z}}$ is iid, independent of $(\eta_n)_{n\in\mathbb{Z}}$. This covers the case when $(\log\sigma_n^2)_{n\in\mathbb{Z}}$ is a causal ARMA process with iid Gaussian noise $(\eta_n)_{n\in\mathbb{Z}}$, the most prominent case being the volatility model of Taylor [28]:

$$Y_n = \sigma_n\varepsilon_n, \quad \log\sigma_n^2 = \alpha_0 + \psi\log\sigma_{n-1}^2 + \eta_n, \quad n\in\mathbb{Z}, \quad |\psi|<1, \tag{4}$$

when the log volatility is a causal Gaussian AR(1) process.

Extreme value analysis for (3) is based on the transformation

$$X_n = \log Y_n^2 = \alpha_0 + \sum_{j=0}^{\infty} c_j\eta_{n-j} + \log\varepsilon_n^2\,, \quad n\in\mathbb{Z}\,, \tag{5}$$

which is a Gaussian linear process plus an iid noise. From [5, 7] we have:

Theorem 1 (Tail behaviour and extremes of the SV model). *Assume the stochastic volatility model* (3) *as above with* X_n *defined by* (5).
(a) If ε_1 *is* $N(0,1)$ *denote* $\widetilde{s}^2 = s^2\sum_{j=0}^{\infty} c_j^2$ *and* $k = \log(2/\widetilde{s}^2)$.
Then the tail of the stationary process $(X_n)_{n\in\mathbb{Z}}$ *satisfies as* $x\to\infty$

$$P(X_1 > x+\alpha_0) = \frac{\widetilde{s}^2}{\sqrt{\pi}}\exp\{-\frac{x^2}{2\widetilde{s}^2} + \frac{x\log x}{\widetilde{s}^2} + \frac{(k-1)x}{\widetilde{s}^2} + \frac{(k+\widetilde{s}^2)\log x}{\widetilde{s}^2} - \frac{(\log x)^2}{2\widetilde{s}^2} - \frac{k^2}{2\widetilde{s}^2} + O(\frac{(\log x)^2}{x})\}\,.$$

The tail of the stationary log price increments is given by

$$P(Y_1 > \sqrt{y}) = \frac{1}{2}P(|Y_1| > \sqrt{y}) = \frac{1}{2}P(X_1 > \log y)\,, \quad n\in\mathbb{Z}\,. \tag{6}$$

If furthermore the autocorrelation function ρ *of* $(\log\sigma_n^2)_{n\in\mathbb{Z}}$ *satisfies*

$$\rho(h) = \operatorname{corr}(\log \sigma_n^2, \log \sigma_{n+h}^2) = o((\log h)^{-1}), \quad h \to \infty, \tag{7}$$

then each of the sequences $(X_n)_{n\in\mathbb{Z}}$ *and* $(Y_n)_{n\in\mathbb{Z}}$ *has extremal index 1, and the extreme value distribution* G *appearing in* (1) *and* (2) *is the Gumbel distribution* Λ *for both* X *and* Y.
(b) Assume that $\varepsilon_1 \in \mathcal{R}(-\alpha)$ *for some* $\alpha > 0$. *Then as* $x \to \infty$ *we have*

$$P(Y_1 > x) \sim E(\sigma_1^\alpha)\, P(\varepsilon_1 > x)\,.$$

If moreover for $p \in (0,1]$ *the tail balance condition for* $x \to \infty$ $P(\varepsilon_1 > x) \sim p\,P(|\varepsilon_1| > x)$ *holds, then* $(Y_n)_{n\in\mathbb{Z}}$ *has extremal index 1 and the extreme value distribution* G *appearing in* (1) *and* (2) *is the Fréchet distribution* Φ_α.

