Strictly Stationary Solutions of Multivariate ARMA and Univariate ARIMA Equations

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Bernd Karl Vollenbröker
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Referent: Prof. Dr. Alexander Lindner, TU Braunschweig
Koreferent: Prof. Dr. Michael Neumann, Universität Jena

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Abstract

The main focus of this thesis is to give a characterization of the existence of strictly stationary solutions of multivariate ARMA and univariate ARIMA equations.
In Chapter 2 we consider the multivariate ARMA(p,q) equation

$$Y_t - \Psi_1 Y_{t-1} - \ldots - \Psi_p Y_{t-p} = \Theta_0 Z_t + \ldots + \Theta_q Z_{t-q}, \quad t \in \mathbb{Z},$$

where $m, d \in \mathbb{N} = \{1, 2 \ldots, \}$, $p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(Z_t)_{t\in\mathbb{Z}}$ is a d-variate independent and identically distributed (i.i.d.) noise sequence of random vectors defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$ are deterministic complex-valued matrices. No a priori moment assumptions on the noise sequence are made.
First we give necessary and sufficient conditions for the existence of a strictly stationary solution to an ARMA(1,q) equation in terms of the Jordan canonical decomposition of Ψ_1 and properties of Z_0 and the coefficients Θ_k. An explicit solution, assuming its existence, is also derived and the question of uniqueness of this solution is solved.
Then, applying this, we give equivalent conditions for the existence of a strictly stationary solution to an ARMA(p,q) equation in terms of finite log-moments of certain linear combinations of the components of Z_0 and the characteristic polynomials

$$P(z) := \mathrm{Id}_m - \sum_{k=1}^{p} \Psi_k z^k \quad \text{and} \quad Q(z) := \sum_{k=0}^{q} \Theta_k z^k \quad \text{for} \quad z \in \mathbb{C}.$$

Again, an explicit solution, assuming its existence, is derived and the question of uniqueness of this solution is solved.
In Chapter 3 we consider the univariate ARIMA(p,D,q) equation

$$\Phi(B) Y_t = \Theta(B)[\nabla^{-D} Z_t], \quad t \in \mathbb{Z},$$

where $(Z_t)_{t\in\mathbb{Z}}$ is a real-valued i.i.d. noise sequence, $\Phi(z) := 1 - \sum_{k=1}^{p} \varphi_k z^k$, $\Theta(z) := 1 + \sum_{k=1}^{q} \theta_k z^k$, $z \in \mathbb{C}$, $\varphi_1, ..., \varphi_p, \theta_1, ..., \theta_q \in \mathbb{C}$, $\varphi_p \neq 0$ and $\theta_q \neq 0$, $\nabla^{-D} = (1-B)^{-D} = \sum_{j=0}^{\infty} (-1)^j \binom{-D}{j} B^j$, B the backwards shift operator. We characterize for which i.i.d.

noise sequences $(Z_t)_{t\in\mathbb{Z}}$ the series defining the fractional noise $\nabla^{-D} Z_t$ converges almost surely and give necessary and sufficient conditions for the existence of a strictly stationary solution to the above ARIMA equation to exist, derive an explicit solution, given its existence, and solve the question of uniqueness of the solution.

Zusammenfassung

Das Hauptaugenmerk dieser Arbeit liegt auf einer Charakterisierung der Existenz von strikt stationären Lösungen multivariater ARMA- und univariater ARIMA-Gleichungen.

In Kapitel 2 betrachten wir die multivariate ARMA(p,q) Gleichung

$$Y_t - \Psi_1 Y_{t-1} - \ldots - \Psi_p Y_{t-p} = \Theta_0 Z_t + \ldots + \Theta_q Z_{t-q}, \quad t \in \mathbb{Z},$$

wobei $m, d \in \mathbb{N} = \{1, 2 \ldots, \}$, $p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(Z_t)_{t\in\mathbb{Z}}$ eine d-variate, unabhängig und identisch verteilte Folge von Zufallsvektoren ist, definiert auf demselben Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, \mathbb{P})$, und $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m\times m}$, $\Theta_0, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$ deterministische, komplexwertige Matrizen sind. A priori wird keine Momentenbedingung an die Folge $(Z_t)_{t\in\mathbb{Z}}$ gestellt.

Zu Beginn geben wir notwendige und hinreichende Bedingungen für die Existenz strikt stationärer Lösungen einer ARMA(1,q)-Gleichung vermittels der Jordanschen Normalform von Ψ_1 und Eigenschaften von Z_0 und den Koeffizienten Θ_k. Im Falle der Existenz wird eine explizite Lösung hergeleitet und die Frage der Eindeutigkeit dieser Lösung beantwortet.

Darauf aufbauend geben wir äquivalente Bedingungen für die Existenz strikt stationärer Lösungen einer ARMA(p,q)-Gleichung vermittels endlicher log-Momente bestimmter Linearkombinationen der Komponenten von Z_0 und vermittels der charakteristischen Polynome

$$P(z) := \mathrm{Id}_m - \sum_{k=1}^{p} \Psi_k z^k \quad \text{und} \quad Q(z) := \sum_{k=0}^{q} \Theta_k z^k \quad \text{for} \quad z \in \mathbb{C}.$$

Im Falle der Existenz wird hier ebenfalls eine explizite Lösung hergeleitet und die Frage der Eindeutigkeit dieser Lösung beantwortet.

In Kapitel 3 betrachten wir die univariate ARIMA(p,D,q)-Gleichung

$$\Phi(B)Y_t = \Theta(B)[\nabla^{-D} Z_t], \quad t \in \mathbb{Z},$$

wobei $(Z_t)_{t\in\mathbb{Z}}$ eine reellwertige, unabhängig und identisch verteilte Folge von Zufallsvariablen ist, $\Phi(z) := 1 - \sum_{k=1}^{p} \varphi_k z^k$, $\Theta(z) := 1 + \sum_{k=1}^{q} \theta_k z^k$, $z \in \mathbb{C}$, $\varphi_1, ..., \varphi_p$,

$\theta_1, ..., \theta_q \in \mathbb{C}$, $\varphi_p \neq 0$ und $\theta_q \neq 0$, $\nabla^{-D} = (1-B)^{-D} = \sum_{j=0}^{\infty}(-1)^j\binom{-D}{j}B^j$, B der Backwards-Shift-Operator. Wir charakterisieren für welche unabhängig und identisch verteilten Folgen $(Z_t)_{t\in\mathbb{Z}}$ die den fraktionalen Noise definierende Reihe $\nabla^{-D}Z_t$ fast sicher konvergiert und geben notwendige und hinreichende Bedingungen für die Existenz strikt stationärer Lösungen der obigen ARIMA-Gleichung, leiten im Falle der Existenz eine explizite Lösung her und beantworten die Frage der Eindeutigkeit dieser Lösung.

Contents

1 Introduction

In everyday life there are numerous examples of time series. For instance, for hundreds of years people have been keeping records of various weather data like the daily maximum temperature, the Department of Labor monthly announces the unemployment rate, or various stock market indices are determined nearly continuously, to name just a few examples. To single out the first of these examples, why do people over such a long time collect certain weather data? One main reason certainly is the will to forecast tomorrow's weather, next week's weather or even next month's weather with the help of the collected data from the past.

Consequently, when wanting to get to know information about future events, techniques to describe and analyze the collected data are needed, and this is where mathematical time series analysis comes into play. It is based on the assumption that the collected data from different points in time is a realization of a random model in order to allow for the unpredictable nature of future. More precisely, it is assumed that the collected data is a realization of an unknown real-valued sequence of random variables $(Y_t)_{t\in\mathbb{Z}}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

One goal of time series analysis now is to describe and analyze chronologies and interdependencies of such above-mentioned data, which involves finding appropriate models for the dependence structure. At this, it seems desirable to sort out such (more or less) deterministic components of the data that reflect a certain trend or are periodically recurring in order to get a purely stochastic model. For instance, when looking at monthly accidental deaths data over the last 30 years, a linear and decreasing trend immediately becomes evident, which is due to technological change, improved traffic infrastructure or improved medical care, among others. After subtracting this trend, there still is a seasonal fluctuation in the data which is, for instance, due to a higher risk of having a car accident on frosted roads in winter.

But then, after eliminating such seasonal components, one can reasonably hope that the remaining process is in a certain state of equilibrium over time. This leads to the following definition of stationarity.

Definition 1.1. (*a*) *A real-valued time series* $(Y_t)_{t\in\mathbb{Z}}$ *is said to be* weakly stationary *if each* Y_t *has finite second moment, and if* $\mathbb{E}Y_t$ *and* $\mathrm{Cov}(Y_t, Y_{t+h})$ *do not depend on* $t \in \mathbb{Z}$ *for each* $h \in \mathbb{Z}$.

(*b*) *A real-valued time series* $(Y_t)_{t\in\mathbb{Z}}$ *is said to be* strictly stationary *if the joint distributions* $(Y_{t_1}, ..., Y_{t_k})$ *and* $(Y_{t_1+h}, ..., Y_{t_k+h})$ *are the same for all positive integers* k *and for all* $t_1, ..., t_k, h \in \mathbb{Z}$.

Loosely speaking, a stationary time series shows similar characteristics over two time intervals of the same length. In classical mathematical time series analysis, weak stationarity plays an important role. In this case, the focus is on first- and second-order moments and on properties that depend only on these. However, financial time series, for instance, often exhibit apparent heavy-tailed behaviour with asymmetric marginal distributions, so that second-order properties are inadequate to account for the data. To deal with such phenomena we concentrate in this thesis on strict stationarity.

