Continuous Time Approximations to GARCH and Stochastic Volatility Models

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Abstract We collect some continuous time GARCH models and report on how they approximate discrete time GARCH processes. Similarly, certain continuous time volatility models are viewed as approximations to discrete time volatility models.

1 Stochastic volatility models and discrete GARCH

Both stochastic volatility models and GARCH processes are popular models for the description of financial time series. Recall that a *discrete time stochastic volatility model* (SV-*model*) is a process $(X_n)_{n\in\mathbb{N}_0}$ together with a non-negative *volatility process* $(\sigma_n)_{n\in\mathbb{N}_0}$, such that

$$X_n = \sigma_n \varepsilon_n, \quad n \in \mathbb{N}_0, \tag{1}$$

where the *noise sequence* $(\varepsilon_n)_{n\in\mathbb{N}_0}$ is a sequence of independent and identically distributed (i.i.d.) random variables, which is assumed to be *independent of* $(\sigma_n)_{n\in\mathbb{N}_0}$. Further information about these processes can be found e.g. in Shephard (2008) and Davis and Mikosch (2008). In contrast to stochastic volatility models, GARCH processes have the property that the volatility process is specified as a function of the past observations. The classical ARCH(1) process by Engle (1982) and the GARCH(1,1) process by Bollerslev (1986), for example, are processes $(X_n)_{n\in\mathbb{N}_0}$ with a non-negative *volatility process* $(\sigma_n)_{n\in\mathbb{N}_0}$, such that

$$X_n = \sigma_n \varepsilon_n, \quad n \in \mathbb{N}_0, \tag{2}$$

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$$\sigma_n^2 = \omega + \lambda X_{n-1}^2 + \delta \sigma_{n-1}^2, \quad n \in \mathbb{N}. \tag{3}$$

Here, $(\varepsilon_n)_{n\in\mathbb{N}_0}$ is again an i.i.d. noise sequence, and the parameters ω, λ, δ satisfy $\omega > 0$, $\lambda > 0$ and $\delta > 0$ (GARCH(1,1)) or $\delta = 0$ (ARCH(1)), respectively. See e.g. Teräsvirta (2008) and Lindner (2008) for further information regarding GARCH processes and their probabilistic properties.

While financial data are usually observed only at discrete times, financial mathematicians often tend to work in continuous time, which appears to be more convenient for option pricing. However, continuous time models may also offer a good approximation to discrete observations. Typical examples are high-frequency data or irregularly observed data. While in discrete time, $(X_n)_{n\in\mathbb{N}_0}$ models the increments of the log price, in continuous time one rather models the log price $(G_t)_{t\geq 0}$ itself. Typically, one has an unobserved volatility process $(\sigma_t)_{t\geq 0}$ modelled as a semimartingale, and the log price is described by

$$G_t = \int_0^t (\mu + b\sigma_s^2)\, ds + \int_0^t \sigma_{s-} dM_s, \tag{4}$$

where $(M_t)_{t\geq 0}$ is a Lévy process and μ, b are real constants. For *continuous time stochastic volatility models*, the process $(M_t)_{t\geq 0}$ is usually independent of $(\sigma_t)_{t\geq 0}$, and more specifically, taken to be a standard Brownian motion. In the latter case, the quadratic variation of G until time t is $\int_0^t \sigma_s^2\, ds$, justifying the name *volatility* for σ_t.

The aim of this paper is to present some continuous time GARCH and SV models and to discuss in which sense they can be seen as approximations to corresponding discrete time models.

2 Continuous time GARCH approximations

If a continuous time model serves as an approximation to a GARCH process, one may ask in which sense the process when sampled at discrete times is close to a GARCH process. An optimal situation would be that the process itself is a GARCH process, whenever sampled at equidistant times $(hn)_{n\in\mathbb{N}_0}$, for each $h > 0$. This however cannot be achieved: Drost and Nijman (1993), Example 3, have shown that GARCH processes are not closed under temporal aggregation, i.e., if $(X_t)_{t\in\mathbb{N}_0}$ is a GARCH process driven by some noise $(\varepsilon_t)_{t\in\mathbb{N}_0}$ with volatility process $(\sigma_t)_{t\in\mathbb{N}_0}$, then – apart from some situations when the noise is degenerate – there does not exist an i.i.d. noise sequence $(\widetilde{\varepsilon}_{2t})_{t\in\mathbb{N}_0}$ and a volatility process $(\widetilde{\sigma}_{2t})_{t\in\mathbb{N}_0}$ such that $(X_{2t})_{t\in\mathbb{N}_0}$ is a GARCH process driven by $(\widetilde{\varepsilon}_{2t})_{t\in\mathbb{N}_0}$ with volatility process $(\widetilde{\sigma}_{2t})_{t\in\mathbb{N}_0}$. In particular, a continuous time process $(Y_t)_{t\geq 0}$ which happens to be a GARCH(1,1) process when sampled at $\{0, h, 2h, \ldots\}$ for some frequency h will not be GARCH when sampled at $\{0, 2h, 4h, \ldots\}$. Similarly, if for some log price process $(G_t)_{t\geq 0}$, the increments $(G_{nh} - G_{(n-1)h})_{n\in\mathbb{N}}$ of length h constitute a GARCH(1,1) process with non-degenerate noise, then the increments $(G_{n2h} - G_{(n-1)2h})_{n\in\mathbb{N}}$ of length $2h$ will usually not be GARCH. Hence one has to work with other concepts of GARCH approximations.

