On lower bounds of exponential frames*

Alexander M. Lindner

Abstract

Lower frame bounds for sequences of exponentials are obtained in a special version of Avdonin's theorem on "1/4 in the mean" (1974) and in a theorem of Duffin and Schaeffer (1952).

Keywords: lower bound, exponential frame, sine-type-function, irregular sampling

Math Subject Classification: 42C15, 30A10, 94A12

1. Introduction

The notion of *frame* has been introduced by Duffin and Schaeffer [3]. A sequence $(\varphi_n)_{n\in\mathbb{Z}}$ in a Hilbert space $(H,(\cdot,\cdot)_H)$ is a frame for H, if there exist positive constants A, B such that for all f in H:

$$A\,\|f\|_H^2 \le \sum_{n\in\mathbb{Z}} |(f,\varphi_n)_H|^2 \le B\,\|f\|_H^2.$$

More specifically, $(\varphi_n)_{n\in\mathbb{Z}}$ is called an (A,B)*-frame.* The constants A and B are called *lower* and *upper frame bounds,* respectively. A frame is *exact* if it is no longer a frame after any of its elements is removed.

Duffin and Schaeffer [3] have given a sufficient condition for a sequence of exponentials $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ to be a frame for $L^2(-\gamma,\gamma)$, $\gamma>0$. Avdonin [1] has given a sufficient condition for a sequence of exponentials $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ to be an exact frame for $L^2(-\pi,\pi)$. In both papers, only the mere existence of a lower bound is proved.

D. Kölzow [6] asked for explicit lower frame bounds, in terms of the data by which the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ is restricted. In this paper we shall obtain a lower bound in a special version of Avdonin's theorem on "1/4 in the mean" (Theorem 1). The result will be used to obtain a lower bound in a theorem of Duffin and Schaeffer (Theorem 2). Finally, an application to irregular sampling is pointed out.

*The results of this paper are part of the author's forthcoming doctoral thesis and were presented, in a preliminary form, at the *1997 International Workshop on Sampling Theory and Applications*, June 16 - 19, 1997, Aveiro, Portugal.

2. Preliminary remarks and main results

PW_σ^2 denotes the *Paley-Wiener space* of entire functions of exponential type at most σ, whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. For a sequence $(\lambda_n)_{n\in\mathbb{Z}}$ of real numbers the classical Paley-Wiener theorem yields that $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is an (A,B)-frame for $L^2(-\sigma,\sigma)$ if and only if

$$A\,\|F\|_{PW_\sigma^2}^2 \le 2\pi \sum_{n\in\mathbb{Z}} |F(\lambda_n)|^2 \le B\,\|F\|_{PW_\sigma^2}^2 \quad \forall\, F \in PW_\sigma^2.$$

A sequence $(\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers is called *separated by* $\delta > 0$, if

$$|\lambda_n - \lambda_m| \ge \delta \quad \forall\, m,n \in \mathbb{Z} : m \ne n.$$

The following lemma asserts an upper bound for separated sequences with bounded imaginary parts. The proof follows the same argument as in the proof of Lemma 2 in Katsnel'son [4], using $\mathrm{card}\{n \in \mathbb{Z} : |z-\lambda_n| \le 1\} \le (1+2/\delta)^2 \ \ \forall\, z \in \mathbb{C}$.

Lemma 1. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers, separated by $\delta > 0$, with imaginary parts bounded by $\tau < \infty$. Then,*

$$\sum_{n\in\mathbb{Z}} |F(\lambda_n)|^2 \le \frac{2}{\pi}\left(\frac{2}{\delta}+1\right)^2 \cdot \frac{e^{2\sigma(\tau+1)}-1}{2\sigma}\,\|F\|_{PW_\sigma^2}^2 \quad \forall\, F \in PW_\sigma^2$$

for any $\sigma > 0$.