In their recent paper, Brockwell and Lindner [4] focus on so called strict ARMA processes, which are strictly stationary solutions of certain linear recurrence equations. There has been an increased interest in these ARMA processes coming along with an increasing importance of heavy-tailed and asymmetric time series models, particularly in mathematical finance, where for example series of daily log returns on assets show such behaviour. Brockwell and Lindner characterize when strictly stationary solutions of ARMA equations exist. In this thesis, we shall examine two generalizations, namely to strict multivariate ARMA processes and to strict ARIMA processes.

However, in this introduction we shall first give a short overview over the main results for weakly stationary processes that are related to our results. For a more detailed exposition on these topics of time series analysis see Kreiß and Neuhaus [16] or Brockwell and Davis [3]. After our short overview we then outline the differences to the results presented in this thesis and give a short overview of the main ideas.

1.1 Preliminary definitions and results

Weak univariate ARMA

One of the most simple and basic examples of a weakly stationary time series is the weak white noise. A real- or complex-valued sequence $(Z_t)_{t\in\mathbb{Z}}$ is called *weak white noise* if $\mathbb{E}Z_t = \mu$ is finite, $\mathbb{E}|Z_t|^2 = \sigma^2 \in [0,\infty)$ for all $t \in \mathbb{Z}$, and $\mathrm{Cov}(Z_t, Z_{t'}) = 0$ for all $t, t' \in \mathbb{Z}$. From this simple example, three important classes of time series can be deduced, autoregressive (short AR) and moving average (short MA) time series, and a composite of both, so called autoregressive moving average (short ARMA) time series.

A *weak moving average process* $(Y_t)_{t\in\mathbb{Z}}$ *of order* q, short MA(q) process, is defined as

$$Y_t = Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q} \quad t \in \mathbb{Z},$$

where $(Z_t)_{t\in\mathbb{Z}}$ is a weak white noise sequence, $\theta_1, ..., \theta_q \in \mathbb{C}$, $\theta_q \neq 0$ and $q \in \mathbb{N} = \{1, 2, \ldots\}$. As this process is a sum of uncorrelated weakly stationary processes, it is clear that it is again weakly stationary.

A weakly stationary time series $(Y_t)_{t\in\mathbb{Z}}$ is called *weakly autoregressive of order* p, short AR(p), if it satisfies the equation

$$Y_t - \varphi_1 Y_{t-1} - \ldots - \varphi_p Y_{t-p} = Z_t \quad t \in \mathbb{Z}, \tag{1.1}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is a weak white noise sequence, $\varphi_1, ..., \varphi_p \in \mathbb{C}$, $\varphi_p \neq 0$ and $p \in \mathbb{N}$. So, besides the noise, the value of the process at time t does linearly depend on p previous values of the process. In contrast to a moving average time series, it is not immediately clear that a weakly stationary solution to Equation (1.1) exists. But we will see as a special case of the well known Theorem 1.3 below that this is the case if and only if the polynomial $\Phi(z) := 1 - \varphi_1 z - ... - \varphi_p z^p$ has no zeros on the unit circle.

A composition of the above two equations now leads us to the following definition.

Definition 1.2. *Let* $(Z_t)_{t\in\mathbb{Z}}$ *be a weak white noise sequence,* $p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\varphi_1, ..., \varphi_p, \theta_1, ..., \theta_q \in \mathbb{C}$, $\varphi_p \neq 0$, $\theta_q \neq 0$, *where* $\varphi_0 := \theta_0 := 1$. *Then any weakly stationary stochastic process* $(Y_t)_{t\in\mathbb{Z}}$ *which satisfies*

$$Y_t - \varphi_1 Y_{t-1} - \ldots - \varphi_p Y_{t-p} = Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q} \quad t \in \mathbb{Z}$$

is called a weak autoregressive moving average process of autoregressive order p and moving average order q, *short* weak ARMA(p,q) process. *Defining polynomials*

$$\Phi(z) = 1 - \varphi_1 z - ... - \varphi_p z^p, \quad \Theta(z) = 1 + \theta_1 z + ... + \theta_q z^q, \quad z \in \mathbb{C},$$

and B the backwards shift operator defined by $B^j Y_t = Y_{t-j}$, $j \in \mathbb{Z}$, (1.2) *can be written more compactly in the form*

$$\Phi(B) Y_t = \Theta(B) Z_t, \quad t \in \mathbb{Z}. \tag{1.2}$$

Apparently, AR(p) and MA(q) processes are special cases of ARMA(p,q) processes. As already mentioned above, it is not immidiately clear that weak ARMA processes exist, i.e. that Equation (1.2) has a weakly stationary solution $(Y_t)_{t\in\mathbb{Z}}$. However, the following Theorem 1.3 gives necessary and sufficient conditions for a weak ARMA process to exist. It is a reformulation of Theorem 7.4 in Kreiß and Neuhaus [16] and its proof relies heavily on the spectral representation of Y_t and shall be omitted here.

Theorem 1.3. *Let $(Z_t)_{t\in\mathbb{Z}}$ be a weak white noise sequence with mean μ and variance $\sigma^2 > 0$. Then the ARMA(p,q) equation* (1.2) *admits a weakly stationary solution if and only if all singularities of $\Theta(z)/\Phi(z)$ on the unit circle are removable.*
In this case, a weakly stationary solution of (1.2) *is given by*

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k} \quad t \in \mathbb{Z}, \tag{1.3}$$

where

$$\sum_{k=-\infty}^{\infty} \psi_k z^k = \frac{\Theta(z)}{\Phi(z)}, \quad 1-\delta < |z| < 1+\delta \text{ for some } \delta \in (0,1),$$

is the Laurent expansion of $\Theta(z)/\Phi(z)$. The sum in (1.3) *converges absolutely almost surely.*
If Φ does not have a zero on the unit circle, the solution is unique.

Strict univariate ARMA

As already mentioned above, the involved second-order properties when examining weak ARMA processes are inappropriate to model heavy-tailed behaviour. To allow for such situations, one can focus on *strict ARMA processes*, by which we mean strictly stationary solutions of (1.2) when $(Z_t)_{t\in\mathbb{Z}}$ is supposed to be an independent and identically distributed (i.i.d.) sequence of random variables, not necessarily with finite variance.

While sufficient conditions for the existence of strict ARMA processes are reasonably straightforward to find and have been given in the past, see for instance Cline and Brockwell [7], necessary conditions are way more sophisticated to find since the basic argument based on the spectral density in the proof of Theorem 1.3 does not apply here. However, Brockwell and Lindner [4] establish necessary and sufficient conditions on the independent white noise and the zeros of the defining polynomials in (1.2) for the existence of a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ of (1.2). Further, they specify a solution when these conditions hold and give necessary and sufficient conditions for its uniqueness. More precisely, they prove the following theorem, see [4], Theorem 1.

Theorem 1.4. *Suppose that* $(Z_t)_{t\in\mathbb{Z}}$ *is a nondeterministic i.i.d. noise sequence. Then the ARMA equation* (1.2) *admits a strictly stationary solution* $(Y_t)_{t\in\mathbb{Z}}$ *if and only if*

(i) all singularities of $\Theta(z)/\Phi(z)$ *on the unit circle are removable and* $\mathbb{E}\log^+|Z_1| < \infty$*, or*

(ii) all singularities of $\Theta(z)/\Phi(z)$ *in* $\mathbb{C}$ *are removable.*

If (i) or (ii) above holds, then a strictly stationary solution of (1.2) *is given by*

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}, \quad t \in \mathbb{Z}, \tag{1.4}$$

where

$$\sum_{k=-\infty}^{\infty} \psi_k z^k = \frac{\Theta(z)}{\Phi(z)}, \quad 1-\delta < |z| < 1+\delta \text{ for some } \delta \in (0,1),$$

is the Laurent expansion of $\Theta(z)/\Phi(z)$*. The sum in* (1.4) *converges absolutely almost surely.*
If Φ *does not have a zero on the unit circle, then* (1.4) *is the unique strictly stationary solution of* (1.2).

Weak multivariate ARMA

Coming back to our initial weather data example, it seems realistic to assume that in practice oftentimes multiple data is collected at one time $t \in \mathbb{Z}$, for example temperature, air pressure and air humidity at one certain place every day at a certain time. For $m \in \mathbb{N}$, this leads to a vector

$$Y_t = (Y_{t,1}, ..., Y_{t,m})^T, \quad t \in \mathbb{Z},$$

and $Y = (Y_t)_{t\in\mathbb{Z}}$ is called an *m-variate time series* or *m-variate sequence of random vectors.*
Now, let $d \in \mathbb{N}$, $p, q \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $(Z_t)_{t\in\mathbb{Z}}$ be a d-variate weak white noise sequence of random vectors and $\Psi_1, \dots, \Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0, \dots, \Theta_q \in \mathbb{C}^{m\times d}$ be deterministic complex-valued matrices. Then any m-variate weakly stationary stochastic process $(Y_t)_{t\in\mathbb{Z}}$ which satisfies almost surely

$$Y_t - \Psi_1 Y_{t-1} - \ldots - \Psi_p Y_{t-p} = \Theta_0 Z_t + \ldots + \Theta_q Z_{t-q}, \quad t \in \mathbb{Z}, \tag{1.5}$$

is called a *weak (multivariate) ARMA(p, q) process.* Such a process is often also called a weak VARMA (vector ARMA) process to distinguish it from the scalar case, but we shall simply use the term ARMA throughout. Denoting the identity matrix in $\mathbb{C}^{m\times m}$ by Id_m, the *characteristic polynomials* $P(z)$ and $Q(z)$ of the ARMA(p, q) equation (1.5) are defined as

$$P(z) := \mathrm{Id}_m - \sum_{k=1}^{p} \Psi_k z^k \quad \text{and} \quad Q(z) := \sum_{k=0}^{q} \Theta_k z^k \quad \text{for} \quad z \in \mathbb{C}.$$

With the aid of the backwards shift operator B, Equation (1.5) can be written more compactly in the form

$$P(B)Y_t = Q(B)Z_t, \quad t \in \mathbb{Z}. \tag{1.6}$$

The following sufficient condition for the existence of a weakly stationary multivariate ARMA process is well known, and we present its short proof.