Here, $f(x) \sim g(x)$ as $x \to \infty$ for two strictly positive functions f and g means that $\lim_{x\to\infty} f(x)/g(x) = 1$. Berman's condition (7) is very weak and for example satisfied, whenever $\log \sigma_n^2$ follows a causal ARMA equation (in particular for the volatility model (4) of Taylor). This means that for most stochastic volatility models (3) with Gaussian η and either light- or heavy-tailed noise ε, the extremal index is 1, so that the point processes of exceedances over high thresholds converge to a Poisson process. The model cannot model clusters of extremes.

Extensions of the η_n to non-Gaussian random variables in (3) have been considered. In most cases the qualitative behaviour remains the same provided that the processes σ and ε are independent and η_1 is light-tailed; see [9, 10, 22].

2.2 The EGARCH model

A model related to stochastic volatility models is the EGARCH model of Nelson [26] given by

$$Y_n = \sigma_n \varepsilon_n, \quad \log \sigma_n^2 = \alpha_0 + \sum_{j=1}^{\infty} c_j g(\varepsilon_{n-j}), \quad n \in \mathbb{Z}, \tag{8}$$

where $(\varepsilon_n)_{n\in\mathbb{Z}}$ is iid normal (or more generally follows a generalised error distribution GED(ν)), the real coefficients α_0 and $(c_j)_{j\in\mathbb{N}}$ decay sufficiently fast, and g is a deterministic function. The infinite moving average representation for $\log \sigma^2$ typically arises from EGARCH(p,q) equations of the form

$$\log \sigma_n^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j g(\varepsilon_{n-j}) + \sum_{j=1}^{q} \beta_j \log(\sigma_{n-j}^2) \tag{9}$$

with real coefficients α_j and β_j. The standard choice for g is $g(x) = \varphi x + \gamma(|x| - E|\varepsilon_0|)$, where φ and γ are real constants, so that g is an affine linear function and $g(\varepsilon_n)$ allows the volatility to respond asymmetrically to negative and positive innovations. Depending on the size of φ and γ, different cases arise, but the most important one for $\gamma - \varphi > \gamma + \varphi > 0$, in which case a negative innovation increases the volatility more than a positive innovation of the same modulus (i.e. it models the leverage effect).

As for stochastic volatility models, extreme value analysis for (8) is based on the transformed process (which is stationary by (8))

$$X_n = \log Y_n^2 = \alpha_0 + \sum_{j=1}^{\infty} c_j g(\varepsilon_{n-j}) + \log \varepsilon_n^2, \quad n \in \mathbb{Z}.$$

It then follows from [9, 10, 22]:

Theorem 2 (Tail behaviour and extremes of EGARCH).
Assume the EGARCH model as above with $(\varepsilon_n)_{n\in\mathbb{N}}$ iid $N(0,1)$ and $\gamma - \varphi > \gamma + \varphi > 0$. Suppose further that the coefficients $(c_j)_{j\in\mathbb{N}}$ are non-negative and that $c_j = O(j^{-\delta})$ as $j \to \infty$ for some $\delta > 1$. Denote

$$A := (\gamma - \varphi)^2 \sum_{j=1}^{\infty} c_j^2 \quad \textit{and} \quad B := -\gamma E|\varepsilon_0| \sum_{j=1}^{\infty} c_j.$$

Then the stationary processes $(X_n)_{n\in\mathbb{Z}}$ and $(Y_n)_{n\in\mathbb{Z}}$ satisfy as $x \to \infty$

$$P(X_1 > \alpha_0 + x) = \exp\left(-[x^2 - x\log x - (B + \log(2/A) - 1)x]/A + o(x)\right),$$

and (6)*, respectively. Both processes $(X_n)_{n\in\mathbb{Z}}$ and $(Y_n)_{n\in\mathbb{Z}}$ have extremal index 1, and the extreme value distribution G appearing in* (1) *and* (2) *is the Gumbel distribution.*

Observe that for an EGARCH(p,q) process as in (9) with $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q \geq 0$ and such that $\sum_{j=1}^{q} \beta_j < 1$, the coefficients $(c_j)_{j\in\mathbb{N}}$ in (8) are automatically non-negative and decay exponentially, so that Theorem 2 applies. In particular, the EGARCH process with normal innovations cannot cluster. Extensions to other light-tailed innovations ε such as GED(ν) distributions with $\nu > 1$ are possible, cf. [10].