The main results of this paper are the following:

Theorem 1 (A lower bound in the Theorem of Avdonin). *Let $(\delta_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers, bounded by $L < \infty$. Suppose $(k+\delta_k)_{k\in\mathbb{Z}}$ is separated by $\delta > 0$, and there are $d \in [0,1/4)$ and a natural number N, such that*

$$\left| \sum_{k=jN+1}^{(j+1)N} \delta_k \right| \le N \cdot d \quad \forall\, j \in \mathbb{Z}.$$

Then, $(e^{i(k+\delta_k)\bullet})_{k\in\mathbb{Z}}$ is an exact frame for $L^2(-\pi,\pi)$, and the following constant is a lower frame bound:

$$A_{Av}(L,\delta,d,N) := e^{-20\pi^2 (2\tilde{L})^{2\tilde{N}}/\tilde{N}^2} \cdot \left(\frac{\tilde{\delta}}{9\tilde{L}}\right)^{240(2\tilde{L})^{\tilde{N}}},$$

where

$$\tilde{N} := N \cdot \lceil \frac{1}{N} \cdot \frac{2(4L+2)^2}{1/4-d} \rceil, \quad \tilde{L} := \frac{3}{2} + 2(3L+1), \quad \tilde{\delta} := \frac{1}{2}(\frac{1}{4}-d)\delta.$$

($\lceil x \rceil$ denotes the smallest integer greater than or equal to x.)

Theorem 2 (A lower bound in the Theorem of Duffin and Schaeffer). *Suppose $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of real numbers, separated by $\delta > 0$. Let $\sigma > 0, L \ge$*

0, $0 < \gamma < \pi\sigma$ *and suppose* $(\lambda_n - n/\sigma)_{n\in\mathbb{Z}}$ *is bounded by* L. *Then,* $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ *is a frame for* $L^2(-\gamma,\gamma)$, *and the following constant is a lower frame bound:*

$$A_{DS}(L,\delta,\sigma,\gamma) := \frac{\gamma}{\pi} \cdot e^{-5\pi^2 \frac{(12M+1)^{4M(M+1)}}{M^2(M+1)^2}} \cdot \left(\frac{\delta\gamma/\pi}{308M}\right)^{240(12M+1)^{2M(M+1)}},$$

where

$$M := \max\left\{13, \lceil \frac{3+2\sigma L}{\pi\sigma/\gamma - 1} \rceil \right\}.$$

While the proofs of Avdonin [1, Theorem 2] and of Duffin and Schaeffer [3, Theorem I] only assured the existence of lower bounds, Theorems 1 and 2 give explicit lower bounds. The proof of Theorem 1 follows by explicating and supplementing that of Avdonin. Theorem 2 will follow from Proposition 1, by explicating a construction of Seip [9, Theorem 2.3]. We remark that Avdonin's theorem was proved for more general sets of zeros of sine-type-functions, more general partitions and complex sequences $(\delta_k)_{k\in\mathbb{Z}}$. The proof of Theorem 1 works for this general situation as well, if suitable conditions on the sine-type-function and partition are posed.

3. Proof of the theorems

For the proof of Theorem 1, we will need the notion of sine-type-function.

Definition. An entire function f of exponential type is called *σ-sine-type-function* ($\sigma > 0$), if the zeros of f are simple and separated and there are C_1, C_2, $\tau > 0$ such that

$$C_1 \cdot e^{\sigma|y|} \le |f(x+iy)| \le C_2 \cdot e^{\sigma|y|} \quad \forall\, x, y \in \mathbb{R} : |y| \ge \tau. \tag{1}$$

Remark. The Theorem of Avdonin rests on the following result of Levin, proved indirectly: *For any sine-type-function f with zeros $(\lambda_n)_{n\in\mathbb{Z}}$, there are constants C_3, $C_4 > 0$ such that*

$$C_3 \le |f'(\lambda_n)| \le C_4 \quad \forall\, n \in \mathbb{Z}$$

(cf. Levin [7], [8, p. 164], Young [10, p. 173]). The proof does not yield any estimates for C_3.

The following lemma plays the central role in the proof of Theorem 1; it supplements Lemmas 1, 2 in Avdonin [1]: the growth of the constructed sine-type-function is determined and estimates for the derivatives at the zeros are found. (Let $1/2\,\mathbb{Z} := \{n/2 : n \in \mathbb{Z}\}$.)

