Lévy Integrals and the Stationarity of Generalised Ornstein-Uhlenbeck Processes*

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Abstract

The generalised Ornstein-Uhlenbeck process constructed from a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ is defined as

$$V_t = e^{-\xi_t} \left(\int_0^t e^{\xi_{s-}} d\eta_s + V_0 \right), \quad t \geq 0,$$

where V_0 is an independent starting random variable. The stationarity of the process is closely related to the convergence or divergence of the Lévy integral $\int_0^\infty e^{-\xi_{t-}} d\eta_t$. We make precise this relation in the general case, showing that the conditions are not in general equivalent, though they are for example if ξ and η are independent. Characterisations are expressed in terms of the Lévy measure of (ξ, η). Conditions for the moments of the strictly stationary distribution to be finite are given, and the autocovariance function and the heavy-tailed behaviour of the stationary solution are also studied.

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1 Introduction

Let $(\xi, \eta) = (\xi_t, \eta_t)_{t \geq 0}$ be a bivariate Lévy process with characteristic triplet $(\gamma, \Sigma, \Pi_{\xi,\eta})$ (see Section 2 for a precise formulation). The *generalised Ornstein-Uhlenbeck (O-U) process* $(V_t)_{t\geq 0}$ is defined as

$$V_t = e^{-\xi_t}\left(\int_0^t e^{\xi_{s-}} d\eta_s + V_0\right), \quad t \geq 0, \tag{1.1}$$

where V_0 is a finite random variable (rv), independent of $(\xi_t, \eta_t)_{t\geq 0}$. Special cases of this process have been of importance and application in a wide variety of areas; see, for example, Carmona et al. [4, 5], Donati-Martin et al. [8], Embrechts et al. [11] and their references. Particular applications are in option pricing (e.g. Yor [30, 31]), insurance and perpetuities (e.g. Dufresne [10], Paulsen and Hove [26]), or risk theory (e.g. Kalashnikov and Norberg [16], Klüppelberg and Kostadinova [18], Nyrhinen [22, 23] or Paulsen [24, 25]). The process $(V_t)_{t\geq 0}$ appears naturally when embedding stochastic difference equations in a continuous time process as studied by de Haan and Karandikar [6]. Further properties of $(V_t)_{t\geq 0}$, however, are well understood in only a few special cases.

Of particular interest are questions of stability and stationarity of the generalised O-U process. These are closely related to the convergence or divergence of the Lévy integral $\int_0^\infty e^{\xi_{t-}} d\eta_t$. For example, Carmona et al. [5] showed, in the special case when ξ and η are independent, that V_t is strictly stationary provided the improper integral $\widetilde{V}_\infty := \int_0^\infty e^{-\xi_{t-}} d\eta_t$ is almost surely (a.s.) finite and ξ_t diverges a.s. to ∞ as $t \to \infty$, and that we choose V_0 to have the distribution of $\widetilde{V}_\infty$. (The tilde here is to distinguish $\widetilde{V}_\infty$ from a V_∞ in (2.7) below, which is different in general.) On the other hand, Erickson and Maller [12] give necessary and sufficient conditions, in terms of the characteristics of the process, for the convergence of the Lévy integral, without making any independence assumptions. (See Proposition 2.4 below.) Following these results it is natural to investigate the relationship between the stationarity of the generalised O-U process and the convergence of the Lévy integral in the general case when ξ and η are not necessarily independent.

This is the topic we take up in the present paper. It turns out, somewhat surprisingly, that the two conditions are not in general equivalent, although they are for example when ξ and η are independent. In fact, the a.s. convergence of the integral $\widetilde{V}_\infty$ is sufficient for the existence of a strictly stationary solution $(V_t)_{t\geq 0}$, but even the convergence in distribution as $t \to \infty$ of $\int_0^t e^{-\xi_{s-}} d\eta_s$ is not in general necessary for stationarity of V_t. Nevertheless, we can characterise the strict stationarity of the generalised O-U process in terms of the convergence of another, closely related, Lévy integral and thence in terms of the characteristic triplet of (ξ, η).

Our main result, the necessary and sufficient condition for stationarity, and some related results, are set out in Section 2. Armed with this characterisation, Section 3 takes up the issue of the connection between the stationarity of the generalised O-U process and the convergence of the associated Lévy integral. In Section 4, sufficient conditions for the moments of the strictly stationary distribution to be finite are given, and the autocovariance function of $(V_t)_{t\geq 0}$ is calculated and shown always to decrease exponentially with the lag. In this section we also consider the tail behaviour of the stationary solution

V_∞, showing that it has heavy (Pareto-like) tails under some conditions. Some discussion of related results and examples is in Section 5. Proofs are in Sections 6–8.

2 Stationarity of the Generalised O-U Process

Our setup is as follows. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ with càdlàg paths and $(\xi_0, \eta_0) = (0, 0)$ is defined on Ω with respect to the probability measure P. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the smallest right-continuous filtration for which $(\xi_t, \eta_t)_{t\geq 0}$ is adapted, and such that V_0, a given rv independent of $(\xi_t, \eta_t)_{t\geq 0}$, is $\mathcal{F}_0$-measurable. $\mathbb{F}$ is completed to contain all P-null sets, making it a filtration satisfying the "usual hypotheses".

The Lévy characteristic exponent, $\psi(\theta) := -(1/t) \log E \exp(i\langle\theta, (\xi_t, \eta_t)\rangle)$, can be written in the form:

$$\psi(\theta) = -i\langle\gamma, \theta\rangle + \tfrac{1}{2}\langle\theta, \Sigma\theta\rangle + \iint_{|(x,y)|\leq 1} \left(1 - e^{i\langle(x,y),\theta\rangle} + i\langle(x, y), \theta\rangle\right) \Pi_{\xi,\eta}(dx, dy) + \iint_{|(x,y)|>1} \left(1 - e^{i\langle(x,y),\theta\rangle}\right) \Pi_{\xi,\eta}(dx, dy), \quad \text{for } \theta \in \mathbb{R}^2. \tag{2.1}$$

In (2.1), the $\langle\cdot, \cdot\rangle$ denotes inner product in $\mathbb{R}^2$, $|\cdot|$ is the Euclidian distance, γ is a nonstochastic vector in $\mathbb{R}^2$, and Σ is a nonstochastic 2×2 non-negative definite matrix. The Lévy measure, $\Pi_{\xi,\eta}$, is a measure on $\mathbb{R}^2\backslash\{0\}$ satisfying $\int \min(|(x, y)|^2, 1)\, \Pi_{\xi,\eta}(dx, dy) < \infty$. Together, $(\gamma, \Sigma, \Pi_{\xi,\eta})$ forms the *characteristic triplet* of the process. We refer to Bertoin [3] and Sato [28] for basic results and representations concerning Lévy processes.

The component processes ξ_t and η_t are Lévy processes in their own right, with Lévy measures given by

$$\Pi_\xi\{\Lambda\} := \int_{\mathbb{R}} \Pi_{\xi,\eta}\{\Lambda, dy\} \quad \text{and} \quad \Pi_\eta\{\Lambda\} := \int_{\mathbb{R}} \Pi_{\xi,\eta}\{dx, \Lambda\}, \tag{2.2}$$

for Λ a Borel subset of $\mathbb{R} \setminus \{0\}$. We set $(\xi_{s-}, \eta_{s-}) := \lim_{u\uparrow s}(\xi_u, \eta_u)$ for $s > 0$ and use the convention $e^{-\xi_{0-}} := \xi_{0-} := \eta_{0-} := 0$. Denote the Brownian part of $(\xi_t, \eta_t)_{t\geq 0}$ by $(B_{\xi,t}, B_{\eta,t})_{t\geq 0}$. In order to avoid trivialities, throughout the paper we shall always assume that ξ and η are different from the zero process $t \mapsto 0$.

Our analysis focusses on stochastic integrals like $\int_t^u e^{-X_{s-}}\, dY_s$, where X and Y are semimartingales (in fact, they will usually be Lévy processes in this paper), for which we take the definition and properties as in Protter [27]. (As usual, we interpret $\int_t^u$ as the integral over the closed interval $[t, u]$, and write $\int_{t+}^u$ for the integral over $(t, u]$.) We note that, in particular, the stochastic integral $\int_0^t e^{-\xi_{s-}} d\eta_s$ is defined with respect to the filtration $\mathbb{F}$. The symbol "$\stackrel{D}{=}$" will be used to denote equality in distribution of two random variables. Similarly, "$\stackrel{D}{\to}$" and "$\stackrel{P}{\to}$" will denote convergence in distribution and convergence in probability, respectively.