2.3 The GARCH(1,1) model

In the GARCH(1,1) model of Engle [12] and Bollerslev [4] the log price increments $(Y_n)_{n\in\mathbb{Z}}$ and the volatilities $(\sigma_n)_{n\in\mathbb{Z}}$ are given by

$$Y_n = \sigma_n \varepsilon_n, \quad \sigma_n^2 = \gamma + \alpha Y_{n-1}^2 + \beta \sigma_{n-1}^2, \quad n \in \mathbb{Z},$$

where $(\varepsilon_n)_{n\in\mathbb{Z}}$ are iid and $\alpha, \beta, \gamma > 0$. Rewriting

$$\sigma_{n+1}^2 = \gamma + (\alpha\varepsilon_n^2 + \beta)\sigma_n^2, \quad n \in \mathbb{Z},$$

it becomes clear that a stationary and causal solution exists if and only if $E\log(\alpha\varepsilon_1^2 + \beta) < 0$. The following result on the tail behaviour is included in Kesten's seminal work; see [8] for further results and references. The calculation of the extremal index can be found in [25, 18].

Theorem 3 (Tail behaviour and extremes of GARCH(1,1)). *Assume the GARCH(1,1) model such that* ε_1 *is symmetric and has a positive density on* $\mathbb{R}$ *such that* $E(|\varepsilon_1|^h) < \infty$ *for* $h < h_0$ *and* $E(|\varepsilon_1|^{h_0}) = \infty$ *for* $h \geq h_0$ *for some* $h_0 \in (0, \infty]$*. Then there exist unique* $\kappa > 0$ *and* $c > 0$ *such that*

$$E(\alpha\varepsilon_1^2 + \beta)^{\kappa/2} = 1$$

and the stationary distributions have tails for $x \to \infty$

$$P(\sigma_1 > x) \sim cx^{-\kappa} \quad \text{and} \quad P(Y_1 > x) \sim \frac{1}{2} cE(|\varepsilon_1|^\kappa)x^{-\kappa} .$$

The extremal indices of $(\sigma_n)_{n\in\mathbb{Z}}$ *and* $(|Y_n|)_{n\in\mathbb{Z}}$ *are given by*

$$\theta_\sigma = \int_1^\infty P\Big(\sup_{n\geq 1} \prod_{j=1}^n (\alpha\varepsilon_j^2 + \beta) \leq y^{-1}\Big) \frac{\kappa}{2} y^{-(\kappa/2)-1} dy \in (0,1) \quad \text{and}$$

$$\theta_{|Y|} = \frac{E(|\varepsilon_1|^\kappa - \sup_{m\geq 1} |\varepsilon_{m+1}|^\kappa \prod_{j=1}^m (\alpha\varepsilon_j^2 + \beta)^{\kappa/2})^+}{E|\varepsilon_1|^\kappa} \in (0,1),$$

respectively. Also the extremal index $\theta_Y \in (0,1)$*, and for all three sequences* σ*,* $|Y|$ *and* Y*, the extreme value distribution* G *appearing in* (1) *and* (2) *is the Fréchet distribution* Φ_κ*.*

We conclude that the GARCH(1,1) process is able to model clusters in the extremes. Extensions to higher order GARCH processes are given in [3, 8].

3 Continuous-time models

While for discrete time volatility models many results on the extremal behaviour are formulated for both the volatility process and the log price increments process, with few exceptions, most of the literature on extremes for continuous time models concentrates on the volatility process. Hence in the following we will often state results concerning the volatility process only.