Lemma 2. *Let the assumptions of Theorem 1 be satisfied. Suppose furthermore that $(k+\delta_k)_{k\in\mathbb{Z}}$ is increasing and that*

$$N > 2L \cdot \frac{3/4 + d}{1/4 - d}.$$

Define

$$L' := 3/2 + 2L, \quad \delta' := (1/4 - d)\delta,$$

$$D_\varepsilon := \left(\left(1 + \frac{L'}{\varepsilon}\right)\left(1 + \frac{8L'}{\delta'}\right)\right)^{4\sqrt{4L'N^2 + 2(2L')^{2N}} + 6N} \cdot e^{(2L' + (2L')^{2N}/N^2)\pi^2} \quad \forall\, \varepsilon > 0,$$

$$C_1 := \left(\frac{1 - e^{-2\pi}}{2}\right)^2 D_1^{-1} \quad , \quad C_2 := D_1,$$

$$C_3 := \frac{\pi^2 \delta' e^{-\pi^2/4}}{32} D_{\delta'/8}^{-1} \quad , \quad C_4 := \frac{2\pi^2 e^{\pi^2/4}}{\delta'} D_{\delta'/8}.$$

Then, there is a sequence $(\delta_{k+1/2})_{k\in\mathbb{Z}}$ *of real numbers and a* 2π*-sine-type-function* Ψ*, real on the real axis, with zeros* $(k + \delta_k)_{k\in 1/2\,\mathbb{Z}}$*, such that* $(\delta_k)_{k\in 1/2\,\mathbb{Z}}$ *is bounded by* L'*,* $(k + \delta_k)_{k\in 1/2\,\mathbb{Z}}$ *is strictly increasing and separated by* δ'*, and*

$$C_1 e^{2\pi|y|} \le |\Psi(x + iy)| \le C_2 e^{2\pi|y|} \quad \forall\, x, y \in \mathbb{R} : |y| \ge 1, \tag{2}$$

$$C_3 \le |\Psi'(k + \delta_k)| \le C_4 \quad \forall\, k \in 1/2\,\mathbb{Z}. \tag{3}$$

Proof. i) For $j \in \mathbb{Z}$ define

$$\alpha^j := \frac{N/2 - 2\sum_{k=jN+1}^{(j+1)N} \delta_k}{N + \delta_{(j+1)N+1} - \delta_{jN+1}}.$$

This is well defined since $N > 2L$. Moreover

$$\alpha^j \ge 1/4 - d, \qquad 1 - \alpha^j \ge 1/4 - d.$$

ii) For any $k \in \mathbb{Z}$ there are unique numbers $l \in \{1, \dots, N\}$ and $j \in \mathbb{Z}$ such that $k = jN + l$. For such k define

$$\delta_{k+1/2} := -1/2 + (1 - \alpha^j)\delta_k + \alpha^j \delta_{k+1} + \alpha^j.$$

It follows that $|\delta_{k+1/2}| \le 3/2 + 2L = L'$, and that $(k + \delta_k)_{k\in 1/2\,\mathbb{Z}}$ is a strictly increasing sequence, separated by δ'.

iii) For $j \in \mathbb{Z}$ define

$$K^j := 1/2\,\mathbb{Z} \cap [jN + 1, (j+1)N + 1).$$

An easy calculation shows

$$\sum_{k\in K^j} \delta_k = 0 \quad \forall\, j \in \mathbb{Z}. \tag{4}$$

iv) Since $(\delta_k)_{k\in\mathbb{Z}}$ is bounded, we have $\delta \le 1$, hence $\delta' \le 1/4$. Choose $\beta \in [\delta'/4, \delta'/2]$ such that

$$|k + \delta_k + \beta| \ge \delta'/8 \quad \forall\, k \in 1/2\,\mathbb{Z}.$$

Define $\lambda_k := k + \beta$ for $k \in 1/2\,\mathbb{Z}$. Then,

$$|\lambda_k + \delta_k| \ge \delta'/8, \;\; |\lambda_k| \ge \delta'/4 \quad \forall\, k \in 1/2\,\mathbb{Z}.$$

Define

$$\Psi_1(z) := \text{P.V.} \prod_{k\in 1/2\,\mathbb{Z}} \left(1-\frac{z}{\lambda_k+\delta_k}\right) \quad \left(:= \lim_{R\to\infty} \prod_{k\in 1/2\,\mathbb{Z}, |k|\le R} \left(1-\frac{z}{\lambda_k+\delta_k}\right)\right),$$
$$\Phi_1(z) := \text{P.V.} \prod_{k\in 1/2\,\mathbb{Z}} \left(1-\frac{z}{\lambda_k}\right).$$