Theorem 1.5. *Let $(Z_t)_{t\in\mathbb{Z}}$ be a weak white noise sequence in $\mathbb{C}^d$. If $\det P(z) \neq 0$ for all $z \in \mathbb{C}$ such that $|z| = 1$, then (1.6) has a weakly stationary solution*

$$Y_t = \sum_{k=-\infty}^{\infty} M_j Z_{t-j}, \tag{1.7}$$

where the matrices M_j are the coefficients of the Laurent expansion of $M(z) = P^{-1}(z)Q(z)$ in a neighborhood of the unit circle.

Proof. Denoting the adjugate matrix of $P(z)$ by $\mathrm{Adj}(P(z))$, it follows from Cramér's inversion rule that the inverse $P^{-1}(z)$ of $P(z)$ may be written as

$$P^{-1}(z) = (\det P(z))^{-1}\mathrm{Adj}(P(z))$$

which is a $\mathbb{C}^{m\times m}$-valued rational function, i.e. all its entries are rational functions. For the matrix-valued rational function $z \mapsto M(z)$ of the form $M(z) = P^{-1}(z)Q(z)$, the *singularities* of $M(z)$ are the zeroes of $\det P(z)$. But as $\det P(z) \neq 0$ for all $|z| = 1$, $M(z)$ can be expanded in a Laurent series $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$, absolutely convergent in a neighborhood of the unit circle. Define $Y = (Y_t)_{t\in\mathbb{Z}}$ by (1.7). Then Y is weakly stationary. We conclude that

$$P(B)Y_t = P(B)P^{-1}(B)Q(B)Z_t = Q(B)Z_t, \quad t \in \mathbb{Z},$$

showing that $(Y_t)_{t\in\mathbb{Z}}$ is a weakly stationary solution of (1.6). □

In contrast to the univariate case, to the best of our knowledge necessary and sufficient conditions for the existence of weak multivariate ARMA processes have not been given in the literature so far. We shall obtain such a condition in terms of the matrix rational function $z \mapsto P^{-1}(z)Q(z)$ in Theorem 2.3, the proof being an easy extension of the corresponding one-dimensional result. However, the main focus of this thesis will be on strictly stationary processes.

1.2 Main results of this thesis

Strict multivariate ARMA

In **Chapter 2** we shall concentrate on *strict multivariate ARMA processes*, by which we mean strictly stationary solutions of (1.6) when $(Z_t)_{t\in\mathbb{Z}}$ is supposed to be an i.i.d. $\mathbb{C}^d$-valued sequence of random vectors, not necessarily with finite variance. While it is known that finite log-moment of Z_0 together with $\det P(z) \neq 0$ for $|z| = 1$ is sufficient for a strictly stationary solution to exist, by the same arguments used for weakly stationary solutions (cf. Theorem 1.5), necessary and sufficient conditions have not been available so far, and we shall obtain a complete solution to this question in Theorem 2.2, thus generalizing the results of Brockwell and Lindner [4] to higher dimensions.

The chapter is organized as follows. After an introduction, we state in Section 2.2 the main results of the chapter. First of all, we consider the multivariate ARMA$(1, q)$ model

$$Y_t - \Psi_1 Y_{t-1} = \sum_{j=0}^{q} \Theta_j Z_{t-j}, \quad t \in \mathbb{Z}, \tag{1.8}$$

where $\Psi_1 \in \mathbb{C}^{m\times m}$ and $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence. Theorem 2.1 gives necessary and sufficient conditions for (1.8) to have a strictly stationary solution. Elementary considerations will show that the question of strictly stationary solutions may be reduced to the corresponding question when Ψ_1 is assumed to be in Jordan block form, and Theorem 2.1 gives a characterization of the existence of strictly stationary ARMA$(1,q)$ processes in terms of the Jordan canonical decomposition of Ψ_1 and properties of Z_0 and the coefficients Θ_k. An explicit solution of (1.8), assuming its existence, is also derived and the question of uniqueness of this solution is solved. The proof of Theorem 2.1 is given in Section 2.3.

In the following we shall consider a special case to illustrate exemplary the main ideas and arguments of Theorem 2.1. To this end we assume that $m = d = 4$. As $\mathbb{C}$ is an algebraically closed field, there is a (necessarily non-singular) matrix $S \in \mathbb{C}^{m\times m}$ such that $S^{-1}\Psi_1 S$ is in Jordan canonical form. Observe that (1.8) has a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ if and only if the corresponding equation for $X_t := S^{-1}Y_t$, namely

$$X_t - S^{-1}\Psi_1 S X_{t-1} = \sum_{j=0}^{q} S^{-1}\Theta_j Z_{t-j}, \quad t \in \mathbb{Z}, \tag{1.9}$$

has a strictly stationary solution. We assume here that

$$S^{-1}\Psi_1 S = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix}, \quad \text{with} \quad \Phi_1 = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \lambda_2 & 0 \\ 1 & \lambda_2 \end{pmatrix},$$

and $|\lambda_1| > 1$, $|\lambda_2| = 1$. With these assumptions, (1.9) has a strictly stationary solution $(X_t)_{t\in\mathbb{Z}}$ if and only if the equation for the first and for the second block

$$X_t^{(l)} - \Phi_l X_{t-1}^{(l)} = \sum_{j=0}^{q} I_l S^{-1}\Theta_j Z_{t-j}, \quad t \in \mathbb{Z}, \quad l = 1, 2 \tag{1.10}$$

with

$$I_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad I_2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

has a strictly stationary solution $X_t^{(1)} = I_1 X_t$ and $X_t^{(2)} = I_2 X_t$, respectively.

If we now additionally assume that Z_0 is symmetric (i.e. that Z_0 has the same distribution as $-Z_0$) then Theorem 2.1 says that (1.8) has a strictly stationary solution if and only if

$$\text{(i)} \qquad \mathbb{E}\log^+ \left\| \left(\sum_{j=0}^{q} \Phi_1^{q-j} I_1 S^{-1}\Theta_j \right) Z_0 \right\| < \infty, \quad \text{and}$$

$$\text{(ii)} \qquad \left(\sum_{j=0}^{q} \Phi_2^{q-j} I_2 S^{-1}\Theta_j \right) Z_0 = 0. \tag{1.11}$$

The sufficiency of these conditions can be derived very similarly to the univariate case in Theorem 1.4, so we don't go into detail here and refer to Section 2.3.2. For the necessity assume that $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of Equation (1.8). As observed above, this implies that both equations in (1.10) admit a strictly stationary solution $(X_t^{(l)})_{t\in\mathbb{Z}}$, $l = 1, 2$. The necessity of condition (i) now can be derived similar to the finite log-moment condition in the univariate case in Theorem 1.4, so we omit it here and refer to the proof in Section 2.3. The essential step when going from the univariate to the multivariate case is the necessity of condition (ii), involving the Jordan blocks with associated eigenvalues of absolute value 1. One difficulty in the multivariate case becomes evident when considering what happens when multiplying the random vector $X_{t-1}^{(2)}$ from the left with the Jordan block Φ_2. Writing $X_t^{(2)} = (X_{t,1}^{(2)}, X_{t,2}^{(2)})^T$, the left hand side of (1.10) with $l = 2$ reads as

$$X_t^{(2)} - \Phi_2 X_{t-1}^{(2)} = \begin{pmatrix} X_{t,1}^{(2)} - \lambda_2 X_{t-1,1}^{(2)} \\ X_{t,2}^{(2)} - X_{t-1,1}^{(2)} - \lambda_2 X_{t-1,2}^{(2)} \end{pmatrix} = \begin{pmatrix} X_{t,1}^{(2)} \\ X_{t,2}^{(2)} \end{pmatrix} - \lambda_2 \begin{pmatrix} X_{t-1,1}^{(2)} \\ X_{t-1,2}^{(2)} \end{pmatrix} - \begin{pmatrix} 0 \\ X_{t-1,1}^{(2)} \end{pmatrix}.$$

While the first row of this vector only depends on the first row $(X_{t,1}^{(2)})_{t\in\mathbb{Z}}$ of the process $(X_t^{(2)})_{t\in\mathbb{Z}}$ and thus can be treated as in the univariate case, the second row, however, contains with $(X_{t-1,1}^{(2)})_{t\in\mathbb{Z}}$ a component of the first row of the vector $(X_t^{(2)})_{t\in\mathbb{Z}}$. Essentially, this problem can be solved by first proving the assertion for the first row and then taking advantage of this in order to prove the assertion for the second row. In the general case this is done via induction on $i = 1, ..., m$, see Section 2.3 for the details.

The next result stated in Section 2.2 is Theorem 2.2, which addresses strict multivariate ARMA(p,q) processes. It is well known that every m-variate ARMA(p, q) process can in general be expressed in terms of a corresponding mp-variate ARMA$(1, q)$ process as specified in Proposition 2.5 of Section 2.5. Questions of existence and uniqueness can thus, in principle, be resolved by Theorem 2.1. However, since the Jordan canonical form of the corresponding $mp \times mp$-matrix $\underline{\Psi}_1$ in the corresponding higher-dimensional ARMA$(1, q)$ representation is in general difficult to handle, another more compact characterization is derived in Theorem 2.2. This characterization is given in terms of properties of the matrix rational function $P^{-1}(z)Q(z)$ and finite log-moments of certain linear combinations of the components of Z_0, extending the corresponding condition obtained in Theorem 1.4 for $m = d = 1$ in a natural way. Although in the statement of Theorem 2.2 no transformation to Jordan canonical forms is needed, its proof makes fundamental use of Theorem 2.1.