2.1 Preserving the random recurrence equation property

One approach to construct continuous time GARCH approximations is to require that certain properties of discrete time GARCH continue to hold. For example, a GARCH(1,1) process has the elegant property that its squared volatility satisfies the random recurrence equation $\sigma_n^2 = \omega + (\delta + \lambda\varepsilon_{n-1}^2)\sigma_{n-1}^2$, where σ_{n-1} is independent of ε_{n-1}. Denoting $A_n^{n-1} = \delta + \lambda\varepsilon_{n-1}^2$ and $C_n^{n-1} = \omega$, this can be written as

$$\sigma_n^2 = A_n^{n-1}\sigma_{n-1}^2 + C_n^{n-1},$$

where σ_{n-1}^2 is independent of (A_n^{n-1}, C_n^{n-1}) and $(A_n^{n-1}, C_n^{n-1})_{n\in\mathbb{N}}$ is i.i.d. Requiring at least this random recurrence equation property to hold for candidates of squared volatility processes, it is natural to look at processes $(Y_t)_{t\geq 0}$ which satisfy

$$Y_t = A_t^s Y_s + C_t^s, \quad 0 \leq s \leq t, \tag{5}$$

for appropriate sequences $(A_t^s, C_t^s)_{0\leq s\leq t}$ of bivariate random vectors. In order to ensure the i.i.d. property of $(A_{nh}^{(n-1)h}, C_{nh}^{(n-1)h})_{n\in\mathbb{N}}$ for every $h > 0$, one rather assumes that for every $0 \leq a \leq b \leq c \leq d$, the families of random variables $(A_t^s, C_t^s)_{a\leq s\leq t\leq b}$ and $(A_t^s, C_t^s)_{c\leq s\leq t\leq d}$ are independent, and that the distribution of $(A_{t+h}^{s+h}, C_{t+h}^{s+h})_{0\leq s\leq t}$ does not depend on $h \geq 0$. Finally, a natural continuity condition seems to be desirable, namely that

$$A_t^0 > 0 \text{ a.s.} \quad \forall\, t \geq 0, \quad \text{and} \quad (A_t, C_t) := (A_t^0, C_t^0) \xrightarrow{P} (1, 0) \quad \text{as } t \downarrow 0,$$

where "$\xrightarrow{P}$" denotes convergence in probability. De Haan and Karandikar (1989) showed that if $(A_t^s, C_t^s)_{0\leq s\leq t}$ are such that they satisfy the properties described above, then $(A_t, C_t)_{t\geq 0}$ admit càdlàg versions, and with these versions chosen, $(Y_t)_{t\geq 0}$ satisfies (5) if and only if there is a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ such that

$$Y_t = e^{-\xi_t}\left(Y_0 + \int_0^t e^{\xi_{s-}}\, d\eta_s\right), \quad t \geq 0. \tag{6}$$

This is a *generalised Ornstein-Uhlenbeck process*, which is discussed in detail in Maller et al. (2008b). From the point of view described above, generalised Ornstein-Uhlenbeck processes are natural continuous time analogues of random recurrence equations, and hence a desirable property of a continuous time GARCH(1,1) approximation is that its squared volatility process is a generalised Ornstein-Uhlenbeck process. As we shall see later, both the diffusion limit of Nelson (1990) as well as the COGARCH(1,1) process of Klüppelberg et al. (2004) satisfy this requirement. Also, the volatility model of Barndorff-Nielsen and Shephard (2001a) and (2001b) falls into this class, even if not constructed as a GARCH(1,1) approximation.

2.2 The diffusion limit of Nelson

A common method to construct continuous time processes from discrete ones is to use a diffusion approximation. Here, one takes a series of discrete time series defined on a grid (such as $h\mathbb{N}_0$) with mesh $h \downarrow 0$, extends the processes between grid points in a suitable way (such as interpolation, or piecewise constancy), and hopes that this sequence of processes defined on $[0,\infty)$ converges weakly to some limit process. Since the processes encountered will typically have sample paths in the Skorokhod space $D([0,\infty),\mathbb{R}^d)$ of $\mathbb{R}^d$-valued càdlàg functions defined on $[0,\infty)$, by *weak convergence* we mean weak convergence in $D([0,\infty),\mathbb{R}^d)$, when endowed with the (J_1-)Skorokhod topology, cf. Jacod and Shiryaev (2003), Sections VI.1 and VI.3. If the limit process has no fixed points of discontinuity (which will be the case in all cases encountered), then weak convergence in $D([0,\infty),\mathbb{R}^d)$ implies weak convergence of the finite dimensional distributions, and the converse is true under an additional tightness condition, cf. Jacod and Shiryaev (2003), Proposition VI.3.14 and VI.3.20.

Nelson (1990) derived a diffusion limit for GARCH(1,1) processes. In the same paper, he also considered the diffusion limit of EGARCH processes. An extension to diffusion limits of a more general class of GARCH processes (called *augmented* GARCH) was obtained by Duan (1997). Here, we shall concentrate on Nelson's diffusion limit of GARCH(1,1): for each $h>0$, let $(\varepsilon_{kh,h})_{k\in\mathbb{N}_0}$ be an i.i.d. sequence of standard normal random variables, let $\omega_h,\lambda_h>0$ and $\delta_h\geq 0$, and let $(G_{0,h},\sigma^2_{0,h})$ be starting random variables, independent of $(\varepsilon_{kh,h})_{k\in\mathbb{N}_0}$. Then $(G_{kh,h}-G_{(k-1)h,h},\sigma_{kh,h})_{k\in\mathbb{N}}$, defined recursively by

$$\begin{aligned} G_{kh,h} &= G_{(k-1)h,h} + h^{1/2}\sigma_{kh,h}\varepsilon_{kh,h}, \quad k\in\mathbb{N},\\ \sigma^2_{kh,h} &= \omega_h + (\lambda_h\varepsilon^2_{(k-1)h,h}+\delta_h)\sigma^2_{(k-1)h,h}, \quad k\in\mathbb{N}, \end{aligned}$$

is a GARCH(1,1) process for every $h>0$. Then $(G_{kh,h},\sigma^2_{kh,h})_{k\in\mathbb{N}_0}$ is embedded into a continuous time process $(G_{t,h},\sigma^2_{t,h})_{t\geq 0}$ by defining

$$G_{t,h} := G_{kh,h}, \quad \sigma^2_{t,h} := \sigma^2_{kh,h}, \quad kh \leq t < (k+1)h.$$