Our main result characterises when stationary versions of the generalised Ornstein-Uhlenbeck process V_t defined in (1.1) exist. To state it, we need to define an auxiliary process $(L_t)_{t\geq 0}$ as follows:

$$L_t := \eta_t + \sum_{0<s\leq t} (e^{-\Delta\xi_s} - 1)\Delta\eta_s - t\,\mathrm{Cov}(B_{\xi,1}, B_{\eta,1}), \quad t \geq 0. \tag{2.3}$$

(Here $\Delta\xi_s = \xi_s - \xi_{s-}$, and similarly for other processes throughout, and $\mathrm{Cov}(B_{\xi,t}, B_{\eta,t}) = E(B_{\xi,t}B_{\eta,t})$ denotes the covariance of $B_{\xi,t}$ and $B_{\eta,t}$, for $t > 0$.) The process L will be shown below (see Proposition 2.3) to be a Lévy process with Lévy measure $\Pi_L := (T\Pi_{\xi,\eta})_{\mathbb{R}\setminus\{0\}}$, where $T(x,y) := e^{-x}y$. In fact, the process $(\xi_t, L_t)_{t\geq 0}$ is a bivariate Lévy process with respect to the filtration $\mathbb{F}$, thus making it amenable to application of the results in [12]. We need some further notation. Define, for $x > 0$, the tail functions

$$\overline{\Pi}_\xi^+(x) := \Pi_\xi((x,\infty)), \quad \overline{\Pi}_\xi^-(x) := \Pi_\xi((-\infty,-x)), \quad \overline{\Pi}_\xi(x) = \overline{\Pi}_\xi^+(x) + \overline{\Pi}_\xi^-(x),$$

and let

$$A_\xi(x) := \max\{\overline{\Pi}_\xi^+(1), 1\} + \int_1^x \overline{\Pi}_\xi^+(z)\,dz, \quad x \geq 1, \tag{2.4}$$

with a similar notation for η, L, and other Lévy processes we will encounter. It is easy to see that $x \mapsto x/A_\xi(x)$ is non-decreasing, as is $x \mapsto A_\xi(x)$. Further, $E\xi_1^+ < \infty$ if and only if $\lim_{x\to\infty} A_\xi(x) < \infty$ (where $\xi_1^+ = \max(0,\xi_1)$). Doney and Maller [9] have shown that $\lim_{t\to\infty} \xi_t = \infty$ a.s. if and only if

$$\int_1^\infty \overline{\Pi}_\xi^+(z)\,dz = \infty \ \text{ and } \int_1^\infty \left(\frac{x}{A_\xi(x)}\right)\left|d\overline{\Pi}_\xi^-(x)\right| < \infty, \quad \text{or} \quad 0 < E\xi_1 \leq E|\xi_1| < \infty. \tag{2.5}$$

Stationarity of $(V_t)_{t\geq 0}$ is characterised in the following theorem, where we use $\mathcal{E}(\cdot)$ to denote the stochastic exponential (as e.g. in Protter [27], page 85).

Theorem 2.1. *Let $(V_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ be as in* (1.1) *and* (2.3)*, respectively. Suppose the process $(V_t)_{t\geq 0}$ is strictly stationary. Then one of the following two conditions (i) or (ii) is satisfied:*

(i) $\int_0^\infty e^{-\xi_{t-}}\,dL_t$ converges a.s. to a finite random variable, or, equivalently, $\lim_{t\to\infty}\xi_t = +\infty$ a.s. and

$$\int_{(e,\infty)} \left(\frac{\log y}{A_\xi(\log y)}\right)\left|d\overline{\Pi}_L(y)\right| < \infty; \tag{2.6}$$

(ii) there is a constant $k \in \mathbb{R}\setminus\{0\}$ such that the process $(V_t)_{t\geq 0}$ is indistinguishable from the constant process $t \mapsto k$ (that is, a.s., $V_t = k\ \forall\, t \geq 0$), or, equivalently, there is a constant $k \in \mathbb{R}\setminus\{0\}$ such that $V_0 = k$ and $e^\xi = \mathcal{E}(\eta/k)$.

Conversely, if (i) or (ii) holds then there is a finite random variable V_∞ (unique in distribution) such that $(V_t)_{t\geq 0}$, started with $V_0 \stackrel{D}{=} V_\infty$, is strictly stationary.
Furthermore, if (i) holds, then the stationary random variable V_∞ satisfies

$$V_\infty \stackrel{D}{=} \int_0^\infty e^{-\xi_{s-}}\,dL_s. \tag{2.7}$$

Remark 2.2. (a) It can be easily checked using the Doléans-Dade formula (e.g. Protter [27], Theorem 37 in Chapter II, page 84), that $e^{\xi} = \mathcal{E}(\eta/k)$ if and only if $\Pi_\xi(\{y \in \mathbb{R} : y/k \le -1\}) = 0$ and

$$\xi_t = k^{-1}\eta_t - k^{-2}\sigma_\eta^2 t/2 + \sum_{0<s\le t} \left(\log(1+k^{-1}\Delta\eta_s) - k^{-1}\Delta\eta_s\right), \quad t > 0, \tag{2.8}$$

where σ_η^2 denotes the lower diagonal element of the matrix Σ appearing in (2.1).
(b) If ξ and η are independent, then they have no jumps in common a.s., in which case we deduce from (2.3) that $L = \eta$. Condition (i) then recovers the sufficient condition of Carmona et al. [5] for stationarity of V_t.
(c) If $0 < E\xi_1 \le E|\xi_1| < \infty$, then $\lim_{x\to\infty} A_\xi(x) < \infty$, and (2.6) is equivalent to $\int_{(1,\infty)} \log y \left|d\overline{\Pi}_L(y)\right| < \infty$. As will follow from Theorem 3.1 and Proposition 2.4 below, this is further equivalent in the case $0 < E\xi_1 \le E|\xi_1| < \infty$ to $\int_{(1,\infty)} \log y \left|d\overline{\Pi}_\eta(y)\right| < \infty$, or equivalently to

$$E \log^+ |\eta_1| < \infty \tag{2.9}$$

(see Sato [28], Theorem 25.3, page 159). That $0 < E\xi_1 \le E|\xi_1| < \infty$ together with (2.9) is *sufficient* for the existence of a strictly stationary solution was already shown by de Haan and Karandikar [6]. From Theorem 2.1 and its proof we can see that if $0 < E\xi_1 \le E|\xi_1| < \infty$, then (2.9) is also *necessary* for the existence of a strictly stationary solution.

Theorem 2.1 states that if $(V_t)_{t\ge0}$ is strictly stationary and not degenerate to a constant process, then ξ_t must diverge a.s. to ∞ as $t \to \infty$. If this is so then $e^{-\xi_t}V_0$ converges a.s. to 0 as $t \to \infty$, and thus, for stationarity, it must be the case that $e^{-\xi_t}\int_0^t e^{\xi_{s-}} d\eta_s$ converges in distribution to V_0 as $t \to \infty$ (see (1.1)). In order to characterise this kind of convergence, our method is to construct $(L_t)_{t\ge0}$ as in (2.3) such that $e^{-\xi_t}\int_0^t e^{\xi_{s-}} d\eta_s \overset{D}{=} \int_0^t e^{-\xi_{s-}} dL_s$, and then apply a criterion for the convergence of the latter integral (Proposition 2.4 below). Key steps in this development are a result on the time-reversal of stochastic integrals (Lemma 6.1 in Section 6) and the following proposition which gives the required connection between the generalised O-U process and a Lévy integral.

We denote the quadratic covariation process of two semimartingales X, Y, by $[X,Y]_t$.

Proposition 2.3. *The process L_t defined in (2.3) is a finite-valued Lévy process; in fact, the process $(\xi_t, L_t)_{t\ge0}$ is a bivariate Lévy process (with respect to the filtration $\mathbb{F}$). Further, for $t > 0$,*

$$\int_0^t e^{-\xi_{s-}} dL_s = \int_0^t e^{-\xi_{s-}} d\eta_s + \left[e^{-\xi}, \eta\right]_t \overset{D}{=} e^{-\xi_t}\int_0^t e^{\xi_{s-}} d\eta_s. \tag{2.10}$$

Let the mappings $S, T : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$S(x,y) = e^x y, \quad T(x,y) = e^{-x}y, \quad x, y \in \mathbb{R}.$$

Then $\Pi_L = (T\Pi_{\xi,\eta})_{|\mathbb{R}\setminus\{0\}}$; that is, the Lévy measure Π_L of L is the restriction to $\mathbb{R}\setminus\{0\}$ of the image measure of $\Pi_{\xi,\eta}$ under the mapping T. Furthermore, if $\Pi_{\xi,L}$ denotes the Lévy measure of (ξ, L), then $\Pi_\eta = (S\Pi_{\xi,L})_{|\mathbb{R}\setminus\{0\}}$.

The convergence of the integral $\int_0^\infty e^{-\xi_{t-}} dL_t$ can be characterised using Theorem 2 of [12], of which the following proposition is a consequence. Since we wish to apply it in a couple of situations, we phrase it in terms of a separate Lévy process (ζ, χ), to which our usual notations and assumptions apply.

Proposition 2.4. *The Lévy integral $\int_0^t e^{-\zeta_{s-}} d\chi_s$ constructed from any bivariate Lévy process (ζ, χ), with bivariate Lévy measure $\Pi_{\zeta,\chi}$, converges a.s. to a finite random variable as $t \to \infty$ if and only if*

$$\lim_{t\to\infty} \zeta_t = +\infty \ \ a.s., \ \ and \ \ I_{\zeta,\chi} := \int_{(e,\infty)} \left(\frac{\log y}{A_\zeta(\log y)} \right) \left| d\overline{\Pi}_\chi(y) \right| < \infty. \tag{2.11}$$

In the case of divergence, we have: suppose $\lim_{t\to\infty} \zeta_t = +\infty$ a.s. but $I_{\zeta,\chi} = \infty$. Then

$$\left| \int_0^t e^{-\zeta_{s-}} d\chi_s \right| \stackrel{P}{\to} \infty \ \ as \ \ t \to \infty. \tag{2.12}$$

If on the other hand ζ_t does not tend to $+\infty$ a.s. as $t \to \infty$, then (2.12) holds, or there exists a constant $k \in \mathbb{R} \setminus \{0\}$ such that

$$P \left(\int_0^t e^{-\zeta_{s-}} d\chi_s = k(1 - e^{-\zeta_t}) \ \ \forall\, t > 0 \right) = 1. \tag{2.13}$$

Consequently, $\int_0^t e^{-\xi_{s-}} \, d\chi_s$ converges in distribution to a finite random variable as $t \to \infty$ if and only if it converges a.s. to a finite random variable.