To show the main ideas of the theorem, we again consider a special case, for the general version see Theorem 2.2. Namely we assume the i.i.d. sequence of $\mathbb{C}^d$-valued random vectors $(Z_t)_{t\in\mathbb{Z}}$ to be such that the distribution of a^*Z_0 is not degenerate to a Dirac measure for every $a \in \mathbb{C}^d \setminus \{0\}$, where $a^* = \overline{a}^T$ is the conjugate transpose vector of a.
Then Theorem 2.2 asserts that a strictly stationary solution to the ARMA(p, q) equation (1.6) exists if and only if the following statements (i)—(iii) hold:

(i) All singularities on the unit circle of $M(z) = P^{-1}(z)Q(z)$ are removable.

(ii) If $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ denotes the Laurent expansion of M in a neighbourhood of the unit circle, then

$$\mathbb{E}\log^+ \|M_j Z_0\| < \infty \quad \forall\, j \in \{mp + q - p + 1, \ldots, mp + q\} \cup \{-p, \ldots, -1\}.$$

Further, if (i) above holds, then condition (ii) can be replaced by

(ii') If $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ denotes the Laurent expansion of M in a neighbourhood of the unit circle, then $\sum_{j=-\infty}^{\infty} M_j Z_{t-j}$ converges a.s. absolutely for every $t \in \mathbb{Z}$.

An explicit solution of (1.6), assuming its existence, is also derived and the question of uniqueness of this solution is solved.
Observe that the general version of Theorem 2.2 involves an additional condition (iii). This condition is automatically satisfied under the special case assumption that the distribution of a^*Z_0 is not degenerate to a Dirac measure for every $a \in \mathbb{C}^d \setminus \{0\}$ and is thus dropped here.

The proof of Theorem 2.2 is given in Section 2.5 and will make use of both Theorem 2.1 and Theorem 2.3. The latter is the corresponding characterization for the existence of weakly stationary solutions of ARMA(p, q) equations, expressed in terms of the characteristic polynomials $P(z)$ and $Q(z)$ as already mentioned above. The proof of Theorem 2.3, which is similar to the proof in the one-dimensional case in Theorem 1.5, will be given in Section 2.4.

In the following we shall have a closer look at the necessity of the above condition (i) in order to outline one main idea in the proof of Theorem 2.2. To this end, suppose that $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of (1.6) and that Z_0 is symmetric. As mentioned above, every m-variate ARMA(p, q) process can in general be expressed in terms of a corresponding mp-variate ARMA$(1,q)$ process. Define $\underline{Y}_t$ as the mp-dimensional strictly stationary solution of the corresponding mp-variate ARMA$(1,q)$ equation, and $\underline{\Theta}_k$ and $\underline{\Phi}$ as the corresponding matrices. For the details see Proposition 2.5. For simplicity we assume here that $\underline{\Phi}$ has only eigenvalues λ with $|\lambda| > 1$ and $|\lambda| = 1$. Let $\underline{\Phi_1}$ denote a matrix in Jordan block form with all Jordan blocks corresponding to the eigenvalues $|\lambda| > 1$, and $\underline{\Phi_2}$ a matrix in Jordan block form with all Jordan blocks corresponding to the eigenvalues $|\lambda| = 1$. Taking an invertible $\underline{S} \in \mathbb{C}^{mp\times mp}$ such that $\underline{S}^{-1}\underline{\Phi}\,\underline{S} = \begin{pmatrix} \underline{\Phi}_1 & 0 \\ 0 & \underline{\Phi}_2 \end{pmatrix}$, it follows analogously to (1.11) from Theorem 2.1, with the obvious definition of $\underline{I}_1$, $\underline{I}_2$, that

$$\sum_{k=0}^{q} \underline{\Phi}_2^{q-k} \underline{I}_2 \underline{S}^{-1} \underline{\Theta}_k Z_0 = 0.$$

But by the assumption that $a^* Z_0$ is not degenerate to a Dirac measure for $a \in \mathbb{C}^d \setminus \{0\}$, this implies

$$\sum_{k=0}^{q} \underline{\Phi}_2^{q-k} \underline{I}_2 \underline{S}^{-1} \underline{\Theta}_k = 0. \tag{1.12}$$

Now let $(Z'_t)_{t\in\mathbb{Z}}$ be an i.i.d. $N(0, \mathrm{Id}_d)$ distributed sequence. Then

$$\mathbb{E}\log^+ \left\| \sum_{k=0}^{q} \underline{\Phi}_1^{q-k} \underline{I}_1 \underline{S}^{-1} \underline{\Theta}_k Z'_0 \right\| < \infty, \quad \text{and} \quad \sum_{k=0}^{q} \underline{\Phi}_2^{q-k} \underline{I}_2 \underline{S}^{-1} \underline{\Theta}_k Z'_0 = 0$$

by (1.12). It then follows from Theorem 2.1 that there is a strictly stationary solution $\underline{Y}'_t$ of the ARMA$(1,q)$ equation $\underline{Y}'_t - \underline{\Phi}\,\underline{Y}'_{t-1} = \sum_{k=0}^{q} \underline{\Theta}_k Z'_{t-k}$. From the explicit representation of the solution obtained in Theorem 2.1 it will follow immediately that $(\underline{Y}_t)_{t\in\mathbb{Z}}$ is a Gauss process. Then going back to the m-variate ARMA(p,q) process, we see that there is a Gauss process $(Y'_t)_{t\in\mathbb{Z}}$ which is a strictly stationary solution of $P(B)Y'_t = Q(B)Z'_t$. In particular, this solution is weakly stationary, too. Hence we can apply Theorem 2.3 which yields that $z \mapsto P^{-1}(z)Q(z)$ has only removable singularities on the unit circle, which is condition (i).

Finally, in Section 2.6 at the end of Chapter 2, the main results are further discussed and, as an application, a result of Bougerol and Picard [2] on non-anticipative strictly stationary solutions is generalized.

Strict univariate ARIMA

In **Chapter 3** we consider another generalization of the strict univariate ARMA model, namely the strict univariate ARMA model with fractional noise, often also called ARIMA model (autoregressive integrated moving average) or FARIMA (fractional ARIMA). The main goal of the chapter is to give necessary and sufficient conditions for the existence of so called strict ARIMA(p,D,q) processes.

Let $(Z_t)_{t\in\mathbb{Z}}$ be a real-valued noise sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define univariate autoregressive and moving average polynomials as above, namely

$$\Phi(z) = 1 - \sum_{k=1}^{p} \varphi_k z^k, \quad \text{and} \quad \Theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k, \quad z \in \mathbb{C}, \tag{1.13}$$

with $p, q \in \mathbb{N}_0$, $\varphi_1, ..., \varphi_p, \theta_1, ..., \theta_q \in \mathbb{C}$, $\varphi_p \neq 0$ and $\theta_q \neq 0$, where $\varphi_0 := \theta_0 := 1$. For any $D \in \mathbb{R} \setminus \{1, 2, ...\}$ and B the backwards shift operator, define the difference operator $\nabla^D = (1 - B)^D$ by means of the binomial expansion,

$$\nabla^D = (1 - B)^D = \sum_{j=0}^{\infty} (-1)^j \binom{D}{j} B^j.$$

Now, for $(Z_t)_{t\in\mathbb{Z}}$ weak white noise, Granger and Joyeux [10] and Hosking [11] introduced *weak ARIMA(p,D,q) processes* as weakly stationary solutions of the equation

$$\Phi(B)\nabla^D Y_t = \Theta(B)Z_t, \quad t \in \mathbb{Z}. \tag{1.14}$$

It is shown in [11] that a sufficient condition for a weak ARIMA process to exist is $D < \frac{1}{2}$ and $\Phi(z)$ having no zeros on the unit circle. Furthermore, they found out that a sufficient condition for a solution of (1.14) to be invertible is $D > -\frac{1}{2}$ and $\Theta(z)$ having no zeros on the unit circle.

A couple of years later, Kokoszka and Taqqu [14] and Kokoszka [15] developed the theory of infinite variance stable fractional ARIMA(p,D,q) time series defined by the equation

$$\Phi(B)Y_t = [\Theta(B)\nabla^{-D}]Z_t, \quad t \in \mathbb{Z}, \tag{1.15}$$

where the noise sequence $(Z_t)_{t\in\mathbb{Z}}$ is i.i.d. symmetric α-stable (in [14]) or belongs to the domain of attraction of an α-stable law (in [15]), respectively, with $0 < \alpha < 2$ and fractional D such that the right hand side of (1.15) converges. Among other results, they obtained a unique strictly stationary solution of (1.15) with this specific noise in terms of the Laurent series of $\Theta(z)(1 - z)^{-D}/\Phi(z)$, provided $\Phi(z) \neq 0$ for all $|z| \leq 1$, and $\Phi(z)$ and $\Theta(z)$ having no roots in common.

In Chapter 3 of this thesis however, we study a slightly different approach by interpreting Equation (1.15) as an ARMA(p,q) equation with fractional noise $\nabla^{-D} Z_t$, i.e.

$$\Phi(B)Y_t = \Theta(B)[\nabla^{-D} Z_t], \quad t \in \mathbb{Z}, \tag{1.16}$$

where $D \in \mathbb{R}\backslash\{-1,-2,...\}$, and $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence of real random variables, not necessarily with finite variance. Here, the fractional noise has a representation $\nabla^{-D} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ with coefficients

$$\psi_j = (-1)^j \binom{-D}{j} = \prod_{0<k\leq j} \frac{k-1+D}{k} = \frac{\Gamma(j+D)}{\Gamma(j+1)\Gamma(D)}, \quad j = 0,1,2,... \tag{1.17}$$

Note that an application of Stirling's formula, according to which $\Gamma(x) \sim \sqrt{2\pi} e^{-x+1} (x-1)^{x-1/2}$ as $x \to \infty$, yields

$$\psi_j \sim \frac{j^{D-1}}{\Gamma(D)} \quad \text{as } j \to \infty. \tag{1.18}$$

Now, we call a complex-valued process $Y := (Y_t)_{t\in\mathbb{Z}}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a *strict ARIMA(p,D,q) process* (or more precisely a *strict ARMA(p,q) process with fractional noise*) if the series $\nabla^{-D} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ converges almost surely and Y is a strictly stationary solution of (1.16).