The latter process has sample paths in $D([0,\infty),\mathbb{R}^2)$, and Nelson (1990) gives conditions for $(G_{t,h},\sigma^2_{t,h})_{t\geq 0}$ to converge weakly to some process $(G_t,\sigma^2_t)_{t\geq 0}$ as $h\downarrow 0$. Namely, suppose that there are constants $\omega\geq 0$, $\theta\in\mathbb{R}$ and $\lambda>0$ as well as starting random variables (G_0,σ^2_0) such that $(G_{0,h},\sigma^2_{0,h})$ converges weakly to (G_0,σ^2_0) as $h\downarrow 0$, such that $P(\sigma^2_0>0)=1$ and

$$\lim_{h\downarrow 0} h^{-1}\omega_h=\omega,\quad \lim_{h\downarrow 0} h^{-1}(1-\delta_h-\lambda_h)=\theta,\quad \lim_{h\downarrow 0} 2h^{-1}\lambda_h^2=\lambda^2. \tag{7}$$

Then $(G_{t,h},\sigma^2_{t,h})_{t\geq 0}$ converges weakly as $h\downarrow 0$ to the unique solution $(G_t,\sigma^2_t)_{t\geq 0}$ of the diffusion equation

$$dG_t=\sigma_t\,dB_t,\quad t\geq 0, \tag{8}$$
$$d\sigma^2_t=(\omega-\theta\sigma^2_t)\,dt+\lambda\sigma^2_t dW_t,\quad t\geq 0, \tag{9}$$

with starting value (G_0,σ^2_0), where $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are independent Brownian motions, independent of (G_0,σ^2_0). Nelson also showed that (9) has a strictly stationary solution $(\sigma^2_t)_{t\geq 0}$ if $2\theta/\lambda^2>-1$ and $\omega>0$, in which case the marginal stationary distribution of σ^2_0 is inverse Gamma distributed with parameters $1+2\theta/\lambda^2$ and $2\omega/\lambda^2$. An example for possible parameter choices to satisfy (7) is given by $\omega_h=\omega h$, $\delta_h=1-\lambda\sqrt{h/2}-\theta h$, and $\lambda_h=\lambda\sqrt{h/2}$.

Observe that the limit volatility process $(\sigma^2_t)_{t\geq 0}$ in (9) is a generalised Ornstein-Uhlenbeck process as defined in (6), with $(\xi_t,\eta_t)=(-\lambda W_t+(\theta+\lambda^2/2)t,\omega t)$, see e.g. Fasen (2008) or Maller et al. (2008b).

A striking difference between discrete time GARCH processes and their diffusion limit is that the squared volatility process $(\sigma^2_t)_{t\geq 0}$ in (9) is independent of the Brownian motion $(B_t)_{t\geq 0}$, driving the log price process. So the volatility model (8), (9) has two independent sources of randomness, namely $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$. On the other hand, discrete time GARCH processes are defined only in terms of a single noise sequence $(\varepsilon_n)_{n\in\mathbb{N}_0}$, rather than two. While it was believed for a long time that Nelson's limit result justified the estimation of stochastic volatility models by GARCH-estimation procedures, Wang (2002) showed that statistical inference for GARCH modelling and statistical inference for the diffusion limit (8), (9) are not asymptotically equivalent, where asymptotic equivalence is defined in terms of Le Cam's deficiency distance (see Le Cam (1986)). As a heuristic explanation, Wang (2002) mentions the different kinds of noise propagations in the GARCH model with one source of randomness and the volatility model with two sources of randomness. It is possible to modify Nelson's approximation to obtain a limit process which is driven by a single Brownian motion only (see Corradi (2000)), but in that case the limiting volatility process is deterministic, an undesirable property of price processes. Observe however that for the latter case, the statistical estimation procedures are equivalent, cf. Wang (2002).

2.3 The COGARCH model

Apart from the fact that Nelson's diffusion limit is driven by two independent sources of randomness, it also has a continuous volatility process. Nowadays, jumps in the volatility of continuous time processes are often considered as a stylised fact, which hence is not met by model (8), (9). This led Klüppelberg et al. (2004) to the introduction of a new continuous time GARCH(1,1) process, called COGARCH(1,1), where "CO" stands for continuous time. Its construction starts from the observation that the recursions (2), (3) can be solved recursively, and σ_n^2 and X_n can be expressed by

$$\begin{aligned}\sigma_n^2 &= \omega \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} (\delta + \lambda\varepsilon_j^2) + \sigma_0^2 \prod_{j=0}^{n-1} (\delta + \lambda\varepsilon_j^2) \qquad (10)\\ &= \left(\omega \int_0^n \exp\left\{ - \sum_{j=0}^{\lfloor s \rfloor} \log(\delta + \lambda\varepsilon_j^2) \right\} ds + \sigma_0^2 \right) \exp\left\{ \sum_{j=0}^{n-1} \log(\delta + \lambda\varepsilon_j^2) \right\},\\ X_n &= \sigma_n \varepsilon_n = \sigma_n \left(\sum_{j=0}^{n} \varepsilon_j - \sum_{j=0}^{n-1} \varepsilon_j \right). \qquad (11)\end{aligned}$$