It follows from general theorems on entire functions that both products converge uniformly on compact subsets of $\mathbb{C}$ and that Ψ_1, Φ_1 are entire. Moreover, Ψ_1 is of exponential type (cf. Levin [8, pp. 30, 33]). The zeros of Ψ_1 are given by $(\lambda_k+\delta_k)_{k\in 1/2\,\mathbb{Z}}$, the zeros of Φ_1 are given by $(\lambda_k)_{k\in 1/2\,\mathbb{Z}}$. For $z\in\mathbb{C}\backslash\{\lambda_k : k\in 1/2\,\mathbb{Z}\}$ we have

$$\left|\frac{\Psi_1(z)}{\Phi_1(z)}\right| = \underbrace{\text{P.V.}\prod_{j\in\mathbb{Z}}\prod_{k\in K^j}\left|1+\frac{\delta_k}{\lambda_k-z}\right|}_{\text{(I)}}\cdot\underbrace{\left(\text{P.V.}\prod_{j\in\mathbb{Z}}\prod_{k\in K^j}\left|1+\frac{\delta_k}{\lambda_k}\right|\right)^{-1}}_{\text{(II)}}.$$

v) Let $\varepsilon>0$, $z=x+iy$, $x,y\in\mathbb{R}$, $\operatorname{dist}(z,\{\lambda_k,\lambda_k+\delta_k : k\in 1/2\,\mathbb{Z}\})\ge\varepsilon$. Let $\lambda^{j,z}$ be an element of $\{\lambda_k : k\in K^j\}$, satisfying

$$|\lambda^{j,z}-x| = \min_{k\in K^j}|\lambda_k-x|.$$

Denote by K^j_p an arbitrary subset of K^j with p elements ($p\in\{2,3,\ldots,2N\}$). Then,

$$\prod_{k\in K^j}\left(1+\frac{\delta_k}{\lambda_k-z}\right) = 1+\sum_{k\in K^j}\frac{\delta_k}{\lambda_k-z}+\sum_{p=2}^{2N}\sum_{K^j_p\subseteq K^j}\prod_{k\in K^j_p}\frac{\delta_k}{\lambda_k-z} =: 1+a_j.$$

¿From (4) we have $\sum_{k\in K^j}\delta_k/(\lambda^{j,z}-z)=0$, hence

$$a_j = \sum_{k\in K^j}\frac{\delta_k(\lambda^{j,z}-\lambda_k)}{(\lambda_k-z)(\lambda^{j,z}-z)}+\sum_{p=2}^{2N}\sum_{K^j_p\subseteq K^j}\prod_{k\in K^j_p}\frac{\delta_k}{\lambda_k-z}.$$

This can be estimated by

$$|a_j| \le \frac{2N\cdot L'\cdot N}{|\lambda^{j,z}-z|^2}+\frac{2^{2N}\cdot(L')^{2N}}{|\lambda^{j,z}-z|^2}. \tag{5}$$

Let j_0 be an integer such that $|\lambda^{j_0,z}-x| = \min\{|\lambda^{j,z}-x| : j\in\mathbb{Z}\}$. Then,

$$|\lambda^{j_0+j,z}-z| \ge \varepsilon+(|j|-1)N \quad \forall\, j\in\mathbb{Z}. \tag{6}$$

Defining

$$j_1 := \sqrt{4L'+2(2L')^{2N}/N^2}+1,$$

(5) and (6) give $\sum_{|j|\ge j_1} |a_{j_0+j}| \le (4L' + 2(2L')^{2N}/N^2)\pi^2/6$
and $|a_{j_0+j}| \le 1/2 \;\forall\, |j| \ge j_1$, hence

$$e^{-(L' + (2L')^{2N}/(2N^2))\pi^2} \le \prod_{|j|\ge j_1} |1 + a_{j_0+j}| \le e^{(L' + (2L')^{2N}/(2N^2))\pi^2}.$$

Using

$$(1 + L'/\varepsilon)^{-2N} \le |1 + a_j| \le (1 + L'/\varepsilon)^{2N} \quad \forall\, j \in \mathbb{Z}$$

gives

$$(1 + \frac{L'}{\varepsilon})^{-2N(2j_1+1)} e^{-(L'+(2L')^{2N}/(2N^2))\pi^2} \le \text{(I)} \le (1 + \frac{L'}{\varepsilon})^{2N(2j_1+1)} e^{(L'+(2L')^{2N}/(2N^2))\pi^2}. \tag{7}$$