3.1 The volatility model of Wiggins

In the volatility model of Wiggins [30], see also [27], the log volatility is modelled as a Gaussian Ornstein-Uhlenbeck process. More precisely, the log price increments Y_t and the volatility σ_t are given by

$$Y_t = \int_{(t-1,t]} \sigma_{s-}\, dB_s, \quad d\log\sigma_t^2 = (b_1 - b_2 \log\sigma_t^2)\, dt + \delta\, dW_t, \quad t \in \mathbb{R}, \tag{10}$$

with two independent standard Brownian motions B and W, and real constants b_1, b_2 and $\delta \neq 0$. The volatility has a stationary solution if and only if $b_2 > 0$, in which case it is given by

$$\log \sigma_t^2 = \int_{-\infty}^{t} e^{-b_2(t-s)}(b_1\, ds + \delta\, dW_s), \quad t \in \mathbb{R}.$$

Sampling the log volatility at integer points results in a causal Gaussian AR(1) process, so that the Euler type approximation $\overline{Y}_n := \sigma_{n-1}(B_n - B_{n-1})$ to (10) is the discrete time volatility model (4) of Taylor.

From the above representation it is clear that $\log \sigma_t^2$ is $N(b_1/b_2, \delta^2/(2b_2))$ distributed. The extremal index function of $\log \sigma^2$ and hence σ^2 follows from results in [24] as shown in [13].

Theorem 4 (Extremal index function of the volatility).
Under the assumptions above, the extremal index function $\theta_\sigma(h)$ for $h \in (0,\infty)$ of the stationary volatility process σ^2 in (10) is identical 1.

We conclude that the volatility in the model (10) does not allow for extremal clusters. This continues to hold if the Gaussian Ornstein-Uhlenbeck process for the log volatility in (10) is replaced by any Gaussian process with continuous sample paths satisfying (7). While it is easy to show that the stationary log price increment Y_1 in (10) is distributed as $\big(\int_0^1 \sigma_t^2\, dt\big)^{1/2} \varepsilon_1$ with ε_1 standard normally distributed and independent of $(\sigma_t)_{t\in\mathbb{R}}$, we are not aware of any explicit expressions for the tail behaviour and the extremal index function of Y_1 or $\log Y_1^2$ as in Theorem 1(a).

3.2 The Barndorff-Nielsen and Shephard (BNS) model

In [1, 2] Barndorff-Nielsen and Shephard model the volatility process as a Lévy driven Ornstein-Uhlenbeck (OU) process, which results in the model

$$Y_t = \int_{(t-1,t]} \sigma_{s-} dB_s\,, \quad \sigma_t^2 = \int_{-\infty}^{t} e^{-\lambda(t-s)} dL_{\lambda s}\,, \quad t \in \mathbb{R}\,, \tag{11}$$

where B is Brownian motion, $\lambda > 0$ and L is a Lévy process with increasing sample paths (i.e. a subordinator), independent of B. The volatility process is stationary and satisfies the SDE $d\sigma_t^2 = -\lambda \sigma_t^2\, dt + dL_{\lambda t}$.

The extremal behaviour of this model depends on the driving Lévy process and has been analysed in [13, 14, 15]. For regularly varying noise, as shown in [14] one obtains the following.

Theorem 5 (Tail and extremes for noise in $\mathcal{R}(-\alpha)$).
Consider the stationary BNS-model (11) and assume that $L_1 \in \mathcal{R}(-\alpha)$ with $\alpha > 0$. Then $\sigma_1 \in \mathcal{R}(-2\alpha)$, $Y_1 \in \mathcal{R}(-2\alpha)$ and we have for $x \to \infty$