Remark 2.5. (a) The equivalence of a.s. and distributional convergence of $\int_0^t e^{-\zeta_{s-}} \, d\chi_s$ as $t \to \infty$ was not stated explicitly in [12], but follows immediately from (2.12) and (2.13). A similar equivalence of convergence in distribution and a.s. convergence was noted by Grincevičius [15], Corollary 1 to Theorem 1, in the discrete case.
(b) Conditions (i) and (ii) in Theorem 2.1 are not exclusive, as is shown by taking $\xi_t = \eta_t = t$ and $V_0 = 1$. However, (ii) can also happen when ξ_t does not tend to $+\infty$ a.s., for example when $\eta_t = -\xi_t = t$ and $V_0 = -1$, or $\eta_t = \xi_t + t/2$, $V_0 = 1$, and ξ is standard Brownian motion. Further non-trivial examples, including processes with non-trivial Lévy measure, can be easily constructed using (2.8).

3 Stationarity versus Convergence

The characterisation of stationarity in Theorem 2.1 relies heavily on the characterisation of a.s. convergence of the integral $\int_0^\infty e^{-\xi_{t-}} dL_t$, with $L = (L_t)_{t\geq 0}$ being defined as in Equation (2.3). When ξ and η are independent, we have $L = \eta$ by Remark 2.2 (b), so in that case $\int_0^\infty e^{-\xi_{s-}} dL_s$ converges a.s. if and only if $\int_0^\infty e^{-\xi_{s-}} d\eta_s$ converges a.s. It is natural to ask if this is true without any independence assumptions. We discuss this in Theorem 3.1, showing that the a.s. convergence of $\int_0^\infty e^{-\xi_{t-}} d\eta_t$ implies that of $\int_0^\infty e^{-\xi_{t-}} dL_t$, but the converse is true only if $E\xi_1^+ < \infty$. Then, in Theorem 3.3 we specialise to the case $\eta = \xi$, showing that, then, $\lim_{t\to\infty} \xi_t = +\infty$ a.s. always implies the existence of a strictly stationary solution of the generalised Ornstein-Uhlenbeck process, but that $\int_0^\infty e^{-\xi_{s-}} d\xi_s$ need not converge a.s., in general, even when $\lim_{t\to\infty} \xi_t = +\infty$ a.s.

Theorem 3.1. *With $(L_t)_{t\geq 0}$ as in Equation (2.3), the following holds:*
(a) If $\int_0^\infty e^{-\xi_{t-}}d\eta_t$ converges a.s., then so does $\int_0^\infty e^{-\xi_{t-}}dL_t$, and hence a stationary version of $(V_t)_{t\geq 0}$ exists.
(b) If $E\xi_1^+ < \infty$ and $\int_0^\infty e^{-\xi_{t-}}dL_t$ converges a.s., then $\int_0^\infty e^{-\xi_{t-}}d\eta_t$ converges a.s.
(c) For every Lévy process $\xi = (\xi_t)_{t\geq 0}$ such that $\lim_{t\to\infty}\xi_t = \infty$ a.s. and $E\xi_1^+ = \infty$, there exists a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ with margin ξ such that $\int_0^\infty e^{-\xi_{t-}}dL_t$ converges a.s., but $\int_0^\infty e^{-\xi_{t-}}d\eta_t$ does not.

Remark 3.2. Proposition 2.4 shows that only the marginal Lévy measures of ξ and η are significant in determining the convergence or otherwise of the Lévy integral $\int_0^\infty e^{-\xi_{t-}}d\eta_t$. By Theorem 3.1, convergence of this integral is sufficient to establish the strict stationarity of V_t, but to calculate Π_L via Proposition 2.3, and the stationary random variable via (2.7), knowledge of the whole bivariate measure $\Pi_{\xi,\eta}$ is required, in general.

We shall pay special attention to the case when $\eta_t = \xi_t \neq 0$.

Theorem 3.3. *(a) Let $\xi = (\xi_t)_{t\geq 0}$ be a one-dimensional Lévy process. Then the generalised Ornstein-Uhlenbeck process*

$$V_t = e^{-\xi_t}\left(\int_0^t e^{\xi_{s-}}d\xi_s + V_0\right), \quad t \geq 0,$$

admits a strictly stationary solution if and only if $\lim_{t\to\infty}\xi_t = +\infty$ a.s., or ξ is of the form $\xi_t = at + bN_t$, where $(N_t)_{t\geq 0}$ is a Poisson process and a and b are real constants subject to the constraint $(e^b - 1 - b)a = 0$.
(b) There exists a Lévy process ξ such that $\lim_{t\to\infty}\xi_t = +\infty$ a.s. but the integral $\int_0^t e^{-\xi_{s-}}d\xi_s$ does not converge (a.s. or in distribution) to a finite random variable as $t \to \infty$.

Remark 3.4. When $\lim_{t\to\infty}\xi_t = +\infty$ a.s., the integral $\int_0^\infty e^{-\xi_{t-}}d\xi_t$ converges a.s. for a large class of distributions; sufficient is that $\Pi_\xi(\cdot)$ have a finite logarithmic moment. The counterexample used to demonstrate part (b) of Theorem 3.3 thus involves an extremely heavy tailed distribution for the margins of ξ.

4 Second Order Behaviour and Tail Behaviour

As before, let (ξ, η) be a bivariate Lévy process and consider the generalised Ornstein-Uhlenbeck process $(V_t)_{t\geq 0}$, as given by (1.1). In this section we shall determine the autocovariance function of $(V_t)_{t\geq 0}$ in a convenient form. Before this we give a sufficient condition for the existence of the moments of the stationary version. Let $\kappa \geq 0$. Provided that $Ee^{-\kappa\xi_1}$ is finite (or, equivalently, $Ee^{\kappa\xi_1^-}$ is finite), set

$$\Psi_\xi(\kappa) = \log Ee^{-\kappa\xi_1}.$$

Then $Ee^{-\kappa\xi_t} = e^{t\Psi_\xi(\kappa)}$ is finite for all $t \geq 0$, see Sato [28], Theorem 25.17, page 165.

Proposition 4.1. *Fix $\kappa > 0$, and assume that there are $p, q > 1$ with $1/p + 1/q = 1$ such that*

$$Ee^{-\max\{1,\kappa\}p\xi_1} < \infty \quad \text{and} \quad E|\eta_1|^{\max\{1,\kappa\}q} < \infty. \tag{4.1}$$

Suppose further that $\Psi_\xi(\kappa) < 0$. Let L be defined as in Equation (2.3). *Then* $\lim_{t\to\infty} \xi_t = +\infty$ *a.s., the integral* $\int_0^\infty e^{-\xi_{s-}} dL_s$ *converges a.s. to a finite random variable, and the stationary version of the generalised Ornstein-Uhlenbeck process* $(V_t)_{t\geq 0}$ *satisfies*

$$E|V_0|^\kappa = E\left|\int_0^\infty e^{-\xi_{s-}} dL_s\right|^\kappa < \infty.$$

Remark 4.2. The integrability conditions in (4.1) are easily expressed in terms of the Lévy measures of ξ and η; see, e.g., Sato [28], Sect. 25.

Next, we shall show that the autocovariance function of $(V_t)_{t\geq 0}$ behaves like an exponential function:

Theorem 4.3. *Let $(V_t)_{t\geq 0}$ be the generalised Ornstein-Uhlenbeck process (not necessarily stationary) constructed from a bivariate Lévy process (ξ, η) as in* (1.1). *Let $0 \leq y < t$ and suppose that $\Psi_\xi(1), \mathrm{Var}(V_y)$ and $\mathrm{Var}(V_t)$ are all finite. Then*

$$\mathrm{Cov}(V_y, V_t) = (\mathrm{Var} V_y) e^{(t-y)\Psi_\xi(1)}. \tag{4.2}$$

In particular, if the assumptions of Proposition 4.1 hold with $\kappa = 2$, and if $(V_t)_{t\geq 0}$ is the stationary version, then

$$\mathrm{Cov}(V_t, V_{t+h}) = (\mathrm{Var} V_0) e^{-h|\Psi_\xi(1)|}, \quad t, h \geq 0. \tag{4.3}$$

Remark 4.4. It should be observed that by (4.2) and (4.3) the autocorrelation function of the stationary, non-degenerate generalised O-U process depends only on ξ and not on η (provided it exists). The autocovariance function, however, depends also on η through $\mathrm{Var}\, V_0$ in (4.3).

Next we consider the tail behaviour of the stationary distribution. As in many studies in this area (c.f. e.g. de Haan et al. [7], Klüppelberg et al. [20], Rivero [29]), we apply results of Goldie [13] and Kesten [17] to deduce heavy tailed behaviour. A similar result in the special case when η is a compound Poisson process with drift and independent of ξ has been obtained by Klüppelberg and Kostadinova [18].