For $z := 0$, $\varepsilon := \delta'/8$, this yields an estimate for (II), which together with (7) results in

$$D_\varepsilon^{-1} \le \left|\frac{\Psi_1(z)}{\Phi_1(z)}\right| \le D_\varepsilon \quad \forall z \in \mathbb{C} : \mathrm{dist}\,(z, \{\lambda_k, \lambda_k + \delta_k : k \in 1/2\,\mathbb{Z}\}) \ge \varepsilon. \tag{8}$$

vi) Define

$$\Psi(z) := \sin\pi\beta \cdot \cos\pi\beta \cdot \Psi_1(z + \beta), \quad \Phi(z) := \sin\pi\beta \cdot \cos\pi\beta \cdot \Phi_1(z + \beta).$$

Then, Ψ is an entire function of exponential type, real on the real axis, with zeros $(k + \delta_k)_{k\in 1/2\,\mathbb{Z}}$. Standard complex analysis shows $\Phi(z) = \sin\pi z \cdot \sin\pi(z - 1/2)$. Since

$$\left(\frac{1 - e^{-2\pi}}{2}\right)^2 \cdot e^{2\pi|y|} \le |\Phi(x + iy)| \le e^{2\pi|y|} \quad \forall\, x, y \in \mathbb{R} : |y| \ge 1,$$

(2) follows from (8) for $\varepsilon = 1$.

Fix $k_0 \in 1/2\,\mathbb{Z}$. It is possible to choose a circle γ with center $k_0 + \delta_{k_0}$ and radius $r \in \{\delta'/4, \delta'/2\}$, such that $\mathrm{dist}\,(\gamma, 1/2\;\mathbb{Z}) \ge \delta'/8$. Then, (8) combined with estimates for $\Phi(z)$ (which follow from the canonical factorization of $\sin\pi z$) gives

$$\pi^2(\delta'/8)^2 e^{-\pi^2/4} D_{\delta'/8}^{-1} \le |\Psi(z)| \le \pi^2/2 \cdot e^{\pi^2/4} D_{\delta'/8} \quad \forall\; z \in \gamma. \tag{9}$$

If Ψ_{k_0} is defined by

$$\Psi_{k_0}(z) := \begin{cases} \frac{\Psi(z)}{z - k_0 - \delta_{k_0}}, & z \ne k_0 + \delta_{k_0} \\ \Psi'(k_0 + \delta_{k_0}), & z = k_0 + \delta_{k_0}, \end{cases}$$

then Ψ_{k_0} is an entire function without zeros in $\{z \in \mathbb{C} : |z - (k_0 + \delta_{k_0})| \le r\}$. Applying the maximum/minimum principle to Ψ_{k_0} and using (9) gives (3). $\square$

The following result represents a special case of Theorem 1.

Proposition 1. *Under the assumptions of Lemma 2, $(e^{i(k+\delta_k)\bullet})_{k\in\mathbb{Z}}$ is an exact frame for $L^2(-\pi, \pi)$ with lower frame bound*

$$A := e^{-20\pi^2(2L')^{2N}/N^2} \cdot (\delta'/(9L'))^{240(2L')^N},$$

where

$$L' := 3/2 + 2L, \quad \delta' := (1/4 - d)\delta.$$

Proof: The proofs of Lemmas 7 and 9 in KATSNEL'SON [4] show that

$$A_1 := \frac{C_1^2 C_3^2}{C_2^4}\left(1 + \frac{C_4}{C_3}\right)^{-1} \cdot \pi \cdot \frac{e^{-8\pi}}{e^{8\pi+4\pi} - 1}\left(1 + \frac{2}{\delta'}\right)^{-2}$$

is a lower bound for $(e^{i(k+\delta_k)\bullet})_{k\in\mathbb{Z}}$, where C_1, C_2, C_3 and C_4 are as in Lemma 2. Inserting the values for C_1 - C_4 and using some suitable estimates one can show that $A_1 \geq A$. The existence of an upper bound follows from Lemma 1. For the proof of the exactness of the frame, we refer to AVDONIN [1]. □

Proof of Theorem 1: For $j \in \mathbb{Z}$ define

$$S_j := \{j\tilde{N} + \tilde{N} - \lceil 2L \rceil + 1, \ldots, j\tilde{N} + \tilde{N} + \lceil 2L \rceil\}.$$

We have $S_j \cap S_{j'} = \emptyset$ for $j \neq j'$. Define sequences $(\vartheta_k)_{k\in\mathbb{Z}}$ and $(\tilde{\delta}_k)_{k\in\mathbb{Z}}$ such that

$\vartheta_k := \delta_k$ if $k \in \mathbb{Z}\setminus\bigcup_{j\in\mathbb{Z}} S_j$,