Here, we do only consider the cases $D \in (-\infty, 0)\backslash\{-1,-2,...\}$ and $0 < D < \frac{1}{2}$. This is because for $D \geq \frac{1}{2}$, the series $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ can only converge for $Z_t \equiv 0$, $t \in \mathbb{Z}$, because in this case the series $\sum_{j=0}^{\infty} \psi_j^2$ does not converge due to the asymptotic behaviour (1.18) of the coefficients ψ_j. But the convergence of this series is necessary for $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ to converge (see Chow and Teicher [6], Theorem 5.1.4), i.e. fractional noise $\nabla^{-D} Z_t$ cannot exist for $D \geq \frac{1}{2}$ unless $Z_t \equiv 0$. In the case $D \in \{0,-1,-2,...\}$, Equation (1.16) reduces to an ARMA equation with i.i.d. noise sequence and the question of existence and uniqueness of strictly stationary solutions to this equation is thus solved by Theorem 1.4.

Before being able to give equivalent conditions for a strict ARIMA process to exist, questions of convergence of the series $\nabla^{-D} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ need to be addressed. These questions are solved in Section 3.2, where Theorem 3.1 gives a necessary and sufficient condition for $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ to converge almost surely in terms of moment conditions on Z_0. More precisely, the theorem states that for $D \in (-\infty, 0) \setminus \{-1, -2, ...\}$ the series $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ converges almost surely if and only if $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$, and for $D \in (0, \frac{1}{2})$ it is additionally required that $\mathbb{E}Z_0 = 0$.

The crucial step in the proof is to show the sufficiency of the conditions. For $D \in (-\infty, 0) \setminus \{-1, -2, ...\}$, we make fundamental use of Kolmogorov's three series criterion (see Kallenberg [13], Theorem 4.18). This criterion states that $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ converges almost surely if and only if the following conditions hold: $\sum_{j=1}^{\infty} \mathbb{P}(|\psi_j Z_{t-j}| > 1) < \infty$, $\sum_{j=1}^{\infty} \mathbb{E}(\psi_j Z_{t-j} \mathbb{1}_{\{|\psi_j Z_{t-j}| \leq 1\}})$ converges, and $\sum_{j=1}^{\infty} \mathbb{V}(\psi_j Z_{t-j} \mathbb{1}_{\{|\psi_j Z_{t-j}| \leq 1\}}) < \infty$. For showing the convergence of these three series, we make use of the integral criterion for convergence and the asymptotic behaviour (1.18) of the coefficients ψ_j. For $D \in (0, \frac{1}{2})$, these arguments do not apply directly which is because of the fact that (1.18) implies that $\sum_{j \in \mathbb{N}} |\psi_j|$ is not finite for $D \in (0, \frac{1}{2})$. This is why we have to impose the additional condition $\mathbb{E}Z_0 = 0$, which ensures that $(S_n)_{n \in \mathbb{N}}$, with $S_n := \sum_{j=1}^{n} \psi_j Z_{t-j}$, is a martingale. In a technical lemma we do then show that $\sup_{n \in \mathbb{N}} \mathbb{E}|S_n| < \infty$. Hence $(S_n)_{n \in \mathbb{N}}$ is an L^1-bounded martingale and so it converges a.s. (see e.g. Kallenberg [13], Theorem 7.18).

Then, after having resolved the question of convergence of $\nabla^{-D} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, we give in Section 3.3 necessary and sufficient conditions for a strict ARIMA process to exist. These conditions are stated in Theorem 3.5, where also an explicit solution to (1.16), given its existence, is derived and the question of uniqueness of this solution is solved. In contrast to the results in [14] and [15], we make no a priori assumptions on the roots of Φ and Θ, and allow for more general noise distributions.

More precisely, let $(Z_t)_{t \in \mathbb{Z}}$ be a nondeterministic i.i.d. sequence of real random variables and let Φ and Θ be defined as in (1.13). Then Theorem 3.5 states that for $D \in [-\frac{1}{2}, 0)$, the ARIMA equation

$$\Phi(B) Y_t = \Theta(B)[\nabla^{-D} Z_t], \quad t \in \mathbb{Z}, \tag{1.19}$$

admits a strictly stationary solution $(Y_t)_{t \in \mathbb{Z}}$ if and only if all singularities of $\Theta(z)/\Phi(z)$ on the unit circle are removable and $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$. For $D \in (0, \frac{1}{2})$ the additional condition $\mathbb{E}Z_0 = 0$ is required. And for $D \in (-\infty, -\frac{1}{2}) \setminus \{-1, -2, ...\}$ the above characterization is shown to hold if additionally $\Phi(1) \neq 0$ is assumed.

In all three cases, a strictly stationary solution of (1.19) is given by

$$Y_t = \sum_{j=-\infty}^{\infty} \xi_j \left(\nabla^{-D} Z_{t-j}\right), \quad t \in \mathbb{Z}, \tag{1.20}$$

where

$$\sum_{j=-\infty}^{\infty} \xi_j z^j = \frac{\Theta(z)}{\Phi(z)}, \quad 1-\delta < |z| < 1+\delta \text{ for some } \delta \in (0,1),$$

is the Laurent expansion of $\Theta(z)/\Phi(z)$ around zero. The sum in (1.20) converges absolutely almost surely, in the sense that $\sum_{j=-\infty}^{\infty} |\xi_j||\nabla^{-D} Z_{t-j}| < \infty$ a.s. If $\Phi(z) \neq 0$ for all $|z| = 1$, then (1.20) is the unique strictly stationary solution of (1.19).

The proof of the theorem uses similar techniques as the proof of Theorem 1.4, though a main difference is that in the present case the sum on the right hand side of the defining Equation (1.19) is not a finite sum like in the ARMA equation. This needs some special handling.

At the end of Chapter 3 we discuss the connection of our results to the results of Kokoszka and Taqqu [14]. In their paper, they study fractional ARIMA processes defined by the equations

$$\Phi(B)Y_t = R(B)Z_t, \tag{1.21}$$

with

$$R(z) := \Theta(z)(1-z)^{-D} = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j \wedge q} \theta_k \psi_{j-k} \right) z^j = \sum_{j=0}^{\infty} \psi'_j z^j,$$

and i.i.d. symmetric α-stable noise $(Z_t)_{t\in\mathbb{Z}}$. Among other results, they obtain a unique strictly stationary solution of (1.21) with this specific noise in terms of the Laurent series of $R(z)/\Phi(z)$, provided $\Phi(z) \neq 0$ for all $|z| \leq 1$, and $\Phi(z)$ and $\Theta(z)$ having no roots in common.
In contrast, we characterized all strictly stationary solutions of the equation

$$\Phi(B)Y_t = \Theta(B)[\nabla^{-D} Z_t], \tag{1.22}$$

that are, strictly speaking, ARMA equations with fractional noise. However, it is not immediately clear that these both approaches are equivalent. But Theorem 3.9 states that if $\Theta(1) \neq 0$, the series $\sum_{j=0}^{\infty} \psi'_j Z_{t-j}$ converges almost surely if and only if $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ converges almost surely, and in this case it follows $R(B)Z_t = \Theta(B)[\nabla^{-D} Z_t]$. Furthermore, it says that (1.21) admits a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ (in the sense that $\sum_{j=0}^{\infty} \psi'_j Z_{t-j}$ converges almost surely and (Y_t) satisfies (1.21) and is strictly stationary) if and only if (1.22) admits a strictly stationary solution. Any strictly stationary solution of (1.21) is a solution of (1.22) and vice versa.

Summing up, the main contributions of this thesis are a characterization of the existence of strictly stationary solutions of multivariate ARMA equations (in Chapter 2) and of univariate ARIMA equations (in Chapter 3).

2 Strictly stationary solutions of multivariate ARMA equations with i.i.d. noise

Based on [5]: Brockwell, P.J., Lindner, A. and Vollenbröker, B. (2011):
Strictly stationary solutions of multivariate ARMA equations with i.i.d. noise.
Annals of the Institute of Statistical Mathematics, to appear.

Abstract. We obtain necessary and sufficient conditions for the existence of strictly stationary solutions of multivariate ARMA equations with independent and identically distributed noise. For general ARMA(p, q) equations these conditions are expressed in terms of the characteristic polynomials of the defining equations and moments of the driving noise sequence, while for $p = 1$ an additional characterization is obtained in terms of the Jordan canonical decomposition of the autoregressive matrix, the moving average coefficient matrices and the noise sequence. No a priori assumptions are made on either the driving noise sequence or the coefficient matrices.