Theorem 4.5. *Let (ξ, η) be a bivariate Lévy process. Suppose there is a constant $\kappa > 0$ such that $\Psi_\xi(\kappa) = \log Ee^{-\kappa\xi_1} = 0$, and constants $p, q > 1$ with $1/p + 1/q = 1$ such that* (4.1) *holds. In the case when ξ is of finite variation, assume additionally that the drift of ξ is non-zero, or that there is no $r > 0$ such that the support of the Lévy measure of ξ is concentrated on $r\mathbb{Z}$. Then there exists a stationary solution $(V_t)_{t\geq 0}$ of the generalised Ornstein-Uhlenbeck process, and constants $C_+ \geq 0$, $C_- \geq 0$ such that*

$$\lim_{x\to\infty} x^\kappa P(V_0 > x) = C_+ \quad \text{and} \quad \lim_{x\to\infty} x^\kappa P(V_0 < -x) = C_-, \tag{4.4}$$

and thus $\lim_{x\to\infty} x^\kappa P(|V_0| > x) = C_+ + C_-$. *If $(V_t)_{t\geq 0}$ is not degenerate to a constant process, then $C_+ + C_- > 0$, and in particular, $E|V_0|^\kappa = \infty$.*

The question of which one of C_+ or C_- is strictly positive, or whether both are, in the situation of Theorem 4.5, is subtle. While Goldie [13] gives explicit expressions for C_+ and C_- in his Theorem 2.3, it is not easy to decide from these whether C_+ and C_- are strictly positive or not in our situation. Similar questions arise for example in studies on the asymptotic behaviour of ruin probabilities in the generalised Ornstein-Uhlenbeck model, and have been investigated in some depth by Kalashnikov and Norberg [16], Nyrhinen [22, 23] and Paulsen [25]; see especially the discussion on page 267 of [23]. These results concern the passage time above a high level of processes like our V_t (in the discrete or continuous time cases), and do not appear to relate directly to the tail behaviour of the stationary distribution of V_∞, although their methods may be adapted for use in this situation. Here, we shall content ourselves with a couple of simple sufficient conditions, which nevertheless cover some useful cases, ensuring strict positivity of C_+.

Corollary 4.6. *Let the assumptions of Theorem 4.5 be satisfied, and let $(V_t)_{t\ge 0}$ be the stationary version. Assume additionally that either*

(i) η is a subordinator,

or

(ii) (ξ,η) is symmetric in the sense that $(\xi_t,\eta_t)_{t\ge0} \stackrel{D}{=} (\xi_t,-\eta_t)_{t\ge0}$, at least one of ξ and η does not have a Brownian part, and $(V_t)_{t\ge0}$ is not degenerate to a constant process. Then the constant C_+ in Proposition 4.5 is strictly positive.

5 Discussion and Examples

In this section we first discuss related discrete time results. Then, we consider some special cases where ξ or η are linear or Brownian motion.

Discrete Time Results

In [12] the Lévy integral $\int_0^t e^{-\xi_{s-}}\,d\eta_s$ is related back to a discrete time "perpetuity" of the form

$$Z_n := \sum_{i=1}^n \Pi_{i-1} Q_i + \Pi_n Z_0, \quad n = 1, 2, \ldots,$$

where $\Pi_i = \prod_{j=1}^i M_j$ ($\prod_{j=k}^{\ell} = 1$ when $k > \ell$) and the $(M_i, Q_i)_{i\ge1}$ are i.i.d. random 2-vectors, with Q_i not necessarily independent of M_i, and Z_0 independent of $(M_i, Q_i)_{i\ge1}$. Suppose $P(Q_1 = 0) < 1$ and $P(M_1 \le 0) = 0$. A corresponding version of a *discrete time generalised Ornstein-Uhlenbeck process* would be

$$\tilde Z_n := \Pi_n \left(\sum_{i=1}^n \left(\prod_{j=1}^{i-1} M_j^{-1} \right) Q_i + \tilde Z_0 \right), \quad n = 1, 2, \ldots,$$

with $\tilde Z_0$ independent of $(M_i, Q_i)_{i\ge1}$. Thinking, by way of analogy, of $-\log M_i$ as an increment of ξ, and of Q_i as an increment of η, then comparing with (1.1), shows the reason for our suggested nomenclature. Writing $\tilde Z_n$ as

$$\tilde Z_n = \sum_{i=1}^n \left(\prod_{j=i+1}^n M_j \right) M_i Q_i + \Pi_n \tilde Z_0, \quad n = 1, 2, \ldots,$$

and assuming $\tilde{Z}_0 \stackrel{D}{=} Z_0$, we see that $\tilde{Z}_n$ has the same marginal distribution as Z_n, but with Q_i replaced by $M_i Q_i$ in the latter. Note that $(\tilde{Z}_n)_{n\geq 1}$ is Markov whereas $(Z_n)_{n\geq 1}$ is not.

Convergence criteria for these kinds of discrete time processes were given in [14], which paper can also be consulted for background information and references.

Corresponding to Theorem 2.1, and using similar reasoning applied to Theorem 2.1 and Remark 2.3 of [14], we obtain: *$\tilde{Z}_n$ is a strictly stationary sequence if and only if (i) $Z_\infty = \sum_{i\geq 1} \Pi_i Q_i$ is an a.s. finite rv, and $\tilde{Z}_0 \stackrel{D}{=} \tilde{Z}_\infty$; or (ii) $(\widetilde{Z}_n)_{n=0,1,\ldots}$ is degenerate to a constant process $\widetilde{Z}_n = \widetilde{Z}_0 = c$ for all $n = 0, 1, \ldots,$ where $c \in \mathbb{R}$.* (Actually, we need only assume $P(M_1 = 0) = 0$ for this.) Corresponding discrete time versions of some of our other results can similarly be written down; we omit further details.

Special Cases: ξ or η linear

The special cases of the general Ornstein-Uhlenbeck process when $(\xi_t)_{t\geq 0}$ or $(\eta_t)_{t\geq 0}$ is a deterministic linear function are of particular importance. For example, when $\xi_t = t$ and $(\eta_t)_{t\geq 0}$ is a subordinator, we obtain (apart from a timing constant λ) the volatility process of Barndorff-Nielsen and Shephard [1, 2]. Equation (2.9) then recovers the well-known stationarity condition for this process (see e.g. Sato [28], Theorems 17.5 and 17.11, pp. 108, 113). The exponential decrease of the autocorrelation function, known in this case, can be recovered from our Theorem 4.3 (see also [2], page 172). Our Theorem 4.5, however, is not applicable to this situation, since $\Psi_\xi(\kappa) = -\kappa \neq 0$ for all $\kappa > 0$. Indeed, in that case the stationary distributions can exhibit various kinds of light or heavy tailed behaviour.

If, on the other hand, $\eta_t = t$ is deterministic, we obtain a process related to the exponential functional process $t \mapsto \int_0^t e^{-\xi_s} ds$; see Carmona et al. [4, 5]. If $(\xi_t)_{t\geq 0}$ is spectrally negative and has a positive drift, (1.1) gives rise to the volatility of the "COGARCH" process of Klüppelberg et al. [19, 20]. Thus our results apply to give practically useful conditions in these models.

Special Cases: ξ or η Brownian motion

Examples when ξ or η is a Brownian motion are also of importance, for example in mathematical finance, and have been widely studied. If ξ is linear and η is a Brownian motion, then $(V_t)_{t\geq 0}$ reduces to the classical Ornstein-Uhlenbeck process, while the case when ξ is a Brownian motion with drift and η is linear has applications to Asian options, see e.g. Donati-Martin et al. [8] and references therein.

If η is a Brownian motion (with or without drift), then Theorem 2.1 shows that the generalised O-U process constructed from (ξ, η) admits a stationary solution if and only if V_0 can be chosen such that $(V_t)_{t\geq 0}$ is degenerate to a constant process (i.e. by (2.8) there is a constant $k \in \mathbb{R} \setminus \{0\}$ such that $\xi_t = k^{-1}\eta_t - k^{-2}\sigma_\eta^2 t/2$, $t \geq 0$), or if $\lim_{t\to\infty} \xi_t = \infty$ a.s. On the other hand, if ξ is a Brownian motion with drift, then a stationary solution exists if and only if V_0 can be chosen such that $(V_t)_{t\geq 0}$ is degenerate to a constant process, or if the drift of ξ is strictly positive and $E \log^+ |\eta_1| < \infty$ (see Remark 2.2).

6 Proofs for Section 2

We first prove Proposition 2.3 and then Theorem 2.1. We need the following lemma:

Lemma 6.1. *Let $(\xi_t, \eta_t)_{t\geq 0}$ be a bivariate Lévy process. For fixed $t > 0$ set $\widehat{\xi}_s := \xi_t - \xi_{(t-s)-}$ and $\widehat{\eta}_s := \eta_t - \eta_{(t-s)-}$ for $0 \leq s \leq t$, so that $(\widehat{\xi}_s, \widehat{\eta}_s)_{0\leq s\leq t} \stackrel{D}{=} (\xi_s, \eta_s)_{0\leq s\leq t}$. Then*

$$e^{-\xi_t} \int_0^t e^{\xi_{s-}}\, d\eta_s = \int_0^t e^{-\widehat{\xi}_{s-}}\, d\widehat{\eta}_s + \left[e^{-\widehat{\xi}}, \widehat{\eta}\right]_t \quad a.s., \tag{6.1}$$

where the integral on the right hand side is taken with respect to the completed natural filtration $\mathbb{H} = (\mathcal{H}_s)_{0\leq s\leq t}$ of $(\widehat{\xi}_s, \widehat{\eta}_s)_{0\leq s\leq t}$. As a consequence, we have

$$e^{-\xi_t} \int_0^t e^{\xi_{s-}}\, d\eta_s \stackrel{D}{=} \int_0^t e^{-\xi_{s-}}\, d\eta_s + \left[e^{-\xi}, \eta\right]_t. \tag{6.2}$$

Proof of Lemma 6.1: Fix $t > 0$. For $0 \leq s \leq t$ define

$$H_s := e^{-\widehat{\xi}_s}, \quad Y_s := \widehat{\eta}_s \quad \text{and} \quad X_s := \int_0^s H_{u-}\, dY_u = \int_{0+}^s e^{-\widehat{\xi}_{u-}}\, d\widehat{\eta}_u,$$

where the integral is taken with respect to $\mathbb{H}$. Further, for a given càdlàg process $(Z_s)_{0\leq s\leq t}$, define the time-reversed process $(\widetilde{Z}_s)_{0\leq s\leq t}$ by

$$\widetilde{Z}_s := \begin{cases} 0, & s = 0 \\ Z_{(t-s)-} - Z_{t-}, & 0 < s < t \\ Z_0 - Z_{t-}, & s = t. \end{cases}$$