$\{\vartheta_k + k : k \in S_j\} = \{\delta_k + k : k \in S_j\}\ \forall\, j \in \mathbb{Z}$ and $\vartheta_k + k < \vartheta_{k'} + k'$ for $k, k' \in S_j$, $k < k'$,

$\{\tilde{\delta}_k + k : k \in \{j\tilde{N} + 1, \ldots, (j+1)\tilde{N}\}\} = \{\vartheta_k + k : k \in \{j\tilde{N} + 1, \ldots, (j+1)\tilde{N}\}\}\ \ \forall\, j \in \mathbb{Z}$ and

$\tilde{\delta}_k + k < \tilde{\delta}_{k+1} + k + 1\ \ \forall\, k \in \mathbb{Z}$.

We obtain $|\tilde{\delta}_k| \leq 3L + 1\ \forall\, k \in \mathbb{Z}$ and

$$\left|\sum_{k=j\tilde{N}+1}^{(j+1)\tilde{N}} \tilde{\delta}_k\right| \leq \frac{d + 1/4}{2} \cdot \tilde{N} \quad \forall\, j \in \mathbb{Z}.$$

An application of Proposition 1 to $(\tilde{\delta}_k)_{k\in\mathbb{Z}}$ completes the proof of Theorem 1. □

Remark. The second part of Lemma VI in DUFFIN AND SCHAEFFER [3] was proved indirectly. Since their proof rests on this lemma, they did not obtain an explicit lower bound. We will prove Theorem 2, following a construction by SEIP [9].

Proof of Theorem 2: Suppose first $\sigma > 1$ and $\gamma = \pi$. Define

$$L_A := 3M - 1/2, \ \ N := 2M(M+1), \ \ d := 1/(M+1).$$

Then

$$N > 2L_A \cdot \frac{3/4 + d}{1/4 - d},$$

and the number of points of the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ in each interval of length M is at least $M + 1$.

Following the proof of Theorem 2.3 in Seip [9], we conclude that there is a sequence $(\delta_k)_{k\in\mathbb{Z}}$ of real numbers, bounded by L_A, such that $(k+\delta_k)_{k\in\mathbb{Z}}$ is strictly increasing, $\{k+\delta_k : k\in\mathbb{Z}\}$ is a subset of $\{\lambda_k : k\in\mathbb{Z}\}$, and

$$\left|\sum_{k=jN+1}^{(j+1)N}\delta_k\right| \le N\cdot d \quad \forall\, j\in\mathbb{Z}.$$

Applying Proposition 1 shows that $(e^{i(k+\delta_k)\bullet})_{k\in\mathbb{Z}}$ is an exact frame for $L^2(-\pi,\pi)$ with lower bound $A \ge A_{DS}(L,\delta,\sigma,\pi)$. Then, $A_{DS}(L,\delta,\sigma,\pi)$ is a lower bound for $(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$ as well. The existence of an upper bound follows from Lemma 1. This proves Theorem 2 for $\sigma>1, \gamma=\pi$. The general case follows by a dilation. □

Remark. Seip [9, Theorem 2.3] proved that, under the conditions of the theorem of Duffin and Schaeffer, there is a subsequence $(\lambda_{n_k})_{k\in\mathbb{Z}}$, such that $(e^{i\lambda_{n_k}\bullet})_{k\in\mathbb{Z}}$ is an exact frame for $L^2(-\gamma,\gamma)$. The proof of Theorem 2 even shows that $A_{DS}(L,\delta,\sigma,\gamma)$ is a lower bound for this exact frame.

Open question. Can *sharp* frame bounds be obtained, for the Theorems of Avdonin and of Duffin and Schaeffer? (The lower bounds, obtained in Theorems 1 and 2, are not sharp.)