2.1 Introduction

Let $m, d \in \mathbb{N} = \{1, 2 \ldots, \}$, $p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(Z_t)_{t \in \mathbb{Z}}$ be a d-variate noise sequence of random vectors defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m \times m}$ and $\Theta_0, \ldots, \Theta_q \in \mathbb{C}^{m \times d}$ be deterministic complex-valued matrices. Then any m-variate stochastic process $(Y_t)_{t \in \mathbb{Z}}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies almost surely

$$Y_t - \Psi_1 Y_{t-1} - \ldots - \Psi_p Y_{t-p} = \Theta_0 Z_t + \ldots + \Theta_q Z_{t-q}, \quad t \in \mathbb{Z}, \tag{2.1}$$

is called a solution of the ARMA(p,q) equation (2.1) (autoregressive moving average equation of autoregressive order p and moving average order q). Such a solution is often called a VARMA (vector ARMA) process to distinguish it from the scalar case, but we shall simply use the term ARMA throughout. Denoting the identity matrix in $\mathbb{C}^{m\times m}$ by Id_m, the *characteristic polynomials* $P(z)$ and $Q(z)$ of the ARMA(p,q) equation (2.1) are defined as

$$P(z) := \mathrm{Id}_m - \sum_{k=1}^{p} \Psi_k z^k \quad \text{and} \quad Q(z) := \sum_{k=0}^{q} \Theta_k z^k \quad \text{for} \quad z \in \mathbb{C}. \tag{2.2}$$

With the aid of the backwards shift operator B, Equation (2.1) can be written more compactly in the form

$$P(B)Y_t = Q(B)Z_t, \quad t \in \mathbb{Z}.$$

There is evidence to show that, although VARMA(p,q) models with $q > 0$ are more difficult to estimate than VARMA$(p,0)$ (vector autoregressive) models, significant improvement in forecasting performance can be achieved by allowing the moving average order q to be greater than zero. See, for example, Athanosopoulos and Vahid [1], where such improvement is demonstrated for a variety of macroeconomic time series.

Much attention has been paid to *weak ARMA processes*, i.e. weakly stationary solutions to (2.1) if $(Z_t)_{t\in\mathbb{Z}}$ is a weak white noise sequence. Recall that a $\mathbb{C}^r$-valued process $(X_t)_{t\in\mathbb{Z}}$ is *weakly stationary* if each X_t has finite second moment, and if $\mathbb{E}X_t$ and $\mathrm{Cov}\,(X_t, X_{t+h})$ do not depend on $t \in \mathbb{Z}$ for each $h \in \mathbb{Z}$. If additionally every component of X_t is uncorrelated with every component of $X_{t'}$ for $t \neq t'$, then $(X_t)_{t\in\mathbb{Z}}$ is called *weak white noise*. In the case when $m = d = 1$ and Z_t is weak white noise having non-zero variance, it can easily be shown using spectral analysis, see e.g. Brockwell and Davis [3], Problem 4.28, that a weak ARMA process exists if and only if the rational function $z \mapsto Q(z)/P(z)$ has only removable singularities on the unit circle in $\mathbb{C}$, see Theorem 1.3 in this thesis. For higher dimensions, it is well known that a sufficient condition for weak ARMA processes to exist is that the polynomial $z \mapsto \det P(z)$ has no zeroes on the unit circle, see Theorem 1.5 in this thesis. However, to the best of our knowledge necessary and sufficient conditions have not been given in the literature so far. We shall obtain such a condition in terms of the matrix rational function $z \mapsto P^{-1}(z)Q(z)$ in Theorem 2.3, the proof being an easy extension of the corresponding one-dimensional result.

Weak ARMA processes, by definition, are restricted to have finite second moments. However financial time series often exhibit apparent heavy-tailed behaviour with asymmetric marginal distributions, so that second-order properties are inadequate to account for the data. To deal with such phenomena we focus in this chapter on *strict ARMA processes*, by which we mean strictly stationary solutions of (2.1) when $(Z_t)_{t\in\mathbb{Z}}$ is supposed to be an independent and identically distributed (i.i.d.) sequence of random vectors, not necessarily with finite variance. A sequence $(X_t)_{t\in\mathbb{Z}}$ is *strictly stationary* if all its finite dimensional distributions are shift invariant. Much less is known about strict ARMA processes, and it was shown only recently for $m = d = 1$ in Brockwell and Lindner [4] that for i.i.d. non-deterministic noise $(Z_t)_{t\in\mathbb{Z}}$, a strictly stationary solution to (2.1) exists if and only if $Q(z)/P(z)$ has only removable singularities on the unit circle and Z_0 has finite log moment, or if $Q(z)/P(z)$ is a polynomial, see Theorem 1.4 in this thesis. For higher dimensions, while it is known that finite log-moment of Z_0 together with $\det P(z) \neq 0$ for $|z| = 1$ is *sufficient* for a strictly stationary solution to exist, by the same arguments used for weakly stationary solutions, necessary and sufficient conditions have not been available so far, and we shall obtain a complete solution to this question in Theorem 2.2, thus generalizing the results of [4] to higher dimensions. A related question was considered by Bougerol and Picard [2] who, using their powerful results on random recurrence equations, showed in Theorem 4.1 of [2] that if $\mathbb{E}\log^+\|Z_0\| < \infty$ and the characteristic polynomials are left-coprime, meaning that the only common left-divisors of $P(z)$ and $Q(z)$ are unimodular (see Section 2.6 for the precise definitions), then a non-anticipative strictly stationary solution to (2.1) exists if and only if $\det P(z) \neq 0$ for $|z| \leq 1$. Observe that for the characterization of the existence of strict (not necessarily non-anticipative) ARMA processes obtained in this chapter, we shall not make any a priori assumptions on log-moments of the noise sequence or on left-coprimeness of the characteristic polynomials, but rather obtain related conditions as parts of our characterization. As an application of our main results, we shall then obtain a slight extension of Theorem 4.1 of Bougerol and Picard [2] in Theorem 2.14, by characterizing all non-anticipative strictly stationary solutions to (2.1) without any moment assumptions, however still assuming left-coprimeness of the characteristic polynomials.

This chapter is organized as follows. In Section 2.2 we state the main results of the chapter. Theorem 2.1 gives necessary and sufficient conditions for the multivariate ARMA$(1, q)$ model

$$Y_t - \Psi_1 Y_{t-1} = \sum_{k=0}^{q} \Theta_k Z_{t-k}, \quad t \in \mathbb{Z}, \tag{2.3}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence, to have a strictly stationary solution. Elementary

considerations will show that the question of strictly stationary solutions may be reduced to the corresponding question when Ψ_1 is assumed to be in Jordan block form, and Theorem 2.1 gives a characterization of the existence of strictly stationary ARMA$(1, q)$ processes in terms of the Jordan canonical decomposition of Ψ_1 and properties of Z_0 and the coefficients Θ_k. An explicit solution of (2.3), assuming its existence, is also derived and the question of uniqueness of this solution is solved.
Strict ARMA(p, q) processes are addressed in Theorem 2.2. Since every m-variate ARMA(p, q) process can in general be expressed in terms of a corresponding mp-variate ARMA$(1, q)$ process, questions of existence and uniqueness can, in principle, be resolved by Theorem 2.1. However, since the Jordan canonical form of the corresponding $mp \times mp$-matrix $\underline{\Psi}_1$ in the corresponding higher-dimensional ARMA$(1, q)$ representation is in general difficult to handle, another more compact characterization is derived in Theorem 2.2. This characterization is given in terms of properties of the matrix rational function $P^{-1}(z)Q(z)$ and finite log-moments of certain linear combinations of the components of Z_0, extending the corresponding condition obtained in [4] for $m = d = 1$ in a natural way. Although in the statement of Theorem 2.2 no transformation to Jordan canonical forms is needed, its proof makes fundamental use of Theorem 2.1.
Theorem 2.3 deals with the corresponding question for weak ARMA(p, q) processes. The proofs of Theorems 2.1, 2.3 and 2.2 are given in Sections 2.3, 2.4 and 2.5, respectively. The proof of Theorem 2.2 makes crucial use of Theorems 2.1 and 2.3.
The main results are further discussed in Section 2.6 and, as an application, the aforementioned characterization of non-anticipative strictly stationary solutions is obtained in Theorem 2.14, generalizing slightly the result of Bougerol and Picard [2].
Throughout the chapter, vectors will be understood as column vectors and e_i will denote the i^{th} unit vector in $\mathbb{C}^m$. The zero matrix in $\mathbb{C}^{m\times r}$ is denoted by $0_{m,r}$ or simply 0, the zero vector in $\mathbb{C}^r$ by 0_r or simply 0. The transpose of a matrix A is denoted by A^T, and its complex conjugate transpose matrix by $A^* = \overline{A}^T$. By $\|\cdot\|$ we denote an unspecific, but fixed vector norm on $\mathbb{C}^s$ for $s \in \mathbb{N}$, as well as the corresponding matrix norm $\|A\| = \sup_{x\in\mathbb{C}^s, \|x\|=1} \|Ax\|$. We write $\log^+(x) := \log\max\{1, x\}$ for $x \in \mathbb{R}$, and denote by $\mathbb{P} - \lim$ limits in probability.