Then $(\widetilde{Z}_s)_{0\leq s\leq t}$ is a càdlàg process. With these definitions, we have

$$\begin{aligned} \widetilde{Y}_s &= \widetilde{\widehat{\eta}}_s = \begin{cases} -\eta_s, & 0 \leq s < t, \\ -\eta_{t-}, & s = t, \end{cases} \\ H_{t-s} &= e^{-\xi_t + \xi_{s-}}, \quad 0 \leq s \leq t, \\ \widetilde{X}_t &= -X_{t-} = -\int_{0+}^{t-} e^{-\widehat{\xi}_{s-}}\, d\widehat{\eta}_s, \\ \widetilde{[H,Y]}_t &= [H,Y]_0 - [H,Y]_{t-} = -\left[e^{-\widehat{\xi}}, \widehat{\eta}\right]_{t-}. \end{aligned}$$

Recall that $t > 0$ is fixed and denote by $\mathbb{G} = (\mathcal{G}_s)_{0\leq s\leq t}$ the smallest filtration containing $(\mathcal{F}_s)_{0\leq s\leq t}$ and such that (ξ_t, η_t) is $\mathcal{G}_0$-measurable. Now $(H_s)_{0\leq s\leq t}$ is $\mathbb{H}$-adapted, and $(\widehat{\xi}_s, \widehat{\eta}_s)_{0\leq s\leq t}$ is an $\mathbb{H}$-semimartingale, implying that $(\widehat{\eta}_s)_{0\leq s\leq t}$ is an $\mathbb{H}$-semimartingale. Since $(\xi_s, \eta_s)_{0\leq s\leq t}$ is a Lévy process, it follows from Protter [27], Theorem 3 in Chapter VI, page 356, that it is a $\mathbb{G}$-semimartingale. Thus $(\widetilde{Y}_s)_{0\leq s<t} = (-\eta_s)_{0\leq s<t}$ is a $\mathbb{G}$-semimartingale, showing that $(Y_s)_{0\leq s\leq t}$ is an $(\mathbb{H}, \mathbb{G})$ reversible semimartingale, see Protter [27], page 378. Further, since ξ_t and $\xi_{(t-s)-}$ are both in $\mathcal{G}_{t-s}$, it follows that H_s is $\mathcal{G}_{t-s}$ measurable, $0 \leq s \leq t$. Thus, all assumptions of Theorem 22 in Chapter VI of [27], page 378, are fulfilled, and we obtain a.s.

$$\widetilde{X}_u + \widetilde{[H,Y]}_u = \int_0^u H_{t-s}\, d\widetilde{Y}_s, \quad 0 \leq u \leq t,$$

giving a.s.

$$\int_0^{t-} e^{-\widehat{\xi}_{s-}}\, d\widehat{\eta}_s + \left[e^{-\widehat{\xi}}, \widehat{\eta}\right]_{t-} = -\int_0^t e^{-\xi_t+\xi_{s-}}\, d\widetilde{\widehat{\eta}}_s = \int_0^{t-} e^{-\xi_t+\xi_{s-}}\, d\eta_s = e^{-\xi_t}\int_0^{t-} e^{\xi_{s-}}\, d\eta_s,$$

where the last equation follows from the fact that ξ_t is $\mathcal{G}_0$-measurable. Note that the integral $\int_0^{t-} e^{\xi_{s-}} d\eta_s$ is the same when taken either with respect to the filtration $\mathbb{G}$ or to the filtration $\mathbb{F}$, see [27], Theorem 16 in Chapter II, page 61. This proves (6.1), since for fixed t, $\Delta\eta_t = \Delta\widehat{\eta}_t = 0$ a.s. Equation (6.2) then follows from the fact that $(\widehat{\xi}_s, \widehat{\eta}_s)_{0\le s\le t}$ has the same distribution as $(\xi_s, \eta_s)_{0\le s\le t}$. □

We can now establish Proposition 2.3.

Proof of Proposition 2.3: With L_t defined by (2.3), it is easy to check that $(\xi_t, L_t)_{t\ge 0}$ has independent and stationary increments, is stochastically continuous, starts at $(0,0)$ a.s. and has càdlàg paths a.s., so is a Lévy process (more precisely, a Lévy process with respect to the filtration $\mathbb{F}$). For convenience define

$$R_t := \sum_{0<s\le t} (e^{-\Delta\xi_s} - 1)\Delta\eta_s, \quad t \ge 0. \tag{6.3}$$

Then it is also easy to see that $(R_t)_{t\ge 0}$ is a Lévy process of finite variation. From this and the fact that both R and $\eta - B_\eta$ are quadratic pure jump processes, it follows that

$$\begin{aligned} \int_0^t e^{-\xi_{s-}} dR_s &= \sum_{0<s\le t} e^{-\xi_{s-}}\Delta R_s \\ &= \sum_{0<s\le t} e^{-\xi_{s-}}(e^{-\Delta\xi_s} - 1)\Delta\eta_s \\ &= \sum_{0<s\le t} \Delta(e^{-\xi_s})\Delta\eta_s = [e^{-\xi}, \eta - B_\eta]_t \\ &= [e^{-\xi}, \eta]_t - [e^{-\xi}, B_\eta]_t. \end{aligned} \tag{6.4}$$

An application of Itô's formula shows that

$$e^{-\xi_t} = -\int_0^t e^{-\xi_{s-}}\, d\xi_s + F_t, \quad t \ge 0,$$

where $(F_t)_{t\ge 0}$ is a process of finite variation on compacts. Hence we obtain

$$\begin{aligned} \left[e^{-\xi}, B_\eta\right]_t &= \left[-\int_0^{\cdot} e^{-\xi_{s-}}\, d\xi_s, B_\eta\right]_t + [F, B_\eta]_t \\ &= -\int_0^t e^{-\xi_{s-}}\, d[\xi, B_\eta]_s + 0 = -\int_0^t e^{-\xi_{s-}} d(\mathrm{Cov}(B_{\xi,1}, B_{\eta,1})s). \end{aligned} \tag{6.5}$$

According to (2.3),

$$\int_0^t e^{-\xi_{s-}}\, dL_s = \int_0^t e^{-\xi_{s-}}\, d\eta_s + \int_0^t e^{-\xi_{s-}}\, dR_s - \int_0^t e^{-\xi_{s-}}\, d(\mathrm{Cov}(B_{\xi,1}, B_{\eta,1})s), \quad t \ge 0,$$

so (2.10) follows from (6.4), (6.5) and (6.2).

Let Λ be a Borel subset of $\mathbb{R}$ such that 0 is not contained in the closure of Λ. Then it follows (see [27], page 26) that

$$\begin{aligned}\Pi_L(\Lambda) &= E\sum_{0<s\leq 1} 1_{\Delta L_s\in\Lambda} = E\sum_{0<s\leq 1} 1_{e^{-\Delta\xi_s}\Delta\eta_s\in\Lambda}\\ &= E\sum_{0<s\leq 1} 1_{(\Delta\xi_s,\Delta\eta_s)\in T^{-1}(\Lambda)} = \Pi_{\xi,\eta}(T^{-1}(\Lambda)),\end{aligned}$$

showing that $\Pi_L = (T\Pi_{\xi,\eta})_{|\mathbb{R}\setminus\{0\}}$. Similarly, noting that $\Delta\eta_s = e^{\Delta\xi_s}\Delta L_s$, we obtain

$$\Pi_\eta(\Lambda) = E\sum_{0<s\leq 1} 1_{\Delta\eta_s\in\Lambda} = E\sum_{0<s\leq 1} 1_{(\Delta\xi_s,\Delta L_s)\in S^{-1}(\Lambda)} = \Pi_{(\xi,L)}(S^{-1}(\Lambda)),$$

showing $\Pi_\eta = (S\Pi_{(\xi,L)})_{|\mathbb{R}\setminus\{0\}}$. □

Before proving Theorem 2.1 we want to observe that $(V_t)_{t\geq 0}$ is a time-homogeneous Markov process. This was known and proved for example by Carmona et al. [4, 5]. We will need an explicit expression for the transition functions, stated as Part (a) of the following lemma (but we omit the proof). The Markov property then allows us to reduce the question of the existence of strictly stationary solutions to the question of convergence in distribution of V_t as $t\to\infty$, as in Part (b) of the lemma.