$$\begin{aligned}
&P(\sigma_1^2 > x) \sim \alpha^{-1} P(L_1 > x)\,,\\
&P(Y_1^2 > x) \sim E(|\varepsilon_1|^{2\alpha}) \left(\frac{(1-e^{-\lambda})^\alpha}{\alpha \lambda^\alpha} + \frac{1}{\lambda^\alpha} \int_0^1 (1-e^{s-\lambda})^\alpha\, ds \right) P(L_1 > x)\,,\\
&P(Y_1 > x) = \frac{1}{2} P(Y_1^2 > x^2),
\end{aligned}$$

respectively, where ε_1 is a standard normal random variable. The extremal index function θ_σ of the volatility process is furthermore given by

$$\theta_\sigma(h) = (h\alpha\lambda)(h\alpha\lambda + 1)^{-1}, \quad h > 0.$$

Observe that for a regularly varying noise process, the tail of σ^2 is of the same order as that of the driving noise process. Also, since $\theta_\sigma(h) < 1$, the process exhibits cluster possiblities. This is in contrast to the case when L has exponential tail as in the next theorem, see [13, 19]:

Theorem 6 (Tail and extremes for exponential type noise).
Consider the stationary BNS-model (11) *as above and assume that L_1 has an exponential type distribution tail:*

$$P(L_1 > x) = c(x) \exp\left\{-\int_0^x (a(y))^{-1}\, dy\right\}, \quad x > 0,$$

where $\lim_{x\to\infty} c(x) = c > 0$ and $a > 0$ is absolutely continuous with $\lim_{x\to\infty} a(x) = \gamma^{-1}$ and $\lim_{x\to\infty} a'(x) = 0$, where $\gamma \in [0,\infty)$. Assume also that $L_1 \in \mathcal{S}(\gamma)$, i.e. $P(L_2 > x) \sim E(e^{\gamma L_1})P(L_1 > x)$ as $x \to \infty$ with $E(e^{\gamma L_1}) < \infty$. Then

$$P(\sigma_1^2 > x) \sim \frac{a(x)}{x} \frac{Ee^{\gamma\sigma_1^2}}{Ee^{\gamma L_1}} P(L_1 > x), \quad x \to \infty.$$

In particular, $P(\sigma_1^2 > x) = o(P(L_1 > x))$ for $x \to \infty$. The extremal index function θ_σ is equal to 1, i.e. $\theta_\sigma(h) = 1$ for all $h > 0$.

Brockwell [6] suggests to model the volatility in (11) by Lévy driven continuous time ARMA (CARMA) processes, with the CAR(1) process being the OU process. As shown in [13, 14, 15], for CARMA processes driven by regularly varying noise processes, clusters occur as in Theorem 5, while for driving Lévy processes as described in Theorem 6, CARMA processes may model clusters or may not, depending on the corresponding kernel function. Todorov and Tauchen [29] suggest to model the volatility by a CARMA(2,1) process with a mixture of gamma distributions as driving noise process. For this model the results presented here do not apply.

3.3 Continuous-time GARCH(1,1) models

As a diffusion limit of GARCH(1,1) processes, Nelson [26] obtained

$$Y_t = \int_{(\max(0,t-1),t]} \sigma_{s-}\, dB_s,\ d\sigma_t^2 = (\beta - \varphi\sigma_t^2)\, dt + \lambda\sigma_t^2\, dW_t, \quad t \geq 0, \quad (12)$$

where B and W are independent standard Brownian motions and $\beta \geq 0, \lambda > 0$ and $\varphi \in \mathbb{R}$ are parameters. It has a strictly stationary solution if and only if $2\varphi/\lambda^2 > -1$ and $\beta > 0$, in which case the marginal distribution is inverse gamma. The two independent driving processes in (12) is in contrast to the situation for discrete time GARCH processes, where price and volatility are both driven by the same noise sequence $(\varepsilon_n)_{n\in\mathbb{Z}}$. Inspired by this, Klüppelberg, Lindner and Maller [23] constructed another continuous time GARCH model, termed COGARCH(1,1), which meets the features of discrete time GARCH better and for which the volatility jumps, unlike for the diffusion limit (12). Let $(L_t)_{t\geq 0}$ be a Lévy process with non-zero Lévy measure and $\eta, \varphi, \beta > 0$ be parameters. Defining the auxiliary Lévy process