4. An application to irregular sampling

Suppose $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of real numbers such that $(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$ is an (A,B)-frame for $L^2(-\sigma,\sigma)$. Define φ_n by

$$\varphi_n(z) := \begin{cases} \frac{\sigma}{\pi}\cdot\frac{\sin\sigma(z-\lambda_n)}{\sigma(z-\lambda_n)},\ z\neq\lambda_n \\ \frac{\sigma}{\pi},\ z=\lambda_n. \end{cases}$$

Then, $\varphi_n\in PW_\sigma^2$ and $(\varphi_n)_{n\in\mathbb{Z}}$ is an $(\frac{A}{2\pi},\frac{B}{2\pi})$-frame for PW_σ^2.
If $S: PW_\sigma^2\to PW_\sigma^2, f\mapsto\sum_{n\in\mathbb{Z}} f(\lambda_n)\varphi_n$, is the *frame operator*, corresponding to this frame, then S is a bijective bounded linear operator satisfying $\|S^{-1}\|\le 2\pi/A$ (cf. Benedetto and Walnut [2, Theorem 3.2], Duffin and Schaeffer [3, Section 3], Kölzow [5, Section II.1]). Define

$$s_n(f) := \sum_{|k|\le n} f(\lambda_k)S^{-1}\varphi_k, \quad f\in PW_\sigma^2,\ n\in\mathbb{N}.$$

Then, $(s_n(f))_{n\in\mathbb{N}}$ converges to f in the PW_σ^2-norm. For every $n\in\mathbb{N}$, $\|s_n(f)-f\|_{PW_\sigma^2}$ is called the nth *truncation error.*

Proposition 2. *Suppose $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of real numbers such that $(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$ is an (A,B)-frame for $L^2(-\sigma,\sigma)$. Then, the nth truncation error satisfies*

$$\|s_n(f)-f\|_{PW_\sigma^2} \le \sqrt{\frac{2\pi}{A}}\cdot\left(\sum_{|k|>n}|f(\lambda_k)|^2\right)^{1/2} \qquad \forall\, f\in PW_\sigma^2.$$

The first version of Proposition 2, proposed by BITTNER (private communication), contained the factor $\sqrt{2\pi B}/A$, instead of $\sqrt{2\pi/A}$. The latter was suggested by the referee, refering to the fact that $T : PW_\sigma^2 \to l^2(\mathbb{Z}),\ f \mapsto (f(\lambda_n))_{n\in\mathbb{Z}}$ is a bounded injective linear operator with bounded inverse on its range satisfying $\|T^{-1}\| \leq \sqrt{2\pi/A}$.

ACKNOWLEDGMENTS. The author would like to thank D. Kölzow for his steady support, and B. Bittner as well as the referee for their suggestions.

References

[1] S. A. AVDONIN. 1974. "On the question of Riesz bases of exponential functions in L^2." *Vestnik Leningrad Univ. Ser. Mat.*, **13**, pp. 5 - 12 (Russian); English translation in *Vestnik Leningrad Univ. Math.*, **7** (1979), pp. 203 - 211.

[2] J. J. BENEDETTO, D. F. WALNUT. 1994. "Gabor frames for L^2 and related spaces", in *Wavelets: Mathematics and Applications*, J. J. Benedetto, M. W. Frazier, eds., CRC Press, Boca Raton, pp. 97 - 162.

[3] R. J. DUFFIN, A. C. SCHAEFFER. 1952. "A class of nonharmonic Fourier series." *Trans. Amer. Math. Soc.*, **72**, pp. 341 - 366.

[4] V. É. KATSNEL'SON. 1971. "Exponential bases in L^2." *Funkcional'. Analiz Prilozenija,* **5**, No. 1, pp. 37 - 47 (Russian); English translation in *Functional Analysis Appl.*, **5** (1971), pp. 31 - 38.

[5] D. KÖLZOW. 1994. *Wavelets. A tutorial and a bibliography.* Rend. Ist. Matem. Trieste, **26**, Suppl.

[6] D. KÖLZOW. 1996. *Analysis-Seminar über Abtastsätze*, Wintersemester 1996/ 97, Univ. Erlangen-Nürnberg.

[7] B. YA. LEVIN. 1961. "On bases of exponential functions in L^2." *Zap. Har'kov. Gos. Univ. i Har'kov. Mat. Obšč.* (4), **27**, pp. 39 - 48. (Russian)

[8] B. YA. LEVIN. 1996. *Lectures on entire functions.* Translations of Mathematical Monographs, **150**, American Mathematical Society.

[9] K. SEIP. 1995. "On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$." *J. Funct. Anal.,* **130** , pp. 131 - 160.

[10] R. M. YOUNG. 1980. *An introduction to nonharmonic Fourier series.* Academic Press, New York.

Alexander M. Lindner
Mathematisches Institut, Universität Erlangen-Nürnberg
Bismarckstraße 1 1/2
D-91054 Erlangen GERMANY
e-mail: lindnera@mi.uni-erlangen.de