Here, $\lfloor z \rfloor$ denotes the largest integer not exceeding z, and both $\sum_{j=0}^{n-1} \log(\delta + \lambda\varepsilon_j^2) = n \log\delta + \sum_{j=0}^{n-1} \log(1 + \lambda\varepsilon_j^2/\delta)$ and $\sum_{j=0}^{n} \varepsilon_j$ are random walks, which are linked in such a way that the first can be reconstructed from the second. The idea of Klüppelberg et al. (2004) was then to replace the appearing random walks by Lévy processes, which are a continuous time analogue of random walks, and to replace the ε_j by the jumps of a Lévy process L. More precisely, they start with positive constants $\omega, \delta, \lambda > 0$ and a Lévy process $L = (L_t)_{t\geq 0}$ which has non-zero Lévy measure ν_L, and define an auxiliary Lévy process ξ_t by

$$\xi_t = -t\log\delta - \sum_{0<s\leq t} \log\left(1 + \frac{\lambda}{\delta}(\Delta L_s)^2\right), \quad t \geq 0,$$

corresponding to the random walk $-(n\log\delta + \sum_{j=0}^{n-1} \log(1+\lambda\varepsilon_j^2/\delta))$ in discrete time. Given a starting random variable σ_0^2, independent of $(L_t)_{t\geq 0}$, a *(right-continuous) volatility process* $(\sigma_t)_{t\geq 0}$ and the COGARCH(1,1) process are then defined by

$$\sigma_t^2 = \left(\omega \int_0^t e^{\xi_{s-}}\, ds + \sigma_0^2 \right) e^{-\xi_t}, \quad t \geq 0, \qquad (12)$$

$$G_t = \int_0^t \sigma_{s-}\, dL_s, \quad t \geq 0, \qquad (13)$$

in complete analogy to (10) and (11). (Originally, Klüppelberg et al. (2004) defined a *left-continuous* version of the volatility by considering σ_{t-}^2 rather than σ_t^2.) The process $(\xi_t)_{t\geq 0}$ is indeed a Lévy process, which is the negative of a subordinator together with drift $-\log\delta$. Hence the squared volatility process is again a generalised Ornstein-Uhlenbeck process as in (6) driven by $(\xi_t, \omega t)$. An application of Itô's formula to (12) shows that $(\sigma_t^2)_{t\geq 0}$ satisfies the stochastic differential equation

$$d\sigma_t^2 = (\omega + \log\delta\,\sigma_{t-}^2)\,dt + \frac{\lambda}{\delta}\,\sigma_{t-}^2\,d[L,L]_t^d,$$

where $[L,L]_t^d = \sum_{0<s\leq t}(\Delta L_s)^2$ denotes the discrete part of the quadratic variation of L. Note that G has only one source of randomness, namely L, which drives both σ_t^2 and G. In particular, if L jumps then so does G, with jump size $\Delta G_t = \sigma_{t-}\Delta L_t$. Stationarity and moment conditions for $(\sigma_t^2)_{t\geq 0}$ are given in Klüppelberg et al. (2004), and it follows that $(\sigma_t^2)_{t\geq 0}$ admits a stationary version if and only if $\int_{\mathbb{R}} \log(1+\lambda x^2/\delta)\,\nu_L(dx) < -\log\delta$, which in particular forces $\delta < 1$. As for discrete time GARCH processes, the stationary COGARCH volatility has Pareto tails under weak assumptions, cf. Klüppelberg et al. (2006). Under appropriate conditions, the increments of G are uncorrelated, while the squares of the increments are correlated. More precisely, the covariance structure of $((G_{nh} - G_{(n-1)h})^2)_{n\in\mathbb{N}}$ is that of an ARMA(1,1) process. Extensions of the COGARCH(1,1) process include the COGARCH(p,q) model by Brockwell et al. (2006), an asymmetric COGARCH(1,1) model by Haug et al. (2007) to include the leverage effect, and a multivariate COGARCH(1,1) model by Stelzer (2008).

The COGARCH(1,1) model was motivated by replacing the innovations in GARCH processes by the jumps of Lévy processes. The question in which sense a COGARCH process is close to a discrete time GARCH process was recently considered independently by Kallen and Vesenmayer (2008) as well as Maller et al. (2008a). In both papers it is shown that the COGARCH(1,1) model is a continuous time limit of certain GARCH(1,1) processes: more precisely, given a COGARCH process $(G_t)_{t\geq 0}$ with volatility process $(\sigma_t)_{t\geq 0}$, Kallen and Vesenmayer (2008) construct a sequence of discrete time GARCH processes $(Y_{k,n})_{k\in\mathbb{N}}$ with volatility $(\sigma_{k,n})_{k\in\mathbb{N}}$, such that the processes $(\sum_{k=1}^{\lfloor nt\rfloor} Y_{k,n}, \sigma_{\lfloor nt\rfloor+1,n})_{t\geq 0}$ converge weakly to $(G_t,\sigma_t)_{t\geq 0}$ as $n\to\infty$. Here, weak convergence in the Skorokhod space $D([0,\infty),\mathbb{R}^2)$ is obtained by computing the semimartingale characteristics of $(G_t,\sigma_t)_{t\geq 0}$ and showing that they are the limit of those of $(\sum_{k=1}^{\lfloor nt\rfloor} Y_{k,n}, \sigma_{\lfloor nt\rfloor+1,n})_{t\geq 0}$ as $n\to\infty$. The infinitesimal generator of the strong Markov process $(G_t,\sigma_t)_{t\geq 0}$ was also obtained. They also showed how a given GARCH(1,1) process can be scaled to converge to a COGARCH(1,1) process. Using completely different methods, given a COGARCH(1,1) process driven by a Lévy process with mean zero and finite variance, Maller et al. (2008a) also obtain a sequence of discrete time GARCH processes $(Y_{k,n})_{k\in\mathbb{N}}$ with volatility processes $(\sigma_{k,n})_{k\in\mathbb{N}}$

such that $(\sum_{k=1}^{\lfloor nt \rfloor} Y_{k,n}, \sigma_{\lfloor nt \rfloor +1,n})_{t\geq 0}$ converges in probability to $(G_t, \sigma_t)_{t\geq 0}$ as $n \to \infty$. Observe that convergence in probability is stronger than weak convergence. The discrete time GARCH processes are constructed using a "first-jump" approximation for Lévy processes as developed by Szimayer and Maller (2007), which divides a compact interval into an increasing number of subintervals and for each subinterval takes the first jump exceeding a certain threshold. Summing up, we have seen that the COGARCH(1,1) model is a limit of GARCH(1,1) processes, although originally motivated by mimicking features of discrete GARCH(1,1) processes without referring to limit procedures.