$$R_t = \eta t - \sum_{0<s\leq t} \log(1 + \varphi(\Delta L_s)^2)\,, \quad t \geq 0\,,$$

the log price increments Y and the volatility σ are given by

$$Y_t = \int_{(\max(0,t-1),t]} \sigma_{s-}\, dL_s\,,\ \sigma_t^2 = \left(\beta \int_0^t e^{R_{s-}} ds + \sigma_0^2\right) e^{-R_t}\,,\ t \geq 0\,, \tag{13}$$

where σ_0^2 is independent of L. A sufficient condition for strict stationarity of (13) is the existence of some $\kappa > 0$ such that

$$|L_1|^\kappa \log^+ |L_1| < \infty \quad \text{and} \quad E(e^{-R_1\kappa/2}) = 1\,. \tag{14}$$

Observe that the volatility in Nelson's diffusion limit (12) has also a solution (13), with R_t defined by

$$R_t := (\varphi + \lambda^2/2)t - \lambda W_t, \quad t \geq 0.$$

For the stationary choice, we have

$$E(e^{-R_1\kappa/2}) = 1 \quad \text{with} \quad \kappa := 2 + 4\varphi/\lambda^2 > 0. \tag{15}$$

The following result is from [16, 19]:

Theorem 7 (Tail and extremes of continuous time GARCH). *Consider the stationary diffusion limit* (12) *or COGARCH(1,1) process as above with $\kappa > 0$ as given by* (15) *or* (14)*, respectively. Then there exists a constant $c > 0$ such that*

$$P(\sigma_1 > x) \sim cx^{-\kappa}\,, \quad x \to \infty\,.$$

In the case of the GARCH(1,1) process, assume further that there is $d > \max\{1, \kappa\}$ such that $E|L_1|^{2d} < \infty$ with $\kappa > 0$ as defined in (14) *and that L is not the negative of a subordinator. Denote $M_t := B_t$ for the diffusion limit* (12) *and $M_t := L_t$ for the COGARCH(1,1) process. Then*

$$P(Y_1 > x) \sim E\left[\left(\int_0^1 e^{-R_{t-}/2}\, dM_t\right)^+\right]^\kappa P(\sigma_1 > x), \quad x \to \infty,$$

and σ has extremal index function

$$\theta_\sigma(h) = \frac{E(\sup_{0\le t\le h} e^{-R_t\kappa/2} - \sup_{t\ge h} e^{-R_t\kappa/2})^+}{E(\sup_{0\le t\le h} e^{-R_t\kappa/2})} < 1\,, \quad h > 0.$$

The extremal index of the discrete time process $(Y_n)_{n\in\mathbb{N}}$ *of the log price increments at integer times is given by*

$$\theta = \frac{E\left(\left[\left(\int_0^1 e^{-R_{t-}/2} dM_t\right)^+\right]^\kappa - \max_{k\ge 2}\left[\left(\int_{k-1}^k e^{-R_{t-}/2} dM_t\right)^+\right]^\kappa\right)^+}{E\left(\left[\left(\int_0^1 e^{-R_{t-}/2} dM_t\right)^+\right]^\kappa\right)} < 1\,.$$

It follows that both the diffusion limit and the COGARCH(1,1) can model extremal clusters. Since the diffusion limit (12) has continuous sample paths, one can also consider its clustering behaviour via epsilon-upcrossings. Choosing such an approach, the diffusion limit of Nelson does not cluster, as reported in [19]. In particular, both notions of extremal clustering for processes with continuous sample paths do not lead to the same interpretation.

Similar to the definition of COGARCH, a continuous time analogue to the EGARCH process has been proposed in [21]. So far, no analogue to Theorem 2 for the continuous time EGARCH process seems to be available.

References

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