Lemma 6.2. (*a*) *The generalised Ornstein-Uhlenbeck process $(V_t)_{t\geq 0}$ is a time-homogeneous Markov process. More precisely, for $0\leq y<t$ define*

$$M_{y,t} := e^{-(\xi_t-\xi_y)}, \quad N_{y,t} := e^{-(\xi_t-\xi_y)}\int_{y+}^t e^{\xi_{s-}-\xi_y}\,d\eta_s.$$

Then $(M_{y,t},N_{y,t})$ is independent of $\mathcal{F}_y$, $(M_{y+h,t+h},N_{y+h,t+h}) \stackrel{D}{=} (M_{y,t},N_{y,t})$ for $h\geq 0$, and

$$V_t = M_{y,t}V_y + N_{y,t}. \tag{6.6}$$

(*b*) *The process $(V_t)_{t\geq 0}$ is strictly stationary if and only if V_t converges in distribution to V_0, as $t\to\infty$.*

Proof of Lemma 6.2 (b): Clearly, if $(V_t)_{t\geq 0}$ is strictly stationary, then $V_t \stackrel{D}{=} V_0$ for all $t\geq 0$, implying $V_t \stackrel{D}{\to} V_0$ as $t\to\infty$. If on the other hand, $V_t \stackrel{D}{\to} V_0$ as $t\to\infty$, then keeping $h>0$ fixed, setting $y := y(t) := t-h$ and letting $t\to\infty$, Equation (6.6) gives

$$V_0 \stackrel{D}{=} M_{0,h}V_0 + N_{0,h} = V_h. \tag{6.7}$$

Since $(V_t)_{t\geq 0}$ is a time homogeneous Markov process, this implies that the finite dimensional distributions are shift-invariant, i.e. $(V_t)_{t\geq 0}$ is strictly stationary. □

Proof of Theorem 2.1: Suppose that $(V_t)_{t\geq 0}$, as defined in (1.1), is strictly stationary. Then $V_t \stackrel{D}{=} V_0$ for all $t>0$, so $V_t \stackrel{D}{\to} V_0$ as $t\to\infty$. We will distinguish whether ξ_t tends a.s. to ∞, or a.s. to $-\infty$, or whether it oscillates, these being the only possibilities for a Lévy process (see e.g. [9], or [28], Proposition 37.10, page 255).

Suppose first that $\lim_{t\to\infty}\xi_t \to +\infty$ a.s. Then, as argued following the statement of Remark 2.2, $V_0 e^{-\xi_t}$ converges a.s. to 0 and $e^{-\xi_t}\int_0^t e^{\xi_{s-}}d\eta_s$ converges in distribution to V_0 as $t\to\infty$. Thus, by (2.10), $\int_0^t e^{-\xi_{s-}}dL_s \stackrel{D}{\to} V_0$ as $t\to\infty$, and Proposition 2.4 implies Alternative (i).

Now suppose that ξ_t oscillates. Then by Proposition 2.4 there are two alternatives. One is that the process $\left|\int_0^t e^{-\xi_{s-}}dL_s\right|$ tends in probability to ∞ as $t\to\infty$. If this occurs, then by (2.10), the process $\left|e^{-\xi_t}\int_0^t e^{\xi_{s-}}d\eta_s\right|$ must tend in probability to ∞ as $t\to\infty$. Since $V_t \stackrel{D}{\to} V_0$ as $t\to\infty$, (1.1) then gives $|V_0|e^{-\xi_t}\stackrel{P}{\to}\infty$, hence $\xi_t \stackrel{P}{\to} -\infty$ as $t\to\infty$. But then $V_0+\int_0^t e^{\xi_{s-}}d\eta_s = e^{\xi_t}V_t \stackrel{P}{\to} 0$ as $t\to\infty$. Since $\int_0^t e^{\xi_{s-}}d\eta_s$ is independent of V_0, so is its probability limit, which is $-V_0$. Hence it follows that V_0 must be equal to a constant, and since $V_t \stackrel{D}{=} V_0$ for all $t>0$ it follows that $V_t=$ const. a.s. for all $t>0$. The other alternative in Proposition 2.4 is that there is some constant $k\in\mathbb{R}\setminus\{0\}$ such that for each $t>0$, $\int_0^t e^{-\xi_{s-}}dL_s = k(1-e^{-\xi_t})$ a.s. With the notations of Lemma 6.1, this implies

$$\begin{aligned}
V_t &= e^{-\widehat{\xi}_t}V_0 + \int_0^t e^{-\widehat{\xi}_{s-}}d\widehat{\eta}_s + \left[e^{-\widehat{\xi}},\widehat{\eta}\right]_t \\
&\stackrel{D}{=} e^{-\xi_t}V_0 + \int_0^t e^{-\xi_{s-}}dL_s \quad \text{(by (2.10))} \\
&= k+(V_0-k)e^{-\xi_t} \ \text{ a.s.}
\end{aligned}$$

Since V_t has a finite limit in distribution and since ξ_t oscillates as $t\to\infty$, we must have $V_0=k$ a.s., in which case $V_t=k$ a.s. by stationarity. Since $(V_t)_{t\geq 0}$ has càdlàg paths, it follows that it is indistinguishable from the constant process $t\mapsto k$.

Finally, if $\lim_{t\to\infty}\xi_t=-\infty$ a.s., then $e^{-\xi_t}\left(V_0+\int_0^t e^{\xi_{s-}}d\eta_s\right) = V_t \stackrel{D}{\to} V_0$ as $t\to\infty$, showing that $V_0+\int_0^t e^{\xi_{s-}}d\eta_s$ converges in probability to 0 as $t\to\infty$. Then, just as for the oscillating part of the proof, it follows that $V_t=$ const. a.s. for all $t>0$. Thus, stationarity of $(V_t)_{t\geq 0}$ implies (i) or (ii).

Now if $(V_t)_{t\geq 0}$ degenerates to a constant k a.s., then $V_0=k$ and $k=e^{-\xi_t}(k+\int_0^t e^{\xi_{s-}}\,d\eta_s)$, or equivalently $V_0=k$ and

$$k(e^{\xi_t}-1)=\int_0^t e^{\xi_{s-}}\,d\eta_s. \tag{6.8}$$

For $k\neq 0$, (6.8) is exactly the defining equation for $e^{\xi}=\mathcal{E}(\eta/k)$ (see e.g. Protter [27], Theorem 37 in Chapter II, page 84), while $k=0$ is impossible in (6.8) by uniqueness of the stochastic differential equation; $\int_0^t X_{s-}\,d\eta_s\equiv 0$ implies $X_s\equiv 0$. Conversely, (6.8) and $V_0=k\neq 0$ a.s. imply $V_t=k$ a.s. So the conditions stated in Alternative (ii) are in fact equivalent.

For the converse, it is clear that Alternative (ii) implies strict stationarity of $(V_t)_{t\geq 0}$. Further, if $V_t'=k'$ and $V_t=k$ for all $t\geq 0$ are two constant solutions, then

$$k'-k=V_t'-V_t=e^{-\xi_t}(V_0'-V_0)=e^{-\xi_t}(k'-k),\quad t>0,$$

implying $k = k'$, so the stationary solution is unique in distribution.

Finally suppose that Alternative (i) holds. Then $\lim_{t\to\infty} e^{-\xi_t} V_0 = 0$ a.s. and $\int_0^\infty e^{-\xi_{t-}} dL_t$ converges a.s. Hence it follows from (2.10) that $e^{-\xi_t} \int_0^t e^{\xi_{s-}} d\eta_s$ converges in distribution as $t \to \infty$ to the finite random variable $V_\infty = \int_0^\infty e^{-\xi_{s-}} dL_s$. Thus, by (1.1), $V_t \stackrel{D}{\to} V_\infty$ as $t \to \infty$, and strict stationarity with $V_0 \stackrel{D}{:=} V_\infty$ follows from Lemma 6.2 (b). □

7 Proofs for Section 3

Proof of Theorem 3.1: (*a*) We first claim that for any $z > 1$ we have

$$\overline{\Pi}_L(z) \le \overline{\Pi}_\eta(\sqrt{z}) + \overline{\Pi}_\xi^-((1/2)\log z). \tag{7.1}$$

To see this, note that $|T(x,y)| > z$ implies $|y| > \sqrt{z}$ or $e^{-x} > \sqrt{z}$. But

$$\Pi_{\xi,\eta}(\{(x,y) \in \mathbb{R}^2 \setminus \{0\} : |y| > \sqrt{z}\}) = \overline{\Pi}_\eta(\sqrt{z}),$$

and, keeping $z > 1$,

$$\Pi_{\xi,\eta}(\{(x,y) \in \mathbb{R}^2 \setminus \{0\} : e^{-x} > \sqrt{z}\}) \le \overline{\Pi}_\xi^-((1/2)\log z),$$

giving (7.1).

For $x \ge 1$ and $y \ge e$, denote

$$f(y) := \frac{\log y}{A_\xi(\log y)}, \quad g(x) := \frac{x}{A_\xi(x)} = f(e^x).$$

Now suppose that $\int_0^\infty e^{-\xi_{s-}} d\eta_s$ converges a.s. By Proposition 2.4, this implies that $\lim_{t\to\infty} \xi_t = +\infty$ a.s., hence, by (2.5),

$$\int_{(1,\infty)} g(x) \left| d\overline{\Pi}_\xi^-(x) \right| < \infty. \tag{7.2}$$

Also by Proposition 2.4,

$$\int_{(e,\infty)} f(y) \left| d\overline{\Pi}_\eta(y) \right| < \infty. \tag{7.3}$$

Note that f and g are continuous and non-decreasing. Observe that

$$\int_{(e,\infty)} \overline{\Pi}_\xi^-((1/2)\log y) df(y) = \int_{(1/2,\infty)} \overline{\Pi}_\xi^-(x) dg(2x).$$

With partial integration and using $g(2z) \le 2g(z)$ we see that (7.2) implies finiteness of the latter integral. Further, integration by parts shows that the integral

$$\int_{(e,\infty)} \overline{\Pi}_\eta(\sqrt{y}) df(y) = \int_{(\sqrt{e},\infty)} \overline{\Pi}_\eta(y) df(y^2)$$

is finite if and only if

$$\int_{(\sqrt{e},\infty)} f(y^2) \left| d\overline{\Pi}_\eta(y) \right|$$

is finite. But since $f(y^2) \leq 2f(y)$ for large y (because A_ξ is non-decreasing), this is indeed the case by (7.3). Thus we conclude from (7.1) that $\int_{(e,\infty)} \overline{\Pi}_L(y)df(y) < \infty$, giving $\int_{(e,\infty)} f(y)|d\overline{\Pi}_L(y)| < \infty$. From Proposition 2.4 and Theorem 2.1 then follows the a.s. convergence of $\int_0^\infty e^{-\xi_{s-}}dL_s$ and the existence of a stationary version of $(V_t)_{t\geq 0}$.