2.2 Main results

Theorems 2.1 and 2.2 give necessary and sufficient conditions for the ARMA$(1, q)$ equation (2.3) and the ARMA(p, q) equation (2.1), respectively, to have a strictly sta-

tionary solution. In Theorem 2.1, these conditions are expressed in terms of the i.i.d. noise sequence $(Z_t)_{t\in\mathbb{Z}}$, the coefficient matrices $\Theta_0, \ldots, \Theta_q$ and the Jordan canonical decomposition of Ψ_1, while in Theorem 2.2 they are given in terms of the noise sequence and the characteristic polynomials $P(z)$ and $Q(z)$ as defined in (2.2).
As background for Theorem 2.1, suppose that $\Psi_1 \in \mathbb{C}^{m\times m}$ and choose a (necessarily non-singular) matrix $S \in \mathbb{C}^{m\times m}$ such that $S^{-1}\Psi_1 S$ is in Jordan canonical form. Suppose also that $S^{-1}\Psi_1 S$ has $H \in \mathbb{N}$ Jordan blocks, $\Phi_1, \ldots, \Phi_H$, the h^{th} block beginning in row r_h, where $r_1 := 1 < r_2 < \cdots < r_H < m+1 =: r_{H+1}$. A Jordan block with associated eigenvalue λ will always be understood to be of the form

$$\begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{pmatrix} \tag{2.4}$$

i.e. the entries 1 are below the main diagonal.
Observe that (2.3) has a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ if and only if the corresponding equation for $X_t := S^{-1}Y_t$ namely

$$X_t - S^{-1}\Psi_1 S X_{t-1} = \sum_{j=0}^{q} S^{-1}\Theta_j Z_{t-j}, \quad t \in \mathbb{Z}, \tag{2.5}$$

has a strictly stationary solution. This will be the case only if the equation for the h^{th} block,

$$X_t^{(h)} := I_h X_t, \quad t \in \mathbb{Z}, \tag{2.6}$$

where I_h is the $(r_{h+1} - r_h) \times m$ matrix with (i, j) components,

$$I_h(i,j) = \begin{cases} 1, & \text{if } j = i + r_h - 1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.7}$$

has a strictly stationary solution for each $h = 1, \ldots, H$. But these equations are simply

$$X_t^{(h)} - \Phi_h X_{t-1}^{(h)} = \sum_{j=0}^{q} I_h S^{-1}\Theta_j Z_{t-j}, \quad t \in \mathbb{Z}, \quad h = 1, \ldots, H, \tag{2.8}$$

where Φ_h is the h^{th} Jordan block of $S^{-1}\Psi_1 S$.
Conversely if (2.8) has a strictly stationary solution $X'^{(h)}$ for each $h \in \{1, \ldots, H\}$, then we shall see from the proof of Theorem 2.1 that there exist (possibly different

if $|\lambda_h| = 1$) strictly stationary solutions $X^{(h)}$ of (2.8) for each $h \in \{1, \dots, H\}$, such that

$$Y_t := S(X_t^{(1)T}, \dots, X_t^{(H)T})^T, \quad t \in \mathbb{Z}, \tag{2.9}$$

is a strictly stationary solution of (2.3).
Existence and uniqueness of a strictly stationary solution of (2.3) is therefore equivalent to the existence and uniqueness of a strictly stationary solution of the Equations (2.8) for each $h \in \{1, \dots, H\}$. The necessary and sufficient condition for each one will depend on the value of the eigenvalue λ_h associated with Φ_h and in particular on whether (a) $|\lambda_h| \in (0, 1)$, (b) $|\lambda_h| > 1$, (c) $|\lambda_h| = 1$ and $\lambda_h \neq 1$, (d) $\lambda_h = 1$ and (e) $\lambda_h = 0$. These cases will be addressed separately in the proof of Theorem 2.1, which is given in Section 2.3. The aforementioned characterization in terms of the Jordan decomposition of Ψ_1 now reads as follows.

Theorem 2.1. [Strict ARMA$(1, q)$ processes]
Let $m, d \in \mathbb{N}$, $q \in \mathbb{N}_0$, and let $(Z_t)_{t \in \mathbb{Z}}$ be an i.i.d. sequence of $\mathbb{C}^d$-valued random vectors. Let $\Psi_1 \in \mathbb{C}^{m \times m}$ and $\Theta_0, \dots, \Theta_q \in \mathbb{C}^{m \times d}$ be complex-valued matrices. Let $S \in \mathbb{C}^{m \times m}$ be an invertible matrix such that $S^{-1}\Psi_1 S$ is in Jordan block form as above, with H Jordan blocks Φ_h, $h \in \{1, \dots, H\}$, and associated eigenvalues λ_h, $h \in \{1, \dots, H\}$. Let $r_1, \dots, r_{H+1}$ be given as above and I_h as defined by (2.7). Then the ARMA$(1, q)$ equation (2.3) has a strictly stationary solution Y if and only if the following statements (i) – (iii) hold:

(i) For every $h \in \{1, \dots, H\}$ such that $|\lambda_h| \neq 0, 1$,

$$\mathbb{E} \log^+ \left\| \left(\sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k \right) Z_0 \right\| < \infty. \tag{2.10}$$

(ii) For every $h \in \{1, \dots, H\}$ such that $|\lambda_h| = 1$, but $\lambda_h \neq 1$, there exists a constant $\alpha_h \in \mathbb{C}^{r_{h+1} - r_h}$ such that

$$\left(\sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k \right) Z_0 = \alpha_h \ \text{ a.s.} \tag{2.11}$$

(iii) For every $h \in \{1, \dots, H\}$ such that $\lambda_h = 1$, there exists a constant $\alpha_h = (\alpha_{h,1}, \dots, \alpha_{h, r_{h+1} - r_h})^T \in \mathbb{C}^{r_{h+1} - r_h}$ such that $\alpha_{h,1} = 0$ and (2.11) holds.

If these conditions are satisfied, then a strictly stationary solution to (2.3) is given *by* (2.9) *with*

$$X_t^{(h)} := \begin{cases} \sum_{j=0}^{\infty} \Phi_h^{j-q} \left(\sum_{k=0}^{j \wedge q} \Phi_h^{q-k} I_h S^{-1} \Theta_k \right) Z_{t-j}, & |\lambda_h| \in (0,1), \\ - \sum_{j=1-q}^{\infty} \Phi_h^{-j-q} \left(\sum_{k=(1-j)\vee 0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k \right) Z_{t+j}, & |\lambda_h| > 1 \\ \sum_{j=0}^{m+q-1} \left(\sum_{k=0}^{j \wedge q} \Phi_h^{j-k} I_h S^{-1} \Theta_k \right) Z_{t-j}, & \lambda_h = 0, \\ f_h + \sum_{j=0}^{q-1} \left(\sum_{k=0}^{j} \Phi_h^{j-k} I_h S^{-1} \Theta_k \right) Z_{t-j}, & |\lambda_h| = 1, \end{cases} \tag{2.12}$$

where $f_h \in \mathbb{C}^{r_{h+1}-r_h}$ *is a solution to*

$$(\mathrm{Id}_h - \Phi_h) f_h = \alpha_h, \tag{2.13}$$

which exists for $\lambda_h = 1$ *by (iii) and, for* $|\lambda| = 1, \lambda \neq 1$*, by the invertibility of* $(\mathrm{Id}_h - \Phi_h)$*. The series in* (2.12) *converge a.s. absolutely.*
If the necessary and sufficient conditions stated above are satisfied, then, provided the underlying probability space is rich enough to support a random variable which is uniformly distributed on $[0,1)$ *and independent of* $(Z_t)_{t\in\mathbb{Z}}$*, the solution given by* (2.9) *and* (2.12) *is the unique strictly stationary solution of* (2.3) *if and only if* $|\lambda_h| \neq 1$ *for all* $h \in \{1, \dots, H\}$.

Special cases of Theorem 2.1 will be treated in Corollaries 2.7, 2.9 and Remark 2.8. It is well known that every ARMA(p,q) process can be embedded into a higher dimensional ARMA$(1,q)$ process as specified in Proposition 2.5 of Section 2.5. Hence, in principle, the questions of existence and uniqueness of strictly stationary ARMA(p,q) processes can be reduced to Theorem 2.1. However, it is generally difficult to obtain the Jordan canonical decomposition of the $(mp \times mp)$-dimensional matrix $\underline{\Phi}$ defined in Proposition 2.5, which is needed to apply Theorem 2.1. Hence, a more natural approach is to express the conditions in terms of the characteristic polynomials $P(z)$ and $Q(z)$ of the ARMA(p,q) equation (2.1). Observe that $z \mapsto \det P(z)$ is a polynomial in $z \in \mathbb{C}$, not identical to the zero polynomial. Hence $P(z)$ is invertible except for a finite number of z. Also, denoting the adjugate matrix of $P(z)$ by $\mathrm{Adj}(P(z))$, it follows from Cramér's inversion rule that the inverse $P^{-1}(z)$ of $P(z)$ may be written as

$$P^{-1}(z) = (\det P(z))^{-1} \mathrm{Adj}(P(z))$$

which is a $\mathbb{C}^{m\times m}$-valued rational function, i.e. all its entries are rational functions. For a general matrix-valued rational function $z \mapsto M(z)$ of the form $M(z) = P^{-1}(z)\widetilde{Q}(z)$ with some matrix polynomial $\widetilde{Q}(z)$, the *singularities* of $M(z)$ are the zeroes of

$\det P(z)$, and such a singularity, z_0 say, is *removable* if all entries of $M(z)$ have removable singularities at z_0. Further observe that if $M(z)$ has only removable singularities on the unit circle in $\mathbb{C}$, then $M(z)$ can be expanded in a Laurent series $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$, convergent in a neighborhood of the unit circle. The characterization for the existence of strictly stationary ARMA(p,q) processes now reads as follows.

Theorem 2.2. [Strict ARMA(p,q) processes]
Let $m, d, p \in \mathbb{N}$, $q \in \mathbb{N}_0$, and let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathbb{C}^d$-valued random vectors. Let $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$ be complex-valued matrices, and define the characteristic polynomials as in (2.2). *Define the linear subspace*

$$K := \{a \in \mathbb{C}^d : \textit{the distribution of } a^* Z_0 \textit{ is degenerate to a Dirac measure}\}$$

of $\mathbb{C}^d$, denote by $K^\perp$ its orthogonal complement in $\mathbb{C}^d$, and let $s := \dim K^\perp$ the vector space dimension of $K^\perp$. Let $U \in \mathbb{C}^{d\times d}$ be unitary such that $U\,K^\perp = \mathbb{C}^s \times \{0_{d-s}\}$ and $U\,K = \{0_s\} \times \mathbb{C}^{d-s}$, and define the $\mathbb{C}^{m\times d}$-valued rational function $M(z)$ by

$$z \mapsto M(z) := P^{-1}(z)Q(z)U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}. \tag{2.14}$$

Then there is a constant $u \in \mathbb{C}^{d-s}$ and a $\mathbb{C}^s$-valued i.i.d. sequence $(w_t)_{t\in\mathbb{Z}}$ such that

$$UZ_t = \begin{pmatrix} w_t \\ u \end{pmatrix} \quad a.s. \quad \forall\, t \in \mathbb{Z}, \tag{2.15}$$

and the distribution of $b^* w_0$ is not degenerate to a Dirac measure for any $b \in \mathbb{C}^s \backslash \{0\}$. Further, a strictly stationary solution to the ARMA(p,q) equation (2.1) *exists if and only if the following statements (i)—(iii) hold:*

(i) All singularities on the unit circle of the meromorphic function $M(z)$ are removable.