2.4 Weak GARCH processes

Another approach to obtain continuous time GARCH processes is to weaken the definition of a GARCH process. Observe that if $(X_n)_{n\in\mathbb{N}_0}$ is a GARCH(1,1) process with finite fourth moment and volatility process $(\sigma_n)_{n\in\mathbb{N}_0}$, driven by i.i.d. noise $(\varepsilon_n)_{n\in\mathbb{N}_0}$ such that $E\varepsilon_0 = 0$ and $E\varepsilon_0^2 = 1$, then

$$PL_n(X_n) = 0 \quad \text{and} \quad PL_n(X_n^2) = \sigma_n^2,$$

where $PL_n(Z)$ denotes the best linear predictor of a square integrable random variable Z with respect to 1, σ_0^2, X_0, ..., X_{n-1}, X_0^2, ..., X_{n-1}^2. Drost and Nijman (1993) use this property to define weak GARCH processes: they call a univariate process $(X_n)_{n\in\mathbb{N}_0}$ a *weak* GARCH(1,1) process with parameter $(\omega, \lambda, \delta)$, if X_n has finite fourth moment and there exists a volatility process $(\sigma_n)_{n\in\mathbb{N}_0}$ such that $(\sigma_n^2)_{n\in\mathbb{N}_0}$ is weakly stationary and satisfies (3) for $n \in \mathbb{N}$, and it holds $PL_n(X_n) = 0$ and $PL_n(X_n^2) = \sigma_n^2$. Here, $\omega > 0$, $\lambda \geq 0$, $\delta \geq 0$, and either $\lambda = \delta = 0$ or $0 < \lambda + \delta < 1$. Unlike GARCH processes, the class of weak GARCH processes is closed under temporal aggregation, i.e. if $(X_n)_{n\in\mathbb{N}_0}$ is a symmetric weak GARCH(1,1) process, then so is $(X_{mn})_{n\in\mathbb{N}_0}$ for every $m \in \mathbb{N}$, see Drost and Nijman (1993), Example 1. Based on this property, Drost and Werker (1996) define a *continuous time weak* GARCH(1,1) *process* to be a univariate process $(G_t)_{t\geq 0}$ such that $(G_{t_0+nh} - G_{t_0+(n-1)h})_{n\in\mathbb{N}}$ is a weak GARCH(1,1) process for every $h > 0$ and $t_0 \geq 0$. They also show that the parameters of the discretised weak GARCH process correspond to certain parameters in the continuous time weak GARCH process, so that estimation methods for discrete time weak GARCH processes carry over to certain parameters of continuous time weak GARCH processes. Examples of continuous time weak GARCH processes include the diffusion limit of Nelson, provided it has finite fourth moment, or more generally processes $(G_t)_{t\geq 0}$ with finite fourth moments of the form $dG_t = \sigma_{t-} dL_t$, where $(\sigma_t^2)_{t\geq 0}$ is supposed to be a stationary solution of the stochastic differential equation

$$d\sigma_t^2 = (\omega - \theta\sigma_{t-}^2)\,dt + \lambda\sigma_{t-}^2\,d\eta_t.$$

Here, $(L_t)_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$ are two independent Lévy processes with finite fourth moment, expectation 0 and variance 1, $(L_t)_{t\geq 0}$ is symmetric and the parameters $\omega > 0$, $\theta > 0$ and $\lambda < 1$ are chosen such that $E\sigma_0^4 < \infty$, see Drost and Werker (1996), Example 4.1.

2.5 Stochastic delay equations

A somewhat different approach to obtain continuous time GARCH processes is taken by Lorenz (2006). He considered a weak limit of scaled GARCH($pn+1,1$) processes when the order $pn+1$ goes to ∞, and in the limit he obtained the solution to a stochastic delay differential equation. More precisely, let $(\varepsilon_k)_{k\in\mathbb{N}_0}$ be a sequence of i.i.d. random variables with finite $(4+\alpha)$-moment for some $\alpha > 0$ such that $E(\varepsilon_1) = E(\varepsilon_1^3) = 0$ and $E(\varepsilon_1^2) = 1$. Let $p \in \mathbb{N}$ and $(\sigma_t)_{t\in[-p,0]}$ be some given strictly positive continuous function on $[-p,0]$, and define $G_t = 0$ for $t \in [-p,0]$. Let $\omega_n > 0$, $\delta_{j,n} \geq 0$ ($j = 0,\ldots,pn$) and $\lambda_n \geq 0$, and consider the discrete time GARCH($pn+1,1$) process $(Y_{k,n})_{k\in\mathbb{N}}$ with volatility $(\sigma_{k,n})_{k\in\mathbb{N}}$ given by

$$\begin{aligned}
Y_{k,n} &= n^{-1/2}\,\sigma_{k,n}\varepsilon_k, \quad k \in \mathbb{N},\\
\sigma_{k,n}^2 &= \omega_n + \sum_{j=0}^{np} \delta_{j,n}\sigma_{k-1-j,n}^2 + \lambda_n\sigma_{k-1,n}^2\varepsilon_{k-1}^2, \quad k \in \mathbb{N},
\end{aligned}$$

where $\sigma_{j,n} := \sigma_{-j/n}$ for $j \in \{-pn,\ldots,0\}$. Define further $(G_{t,n},\sigma_{t,n})_{t\geq -p} := (\sum_{k=1}^{\lfloor nt\rfloor} Y_{k,n}, \sigma_{\lfloor nt\rfloor+1,n})_{t\geq -p}$. Assuming that

$$\lim_{n\to\infty} n\omega_n = \omega > 0, \quad \lim_{n\to\infty} n(1-\delta_{0,n}-\lambda_n) = \theta \in \mathbb{R}, \quad \lim_{n\to\infty}(E\varepsilon_1^4-1)n\lambda_n^2 = \lambda^2 \tag{14}$$