(b) Note that in analogy to (7.1) we have for $z > 1$ from Proposition 2.3, using the mapping S instead of the mapping T,

$$\overline{\Pi}_\eta(z) \leq \overline{\Pi}_L(\sqrt{z}) + \overline{\Pi}_\xi^+((1/2)\log z).$$

The proof then follows similarly as above, with (7.2) being replaced by

$$\int_{(1,\infty)} g(x) \left| d\overline{\Pi}_\xi^+(x) \right| < \infty,$$

which is finite since $E\xi_1^+ < \infty$.

(c) Given a Lévy process $(\xi_t)_{t\geq 0}$ with $\lim_{t\to\infty} \xi_t = +\infty$ a.s. and $E\xi_1^+ = \infty$, construct a bivariate Lévy process (ξ_t, η_t) such that the bivariate Lévy measure $\Pi_{\xi,\eta}$ is concentrated on $((-\infty, 1] \times \{0\}) \cup \{(x, e^x) : x > 1\}$. Then $\Pi_L(\{1\}) = \Pi_{\xi,\eta}((x,y) : e^{-x}y = 1) = \Pi_{\xi,\eta}((x, e^x) : x > 1) = \overline{\Pi}_\xi^+(1)$, while $\Pi_L(\mathbb{R} \setminus \{1\}) = 0$, so $\int_0^\infty e^{-\xi_{s-}}dL_s$ converges a.s. by Proposition 2.4. On the other hand, $\Pi_\eta((-\infty, 0)) = 0$, and for $z > e$ we have

$$\overline{\Pi}_\eta^+(z) = \Pi_{\xi,\eta}(\{(x, e^x) : e^x > z\}) = \overline{\Pi}_\xi^+(\log z),$$

giving

$$\int_{(e,\infty)} \left(\frac{\log y}{A_\xi(\log y)} \right) |d\overline{\Pi}_\eta(y)| = \int_{(1,\infty)} \left(\frac{x}{A_\xi(x)} \right) \left| d\overline{\Pi}_\xi^+(x) \right|.$$

But the latter integral is infinite by the Abel-Dini Theorem, since $E\xi_1^+ = \infty$. Proposition 2.4 then shows that $\int_0^\infty e^{-\xi_{s-}}d\eta_s$ does not converge a.s. or in distribution. □

Proof of Theorem 3.3: (a) That $\xi = \mathcal{E}(\xi/k)$ for some $k \neq 0$ is equivalent to $\xi_t = at + bN_t$ with the required constraint on a and b follows easily from (2.8) and a small calculation. Hence, by Theorem 2.1 we only have to show that $\lim_{t\to\infty} \xi_t = +\infty$ a.s. implies convergence of the integral (2.6) when $\xi = \eta$. Note that the Lévy measure $\Pi_{\xi,\xi}$ is concentrated on the diagonal. For the Lévy measure Π_L, we obtain for sufficiently large z

$$\overline{\Pi}_L^+(z) = \Pi_{\xi,\xi}(\{(x,x) \in \mathbb{R}^2 : e^{-x}x > z\}) = 0.$$

Since $e^{-x}x \geq -e^{-2x}$ for $x < 0$ such that $|x|$ is large, for sufficiently large z we have

$$\overline{\Pi}_L^-(z) \leq \Pi_{\xi,\xi}(\{(x,x) \in (-\infty,0)^2 : -e^{-2x} \leq -z\}) = \overline{\Pi}_\xi^-((1/2)\log z).$$

But in the proof of Theorem 3.1(a) we showed that

$$\int_{(e,\infty)} \overline{\Pi}_\xi^-((1/2)\log y)\, d\left(\frac{\log y}{A_\xi(\log y)} \right)$$

is finite when $\lim_{t\to\infty} \xi_t = +\infty$ a.s., giving finiteness of (2.6) by partial integration.

(b) Let $a_1 := 1$ and $A(a_1) := 1$. Define recursively, for $n \in \mathbb{N}$,

$$\begin{aligned} c_n &:= \frac{A(a_n)}{2a_n}, \\ a_{n+1} &:= e^{a_n}, \\ A(a_{n+1}) &:= 1 + \sum_{j=1}^{n} (a_{j+1} - a_j) c_j. \end{aligned}$$

Then $a_n \uparrow \infty$ as $n \to \infty$, and, since

$$A(a_{n+1}) = A(a_n) + (a_{n+1} - a_n)c_n = \frac{1}{2}\left(\frac{e^{a_n}}{a_n} + 1\right) A(a_n),$$

also $A(a_n) \uparrow \infty$ as $n \to \infty$. Further,

$$\frac{c_{n+1}}{c_n} = \frac{A(a_{n+1})a_n}{A(a_n)a_{n+1}} = \frac{1}{2}\left(1 + \frac{a_n}{a_{n+1}}\right) < 3/4 \quad \forall\, n \in \mathbb{N}.$$

We conclude that $(c_n)_{n\in\mathbb{N}}$ is a decreasing sequence, tending to 0 as $n \to \infty$. Now define the compound Poisson process ξ (without drift) in terms of its Lévy measure Π_ξ, such that $\Pi_\xi((-\infty, 1)) = 0$ and

$$\overline{\Pi}_\xi^+(y) = c_n \quad \text{for } \; y \in [a_n, a_{n+1}), \quad n \in \mathbb{N}.$$

It is easy to check that $A_\xi(a_n) = A(a_n)$ for all $n \in \mathbb{N}$, where A_ξ is as in definition (2.4). From (2.5) we conclude that $\xi_t \to \infty$ a.s. Further, we calculate

$$\begin{aligned} \int_{[a_1,\infty)} \left(\frac{\log y}{A_\xi(\log y)}\right) \left|d\overline{\Pi}_\xi(y)\right| &= \sum_{i=2}^{\infty} \frac{\log a_i}{A_\xi(\log a_i)} (c_{i-1} - c_i) \\ &= \frac{1}{2}\sum_{i=2}^{\infty} \frac{1}{c_{i-1}} (c_{i-1} - c_i) = \frac{1}{2}\sum_{i=2}^{\infty}\left(1 - \frac{c_i}{c_{i-1}}\right), \end{aligned}$$

and the latter sum diverges since $c_i/c_{i-1} < 3/4$. An application of Proposition 2.4 then shows that $\int_0^t e^{-\xi_{s-}} d\xi_s$ does not converge a.s. or in distribution to a finite random variable. $\square$

8 Proofs for Section 4

Proof of Proposition 4.1: Define L as in (2.3). Assume (4.1), and that $\Psi_\xi(\kappa) < 0$ for some $\kappa > 0$, and take $p > 1$, $q > 1$, with $1/p + 1/q = 1$. Let $k := \max\{1, \kappa\}$. Then we have $Ee^{-pk\xi_1} < \infty$ and $E|\eta_1|^{qk} < \infty$. The former is equivalent to $Ee^{pk\xi_1^-} < \infty$ and the latter implies $E|\eta_1| < \infty$ (since $qk > 1$). Thus we have $E\xi_1^- < \infty$. If in addition $E\xi_1^+ = \infty$ then by (2.5) we see that $\lim_{t\to\infty} \xi_t = +\infty$ a.s. If, alternatively, $E\xi_1^+ < \infty$ then $E|\xi_1| < \infty$ and $\Psi_\xi(\kappa) < 0$ implies $E\xi_1 > 0$. Then (2.5) again gives $\lim_{t\to\infty} \xi_t = +\infty$ a.s. Also, $E|\eta_1| < \infty$

implies $E\log^+|\eta_1| < \infty$ and since $E\xi_1^- < \infty$ we get from (7.1) that $E\log^+|L| < \infty$. So by (2.11), $\int_0^\infty e^{-\xi_{t-}}dL_t$ converges a.s., and a stationary version (unique in distribution) exists by Theorem 2.1.

We next show that

$$E\left|\int_0^1 e^{-\xi_{s-}}dL_s\right|^k < \infty. \tag{8.1}$$

To show this, recall that $E|\eta_1| < \infty$. Assume first that $E\eta_1 = 0$. Then it follows from the Burkholder-Davis-Gundy inequality (e.g. Liptser and Shiryayev [21], pages 70 and 75) and from Hölder's inequality, that there exists a universal constant $C_k > 0$ such that

$$E\left|\int_0^1 e^{-\xi_{s-}}d\eta_s\right|^k \le C_k \left(E\sup_{0\le s\le 1} e^{-pk\xi_s}\right)^{1/p}\left(E[\eta,\eta]_1^{qk/2}\right)^{1/q}. \tag{8.2}$$

An application of Doob's inequality to $(e^{-\xi_t - t\Psi_\xi(1)})_{t\ge 0}$ and of the Burkholder-Davis-Gundy inequality to $([\eta,\eta]_t)_{t\ge0}$ then shows finiteness of (8.2). Finiteness of $E\left|\left[e^{-\xi},\eta\right]_1\right|^k$ follows similarly, after an application of the Kunita-Watanabe inequality (e.g. [27], page 69), and from (8.2) and (2.10) we conclude that (8.1) holds if $E\eta_1 = 0$. The general case follows by replacing η by $(\eta_t - tE\eta_1)_{t\ge 0}$, and observing that

$$E\left|\int_0^1 e^{-\xi_{s-}}ds\right|^k \le E\sup_{0\le s\le 1} e^{-k\xi_s} < \infty.$$