and that the sequence $(\gamma_n)_{n\in\mathbb{N}}$ of discrete measures γ_n on $[-p,0]$ defined by $\gamma_n(\{-j/n\}) = n\delta_{j,n}$ for $1 \leq j \leq pn$ and $\gamma_n(\{0\}) = 0$ converges vaguely to some finite measure γ on $[-p,0]$ such that $\gamma(\{0\}) = 0$, Lorenz (2006), Theorem 2.5.10, showed that $(G_{t,n},\sigma_{t,n})_{t\geq 0}$ converges weakly as $n \to \infty$ to the unique weak solution $(G_t,\sigma_t)_{t\geq 0}$ of the stochastic delay differential equation

$$dG_t = \sigma_t\,dB_t, \quad t \geq 0, \tag{15}$$

$$d\sigma_t^2 = (\omega - \theta\sigma_t^2)\,dt + \left(\int_{[-p,0]} \sigma_{t+u}^2\,d\gamma(u)\right)dt + \lambda\sigma_t^2\,dW_t, \tag{16}$$

with starting values as given, and where $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are two independent Brownian motions. A sufficient condition for a stationary solution of the stochastic delay equation (16) to exist is also given in Lorenz (2006). Observe that if $\delta_{j,n} = 0$ for $j = 1, \dots, pn$, the discrete GARCH$(pn+1, 1)$ processes are actually GARCH(1,1) processes, the limit measure γ is zero, and (14), (15), (16) reduce to the corresponding equations (7), (8), (9) for Nelson's diffusion limit.

A related paper regarding limits of HARCH processes which give rise to stochastic delay equations is Zheng (2005).

2.6 A continuous time GARCH model designed for option pricing

The previous continuous time GARCH models have been mainly designed as limits of discrete time GARCH processes or as processes with properties similar to GARCH. Option pricing for such models may be demanding, since they often give rise to incomplete markets. Inspired by this, Kallsen and Taqqu (1998) developed a continuous time process which is a GARCH process when sampled at integer times. Their process is also driven by a single Brownian motion only. More specifically, let $\omega, \lambda > 0$, $\delta \geq 0$ and $(B_t)_{t \geq 0}$ be a standard Brownian motion. For some starting random variable σ_0^2, define the volatility process $(\sigma_t)_{t\geq 0}$ by $\sigma_t^2 = \sigma_0^2$ for $t \in [0,1)$ and

$$\sigma_t^2 = \omega + \lambda \left(\int_{\lfloor t \rfloor - 1}^{\lfloor t \rfloor} \sigma_{s-} dB_s \right)^2 + \delta \sigma^2_{\lfloor t \rfloor - 1}, \quad t \geq 1. \tag{17}$$

The continuous time GARCH process $(G_t)_{t\geq 0}$ then models the log-price process, and is given by

$$G_t = G_0 + \int_0^t (\mu(\sigma_{s-}) - \sigma_{s-}^2/2)\, ds + \int_0^t \sigma_s \, dB_s.$$

Here, the drift function μ is assumed to have continuous derivatives. Observe that the volatility process $(\sigma_t)_{t\geq 0}$ given by (17) is constant on intervals $[n, n+1)$ for $n \in \mathbb{N}_0$. Also observe that the process $(G_t - G_{t-1}, \sigma_{t-1})_{t\geq 1}$, when sampled at integer times, gives rise to a discrete time GARCH(1,1)-M process

$$G_n - G_{n-1} = \mu(\sigma_{n-1}) - \sigma_{n-1}^2/2 + \sigma_{n-1}(B_n - B_{n-1}), \quad n \in \mathbb{N},$$
$$\sigma_n^2 = \omega + \lambda \sigma_{n-1}^2 (B_n - B_{n-1})^2 + \delta \sigma_{n-1}^2, \quad n \in \mathbb{N}.$$

This differs from a usual GARCH(1,1) process only by the term $\mu(\sigma_{n-1}) - \sigma_{n-1}^2/2$, which vanishes if the function μ is chosen as $\mu(x) = x^2/2$. If we are not in the classical GARCH situation but rather have $\limsup_{x\to\infty} \mu(x)/x < \infty$, then Kallsen and Taqqu (1998) show that the continuous time model is arbitrage free and complete. This is then used to derive pricing formulas for contingent claims such as European options.

3 Continuous time stochastic volatility approximations

Recall from Section 1 that by a discrete time stochastic volatility model we mean a process $(X_n)_{n\in\mathbb{N}_0}$ satisfying (1), where $(\varepsilon_n)_{n\in\mathbb{N}_0}$ is i.i.d. and $(\sigma_n)_{n\in\mathbb{N}_0}$ is a stochastic volatility process, independent of $(\varepsilon_n)_{n\in\mathbb{N}_0}$. Here, we shall usually restrict ourselves to the case when $(\varepsilon_n)_{n\in\mathbb{N}_0}$ is i.i.d. normally distributed with expectation zero. Also recall that we defined continuous time stochastic volatility models by (4). Now, we shall further restrict ourselves to the case where $\mu = b = 0$ and M in (4) is Brownian motion, i.e. we consider models of the form

$$G_t = \int_0^t \sigma_{s-}\, dB_s, \quad t \geq 0, \tag{18}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, independent of the volatility process $\sigma = (\sigma_t)_{t\geq 0}$. The latter is assumed to be a strictly positive semimartingale, in particular it has càdlàg paths.

3.1 Sampling a continuous time SV model at equidistant times

In the setting as given above, it is easy to see that discrete time SV models are closed under temporal aggregation, however, with a possibly unfamiliar volatility process after aggregation. Similarly, the continuous time SV model (18) gives rise to a discrete time SV model, when sampled at equidistant time points. To see the latter, let G be given by (18), $h > 0$, and define

$$\varepsilon_k := \frac{G_{kh} - G_{(k-1)h}}{(\int_{(k-1)h}^{kh} \sigma_s^2\, ds)^{1/2}}, \quad k \in \mathbb{N}.$$

Since conditionally on $(\sigma_t)_{t\geq 0}$, $G_{kh} - G_{(k-1)h}$ is normally distributed with expectation zero and variance $\int_{(k-1)h}^{kh} \sigma_s^2\, ds$, it follows that conditionally on $(\sigma_t)_{t\geq 0}$, ε_k is standard normally distributed, and since this distribution does not depend on $(\sigma_t)_{t\geq 0}$, ε_k itself is $N(0,1)$ distributed. With similar arguments one sees that ε_k is independent of σ and that $(\varepsilon_k)_{k\in\mathbb{N}}$ is i.i.d. Then $(G_{kh} - G_{(k-1)h} = \widetilde{\sigma}_k \varepsilon_k)_{k\in\mathbb{N}}$ is a discrete time stochastic volatility model, with discrete time volatility process

$$\widetilde{\sigma}_k := \left(\int_{(k-1)h}^{kh} \sigma_s^2 \, ds \right)^{1/2}, \quad k \in \mathbb{N}. \tag{19}$$

Hence we see that unlike GARCH processes, continuous time SV models yield discrete time SV models when sampled, i.e. they stay in their own class. Unfortunately, the volatility process $(\widetilde{\sigma}_k)_{k\in\mathbb{N}}$ obtained by this method is not always in a very tractable form, and often it might be desirable to retain a particular structure on the volatility process. As an illustration of a process, where most but not all of the structure is preserved, consider the stochastic volatility model of Barndorff-Nielsen and Shephard (2001a) and (2001b). Here, the volatility process $(\sigma_t)_{t\geq 0}$ is modeled via $d\sigma_t^2 = -\lambda \sigma_t^2 \, dt + dL_{\lambda t}$, where $\lambda > 0$ and L is a subordinator, i.e. a Lévy process with increasing sample paths. The solution to this Lévy driven Ornstein-Uhlenbeck process is given by

$$\sigma_t^2 = (\sigma_0^2 + \int_0^t e^{\lambda s} \, dL_{\lambda s}) e^{-\lambda t}, \quad t \geq 0. \tag{20}$$

Taking $\lambda = 1$ and $h = 1$ for simplicity, it follows that σ_t^2 satisfies

$$\sigma_t^2 = e^{-1} \sigma_{t-1}^2 + \int_{t-1}^t e^{u-t} \, dL_u, \quad t \geq 1, \tag{21}$$

so that

$$\widetilde{\sigma}_k^2 = \int_{k-1}^k \sigma_s^2 \, ds = e^{-1} \widetilde{\sigma}_{k-1}^2 + \int_{k-1}^k \int_{s-1}^s e^{u-s} \, dL_u \, ds, \quad k \in \mathbb{N} \setminus \{1\}. \tag{22}$$

Like the Ornstein-Uhlenbeck process, which is a continuous time AR(1) process, (22) yields a discrete time AR(1) process for $(\widetilde{\sigma}_k^2)_{k\in\mathbb{N}}$. However, (20) is driven by a Lévy process which can be interpreted as a continuous time analogue to i.i.d. noise, while (22) has 1-dependent noise given by $(\int_{k-1}^k \int_{s-1}^s e^{u-s} \, dL_u \, ds)_{k\in\mathbb{N}\setminus\{1\}}$.

We have seen that sampled continuous time SV models give rise to discrete time SV models, however the discrete time volatility may lose certain structural features. Allowing more general definitions of discrete and continuous time stochastic volatility models, in a spirit similar to the weak GARCH processes by Drost and Nijman (1993) and Drost and Werker (1996), Meddahi and Renault (2004) consider many continuous time SV models which keep the same structure when sampled at equidistant time points. We do not go into further detail, but refer to Meddahi and Renault (2004) and the overview article by Ghysels et al. (1996).

3.2 Approximating a continuous time SV model

Rather than working with the unfamiliar discrete time volatility $(\widetilde{\sigma}_k)_{k\in\mathbb{N}}$ as given in (19), one might try to use an Euler type approximation for the process G, and sample the (unobserved) volatility process $(\sigma_t^2)_{t\geq 0}$ directly at equidistant times. More precisely, consider G_t as defined in (18), where $(\sigma_t)_{t\geq 0}$ is a strictly positive semimartingale, independent of $(B_t)_{t\geq 0}$. For $h > 0$, define

$$Y_{k,h} := \sigma_{(k-1)h}(B_{kh} - B_{(k-1)h}), \quad k \in \mathbb{N}, \tag{23}$$

where $\sigma_{(k-1)h}$ is the continuous time volatility process $(\sigma_t)_{t\geq 0}$ taken at times $(k-1)h$. Then $(Y_{k,h})_{k\in\mathbb{N}}$ defines a discrete time SV model, which approximates $(G_{kh} - G_{(k-1)h})_{k\in\mathbb{N}}$. Indeed, since $(\sigma_t(\omega))_{t\geq 0}$ is a càdlàg function for almost every ω in the underlying probability space, one can easily show that the sequence of processes $\sigma^{(n)} = \sum_{k=1}^{\infty} \sigma_{k/n} \mathbf{1}_{[(k-1)/n,k/n)}$ converges almost surely on every compact interval $[0,T]$ with $T \in \mathbb{N}$ in the Skorokhod topology of $D([0,T],\mathbb{R})$ to $(\sigma_t)_{0\leq t\leq T}$, as $n \to \infty$. On the other hand, the process $(\sum_{k=1}^{\lfloor nt\rfloor+1} Y_{k,1/n})_{0\leq t\leq T}$ converges uniformly in probability to $(\int_0^t \sigma_{s-}\, dB_s)_{0\leq t\leq T}$ as $n \to \infty$, see Protter (2004), Theorem II.21. Using the continuity of $(G_t)_{t\geq 0}$, it is then easy to deduce that the bivariate process $(\sigma^{(n)}(t), \sum_{k=1}^{\lfloor nt\rfloor+1} Y_{k,1/n})_{0\leq t\leq T}$ converges in probability to $(\sigma_t, G_t)_{0\leq t\leq T}$ in the Skorokhod space $D([0,T],\mathbb{R}^2)$, from which convergence in probability on the whole space $D([0,\infty),\mathbb{R}^2)$ can be deduced. Hence the continuous time SV model is a limit of the discrete time SV models (23), as $h = 1/n \to 0$. The structure of $(\sigma_t)_{t\geq 0}$ is usually much more compatible with the structure of $(\sigma_{kh})_{k\in\mathbb{N}}$ than with the structure of $(\widetilde{\sigma}_k)_{k\in\mathbb{N}}$ of (19), and often the discretisation (23) leads to popular discrete time SV models. We give some examples.