Now define

$$Q_j := \int_{j+}^{j+1} e^{-(\xi_{s-}-\xi_j)}d(L_s - L_j)$$

for $j = 0, 1, \ldots$ Then the $(Q_j)_{j=0,1,\ldots}$ are independent and identically distributed, ξ_j and Q_j are independent for fixed j, and $E|Q_j|^\kappa < \infty$ by (8.1). Furthermore, for any $n = 1, 2, \ldots$,

$$\left|\int_0^n e^{-\xi_{s-}}dL_s\right| = \left|\sum_{j=0}^{n-1}\int_{j+}^{j+1} e^{-\xi_{s-}}dL_s\right| = \left|\sum_{j=0}^{n-1} e^{-\xi_j}Q_j\right| \le \sum_{j=0}^{n-1} e^{-\xi_j}|Q_j|. \tag{8.3}$$

Let $\alpha := \lfloor\kappa\rfloor$, the integer part of κ. Suppose $\kappa > \alpha$. Then

$$\left(\sum_{j=0}^{n-1} e^{-\xi_j}|Q_j|\right)^\kappa \le \sum_{j_1=0}^{n-1}\cdots\sum_{j_\alpha=0}^{n-1}|e^{-\xi_{j_1}}Q_{j_1}|\cdots|e^{-\xi_{j_\alpha}}Q_{j_\alpha}|\sum_{j_{\alpha+1}=0}^{n-1}|e^{-\xi_{j_{\alpha+1}}}Q_{j_{\alpha+1}}|^{\kappa-\alpha},$$

and an application of Hölder's inequality with $\frac{1}{\kappa} + \ldots + \frac{1}{\kappa} + \frac{\kappa-\alpha}{\kappa} = 1$ gives

$$\begin{aligned}
&E\left|\int_0^n e^{-\xi_{s-}}dL_s\right|^\kappa \\
&\le \sum_{j_1=0}^{n-1}\cdots\sum_{j_\alpha=0}^{n-1}\sum_{j_{\alpha+1}=0}^{n-1}\left(E\left|e^{-\xi_{j_1}}Q_{j_1}\right|^\kappa\right)^{1/\kappa}\cdots\left(E\left|e^{-\xi_{j_\alpha}}Q_{j_\alpha}\right|^\kappa\right)^{1/\kappa}\left(E|e^{-\xi_{j_{\alpha+1}}}Q_{j_{\alpha+1}}|^\kappa\right)^{\frac{\kappa-\alpha}{\kappa}} \\
&= \left(\sum_{j=0}^{n-1}\left(Ee^{-\kappa\xi_j}\right)^{1/\kappa}\left(E|Q_j|^\kappa\right)^{1/\kappa}\right)^\alpha\left(\sum_{j=0}^{n-1}(Ee^{-\kappa\xi_j})^{(\kappa-\alpha)/\kappa}(E|Q_j|^\kappa)^{(\kappa-\alpha)/\kappa}\right) \\
&= E|Q_0|^\kappa\left(\sum_{j=0}^{n-1} e^{j\Psi_\xi(\kappa)/\kappa}\right)^\alpha\left(\sum_{j=0}^{n-1} e^{j\Psi_\xi(\kappa)(\kappa-\alpha)/\kappa.}\right).
\end{aligned} \tag{8.4}$$

If $\kappa = \alpha$, the last factor can be omitted (by a similar calculation). Since $\Psi_\xi(\kappa) < 0$, the last expression converges absolutely as $n \to \infty$, and since we showed in the first part of the proof that a stationary version exists, with $V_0 \stackrel{D}{=} \int_0^\infty e^{-\xi_{s-}} dL_s$, $E|V_0|^\kappa < \infty$ then follows from (8.3) and (8.4). □

Remark 8.1. If the processes η and ξ are independent, then the moment conditions in Proposition 4.1 can be relaxed. In fact, in that case the assertions of Proposition 4.1 hold if $Ee^{-\max\{1,\kappa\}\xi_1} < \infty$, $\Psi_\xi(\kappa) < 0$ and $E|\eta_1|^{\max\{1,\kappa\}} < \infty$. This follows by circumventing Hölder's inequality in the proof of (8.1) and using instead $E(XY) = (EX)(EY)$ for independent random variables.

Proof of Theorem 4.3: To show (4.2), let $M_{y,t}$ and $N_{y,t}$ be as in Lemma 6.2. Using the independence of $M_{y,t}$ and V_y and (6.6) we get

$$\begin{aligned} E(V_t|\mathcal{F}_y) &= (EM_{y,t})V_y + EN_{y,t} \\ &= (EM_{y,t})(V_y - EV_y) + E(M_{y,t}V_y + N_{y,t}) \\ &= e^{(t-y)\Psi_\xi(1)}(V_y - EV_y) + EV_t. \end{aligned}$$

Hence we conclude that

$$E(V_yV_t) = E(V_y\, E(V_t|\mathcal{F}_y)) = e^{(t-y)\Psi_\xi(1)}(EV_y^2 - (EV_y)^2) + (EV_y)(EV_t),$$

giving (4.2). Equation (4.3) then follows immediately from Proposition 4.1, noting that $\Psi_\xi(2) < 0$ implies $\Psi_\xi(1) < 0$ by convexity of Ψ_ξ, see Sato [28], Lemma 26.4, page 169. □

Proof of Theorem 4.5: Assume the existence of κ, p and q as specified. Since $Ee^{-\kappa\xi_1} < \infty$ by (4.1), it follows that $Ee^{-u\xi_1}$ is finite for every $u \in [0,\kappa]$ and that the function $\Psi_\xi : [0,\kappa] \to \mathbb{R}, u \mapsto \log Ee^{-u\xi_1}$ is strictly convex ([28], Lemma 26.4, page 169). Since $\Psi_\xi(0) = \Psi_\xi(\kappa) = 0$, it follows that there is a constant $\kappa' \in (0,\kappa)$ such that $\Psi_\xi(\kappa') < 0$. From Proposition 4.1 we conclude that $\lim_{t\to\infty} \xi_t = +\infty$ a.s. and that a stationary version $(V_t)_{t\geq 0}$ exists. Now define, for arbitrary $t > 0$,

$$U_t := e^{-\xi_t}, \quad W_t := e^{-\xi_t} \int_0^t e^{\xi_{s-}} d\eta_s.$$

Then, for the stationary version,

$$V_0 \stackrel{D}{=} V_t = e^{-\xi_t}V_0 + e^{-\xi_t} \int_0^t e^{\xi_{s-}} d\eta_s = U_tV_0 + W_t, \tag{8.5}$$

where V_0 is independent of (U_t, W_t). We assert that there is a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to \infty$ and the support of the law of ξ_{t_n} is not concentrated on $r\mathbb{Z}$ for any $r > 0$. If ξ is not of finite variation this is clear by Sato [28], Corollary 24.6, page 149. If ξ is of finite variation, this follows also from Sato [28], Corollary 24.6, which allows us to conclude that ξ_t and $\xi_{t'}$ cannot simultaneously be concentrated on a lattice if $t/t' \in \mathbb{R} \setminus \mathbb{Q}$ when ξ

has nonzero drift or Π_ξ is not concentrated on a lattice (using similar reasoning as in the proof of Theorem 5.2 in [20]). Now fix such a sequence. Then for each $n \in \mathbb{N}$,

$$E|U_{t_n}|^\kappa = e^{t_n \Psi_\xi(\kappa)} = e^0 = 1,$$

$E(|U_{t_n}|^\kappa \log^+ |U_{t_n}|) < \infty$ (since $E|U_{t_n}|^{p\kappa}$ is finite by assumption), and $E|W_{t_n}|^\kappa < \infty$, as was proved in Equation (8.1) (there, t_n was taken to be 1, but the proof holds for arbitrary t_n). So V_0 satisfies for every t_n the distributional fixed point Equation (8.5), and it follows from Theorem 4.1 in Goldie [13] that there are constants $C_+, C_- \geq 0$ such that (4.4) holds. The constants C_+ and C_- do not depend on t_n. In [13], Theorem 4.1 it is further shown that if $C_+ + C_- = 0$, then there necessarily exist real constants c_n, $n \in \mathbb{N}$, such that for each $n \in \mathbb{N}$,

$$W_{t_n} = (1 - U_{t_n})c_n \ \text{ a.s.}$$

Since W_{t_n} converges in distribution to V_0, and U_{t_n} converges to 0 a.s., as $n \to \infty$, the sequence c_n must converge in distribution to V_0, which is impossible if $(V_t)_{t\geq 0}$ is not degenerate to a constant process. □

Proof of Corollary 4.6: For the proof of (i), observe that if η is a subordinator, then so is L (c.f. Equation (2.3)), and hence $V_\infty \geq 0$ a.s., so that $C_- = 0$. The case that V_∞ is a constant cannot occur when η is a subordinator, since (2.8) would then imply that either ξ or $-\xi$ must be a subordinator, which is impossible since $\psi_\xi(\kappa) = 0$ for some $\kappa \neq 0$. Consequently, we conclude that $C_+ > 0$.

To show (ii), suppose that (ξ, η) is symmetric in the given sense, and that ξ or η does not have a Brownian part. Then it follows easily from the definition of L in (2.3) that (ξ, L) is symmetric, too, i.e. $(\xi_t, L_t)_{t\geq 0} \stackrel{D}{=} (\xi_t, -L_t)_{t\geq 0}$. This implies that V_0 is symmetric by (2.7), i.e. $V_0 \stackrel{D}{=} -V_0$, and hence $C_+ = C_-$. Then if V_0 is not a constant, the claim follows from Theorem 4.5. □

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