Example 1 In the volatility model of Hull and White (1987) the continuous time volatility process $(\sigma_t)_{t\geq 0}$ follows a geometric Brownian motion, i.e. $d\sigma_t^2 = \sigma_t^2(b\, dt + \delta\, dW_t)$, where $(W_t)_{t\geq 0}$ is a Brownian motion. Then $\sigma_t^2 = \exp\{(b - \delta^2/2)t + \delta W_t\}$, so that for each $h > 0$, $\log \sigma_{kh}^2 - \log \sigma_{(k-1)h}^2 = (b-\delta^2/2)h + \delta(W_{kh} - W_{(k-1)h})$, meaning that $(\log \sigma_{kh}^2)_{k\in\mathbb{N}_0}$ is a random walk with i.i.d. $N((b - \delta^2/2)h, \delta^2)$ innovations.

Example 2 In the volatility model of Wiggins (1987), see also Scott (1987), the log-volatility is modelled as a Gaussian Ornstein-Uhlenbeck process, i.e. σ_t^2 satisfies the stochastic differential equation $d \log \sigma_t^2 = (b_1 - b_2 \log \sigma_t^2)\, dt + \delta\, dW_t$ with a Brownian motion $(W_t)_{t\geq 0}$. The solution to this equation is

$$\log \sigma_t^2 = e^{-b_2 t}\left(\log \sigma_0^2 + \int_0^t e^{b_2 s}(b_1\, ds + \delta\, dW_s)\right), \quad t \geq 0,$$

so that for each $h > 0$ we obtain

$$\log \sigma_{kh}^2 = e^{-b_2 h} \log \sigma_{(k-1)h}^2 + \int_{(k-1)h}^{kh} e^{b_2(s-kh)} (b_1\, ds + \delta\, dW_s), \quad k \in \mathbb{N},$$

which is an AR(1) process with i.i.d. normal noise. So in this case we recognize model (23) as the volatility model of Taylor (1982). Unlike for Nelson's GARCH(1,1) diffusion limit, the continuous time SV model of Wiggins and its diffusion approximation (23) are statistically equivalent, as investigated by Brown et al. (2003).

Example 3 In the volatility model of Barndorff-Nielsen and Shephard (2001a) and (2001b), where the squared volatility is modelled as a subordinator driven Ornstein-Uhlenbeck process, one obtains similarly to (21),

$$\sigma_{kh}^2 = e^{-\lambda h} \sigma_{(k-1)h}^2 + \int_{(k-1)h}^{kh} e^{\lambda(u-kh)}\, dL_{\lambda u}, \quad k \in \mathbb{N},$$

so that the discretised squared volatility satisfies an AR(1) process with non-Gaussian but positive i.i.d. noise.

Example 4 If one models the squared volatility $(\sigma_t^2)_{t\geq 0}$ by a subordinator driven continuous time ARMA process (CARMA) as suggested by Brockwell (2004), then the discretised squared volatility follows a discrete time ARMA process, but not necessarily with i.i.d. noise, see Brockwell (2008). If one models instead the log-volatility $(\log \sigma_t^2)_{t\geq 0}$ by a Lévy driven CARMA process, similarly to the method of Haug and Czado (2007) who specify the volatility of an exponential continuous time GARCH process in this way, then the discretised log-volatility $(\log \sigma_{kh}^2)_{k\in\mathbb{N}}$ follows a discrete time ARMA process. If the driving Lévy process is a Brownian motion, then the discrete time ARMA process also has i.i.d. Gaussian noise.

Example 5 If one approximates the GARCH diffusion limit (8), (9) via (23), the resulting discretised squared volatility process $(\sigma_{kh}^2)_{k\in\mathbb{N}_0}$ satisfies a random recurrence equation $\sigma_{kh}^2 = A_{kh}^{(k-1)h} \sigma_{(k-1)h}^2 + C_{kh}^{(k-1)h}$ with i.i.d. $(A_{kh}^{(k-1)h}, C_{kh}^{(k-1)h})_{k\in\mathbb{N}}$, where

$$A_{kh}^{(k-1)h} = e^{\lambda(W_{kh} - W_{(k-1)h}) - (\theta + \lambda^2/2)h},$$
$$C_{kh}^{(k-1)h} = \omega \int_{(k-1)h}^{kh} e^{\lambda(W_{kh} - W_s) - (\theta+\lambda^2/2)(kh-s)}\, ds.$$

This follows from the fact that the squared volatility process satisfies a generalised Ornstein-Uhlenbeck process as pointed out in Section 2. Also observe that a random recurrence equation may be viewed as kind of an AR(1) process with random coefficients.

Summing up, we have seen that many of the popular continuous time stochastic volatility models can be approximated by corresponding discrete time stochastic volatility models. Similarly, one can understand a continuous time SV model as an approximation to corresponding discrete time SV models. One could further consider diffusion limits of specific given discrete time SV models after proper scaling, but we will not report on such results for stochastic volatility models, since the discrete time SV models obtained from continuous time SV models via (23) already cover a wide range of popular volatility models.

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