(ii) If $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ denotes the Laurent expansion of M in a neighbourhood of the unit circle, then

$$\mathbb{E} \log^+ \|M_j U Z_0\| < \infty \quad \forall\, j \in \{mp+q-p+1, \ldots, mp+q\} \cup \{-p, \ldots, -1\}. \tag{2.16}$$

(iii) There exist $v \in \mathbb{C}^s$ and $g \in \mathbb{C}^m$ such that g is a solution to the linear equation

$$P(1)g = Q(1)U^*(v^T, u^T)^T. \tag{2.17}$$

Further, if (i) above holds, then condition (ii) can be replaced by

(ii') If $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ denotes the Laurent expansion of M in a neighbourhood of the unit circle, then $\sum_{j=-\infty}^{\infty} M_j U Z_{t-j}$ converges almost surely absolutely for every $t \in \mathbb{Z}$,

and condition (iii) can be replaced by

(iii') For all $v \in \mathbb{C}^s$ there exists a solution $g = g(v)$ to the linear equation (2.17).

If the conditions (i)–(iii) given above are satisfied, then a strictly stationary solution Y of the ARMA(p,q) equation (2.1) is given by

$$Y_t = g + \sum_{j=-\infty}^{\infty} M_j (U Z_{t-j} - (v^T, u^T)^T), \quad t \in \mathbb{Z}, \tag{2.18}$$

the series converging almost surely absolutely. Further, provided that the underlying probability space is rich enough to support a random variable which is uniformly distributed on $[0,1)$ and independent of $(Z_t)_{t\in\mathbb{Z}}$, the solution given by (2.18) is the unique strictly stationary solution of (2.1) if and only if $\det P(z) \neq 0$ for all z on the unit circle.

Special cases of Theorem 2.2 are treated in Remarks 2.10, 2.12 and Corollary 2.11. Observe that for $m = 1$, Theorem 2.2 reduces to the corresponding result in Brockwell and Lindner [4], stated as Theorem 1.4 in this thesis. Also observe that condition (iii) of Theorem 2.2 is not implied by condition (i), which can be seen e.g. by allowing a deterministic noise sequence $(Z_t)_{t\in\mathbb{Z}}$, in which case $M(z) \equiv 0$. The proof of Theorem 2.2 will be given in Section 2.5 and will make use of both Theorem 2.1 and Theorem 2.3 given below. The latter is the corresponding characterization for the existence of weakly stationary solutions of ARMA(p,q) equations, expressed in terms of the characteristic polynomials $P(z)$ and $Q(z)$. That $\det P(z) \neq 0$ for all z on the unit circle together with $\mathbb{E}(Z_0) = 0$ is sufficient for the existence of weakly stationary solutions is well known (cf. Theorem 1.5 in this thesis), but that the conditions given below are necessary and sufficient in higher dimensions seems not to have appeared in the literature so far. The proof of Theorem 2.3, which is similar to the proof in the one-dimensional case (cf. Theorem 1.3), will be given in Section 2.4.

Theorem 2.3. [Weak ARMA(p,q) processes]
Let $m, d, p \in \mathbb{N}$, $q \in \mathbb{N}_0$, and let $(Z_t)_{t\in\mathbb{Z}}$ be a weak white noise sequence in $\mathbb{C}^d$ with expectation $\mathbb{E}Z_0$ and covariance matrix Σ. Let $\Psi_1, \dots, \Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0, \dots, \Theta_q \in$

$\mathbb{C}^{m\times d}$, *and define the matrix polynomials* $P(z)$ *and* $Q(z)$ *by* (2.2). *Let* $U \in \mathbb{C}^{d\times d}$ *be unitary such that* $U\Sigma U^* = \begin{pmatrix} D & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}$, *where* D *is a real* $(s\times s)$*-diagonal matrix with the strictly positive eigenvalues of* Σ *on its diagonal for some* $s \in \{0,\ldots,d\}$. *(The matrix* U *exists since* Σ *is positive semidefinite). Then the ARMA*(p,q) *equation* (2.1) *admits a weakly stationary solution* $(Y_t)_{t\in\mathbb{Z}}$ *if and only if the* $\mathbb{C}^{m\times d}$*-valued rational function*

$$z \mapsto M(z) := P^{-1}(z)Q(z)U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}$$

has only removable singularities on the unit circle and if there is some $g \in \mathbb{C}^m$ *such that*

$$P(1)\, g = Q(1)\, \mathbb{E}Z_0. \tag{2.19}$$

In that case, a weakly stationary solution of (2.1) *is given by*

$$Y_t = g + \sum_{j=-\infty}^{\infty} M_j\, U(Z_{t-j} - \mathbb{E}Z_0), \quad t \in \mathbb{Z}, \tag{2.20}$$

where $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ *is the Laurent expansion of* $M(z)$ *in a neighbourhood of the unit circle, which converges absolutely there.*

It is easy to see that if Σ in the theorem above is invertible, then the condition that all singularities of $M(z)$ on the unit circle are removable is equivalent to the condition that all singularities of $P^{-1}(z)Q(z)$ on the unit circle are removable.

2.3 Proof of Theorem 2.1

In this section we give the proof of Theorem 2.1. In Section 2.3.1 we show that the conditions (i) — (iii) are necessary. The suffiency of the conditions is proven in Section 2.3.2, while the uniqueness assertion is established in Section 2.3.3.

2.3.1 The necessity of the conditions

Assume that $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of Equation (2.3). As observed before Theorem 2.1, this implies that each of the Equations (2.8) admits a strictly stationary solution, where $X_t^{(h)}$ is defined as in (2.6). Equation (2.8) is itself an ARMA$(1,q)$ equation with i.i.d. noise, so that for proving (i) – (iii) we may assume that $H = 1$, that $S = \mathrm{Id}_m$ and that $\Phi := \Psi_1$ is an $m \times m$ Jordan block corresponding to an eigenvalue λ. Hence we assume throughout Section 2.3.1 that

$$Y_t - \Phi Y_{t-1} = \sum_{k=0}^{q} \Theta_k Z_{t-k}, \quad t \in \mathbb{Z}, \tag{2.21}$$

has a strictly stationary solution with $\Phi \in \mathbb{C}^{m\times m}$ of the form (2.4), and we have to show that this implies (i) if $|\lambda| \neq 0, 1$, (ii) if $|\lambda| = 1$ but $\lambda \neq 1$, and (iii) if $\lambda = 1$. Before we do this in the next subsections, we observe that iterating the ARMA$(1, q)$ equation (2.21) gives for $n \geq q$

$$\begin{aligned} Y_t \;=\; & \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k}\Theta_k \right) Z_{t-j} + \sum_{j=q}^{n-1} \Phi^j \left(\sum_{k=0}^{q} \Phi^{-k}\Theta_k \right) Z_{t-j} \\ & + \sum_{j=0}^{q-1} \Phi^{n+j} \left(\sum_{k=j+1}^{q} \Phi^{-k}\Theta_k \right) Z_{t-(n+j)} + \Phi^n Y_{t-n}. \end{aligned} \tag{2.22}$$

The case $|\lambda| \in (0, 1)$.

Suppose that $|\lambda| \in (0, 1)$ and let $\varepsilon \in (0, |\lambda|)$. Then there are constants $C, C' \geq 1$ such that

$$\left\| \Phi^{-j} \right\| \leq C \cdot |\lambda|^{-j} \cdot j^m \leq (C')(|\lambda| - \varepsilon)^{-j} \quad \text{for all } j \in \mathbb{N},$$

as a consequence of Theorem 11.1.1 in [9]. Hence, we have for all $j \in \mathbb{N}_0$ and $t \in \mathbb{Z}$

$$\left\| \left(\sum_{k=0}^{q} \Phi^{-k}\Theta_k \right) Z_{t-j} \right\| \;\leq\; C'(|\lambda| - \varepsilon)^{-j} \left\| \Phi^j \left(\sum_{k=0}^{q} \Phi^{-k}\Theta_k \right) Z_{t-j} \right\|. \tag{2.23}$$

Now, since $\lim_{n\to\infty} \Phi^n = 0$ and since $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ are strictly stationary, an application of Slutsky's lemma to Equation (2.22) shows that

$$Y_t \;=\; \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k}\Theta_k \right) Z_{t-j} + \mathbb{P}\text{-}\lim_{n\to\infty} \sum_{j=q}^{n-1} \Phi^j \left(\sum_{k=0}^{q} \Phi^{-k}\Theta_k \right) Z_{t-j}. \tag{2.24}$$

Hence the limit on the right hand side exists and, as a sum with independent summands, it converges almost surely (see Kallenberg [13], Theorem 4.18). Thus it follows from Equation (2.23) and the Borel-Cantelli lemma that

$$\begin{aligned} & \sum_{j=q}^{\infty} \mathbb{P} \left(\left\| \sum_{k=0}^{q} \Phi^{-k}\Theta_k Z_0 \right\| > C'(|\lambda| - \varepsilon)^{-j} \right) \\ \leq\; & \sum_{j=q}^{\infty} \mathbb{P} \left(\left\| \Phi^j \left(\sum_{k=0}^{q} \Phi^{-k}\Theta_k \right) Z_{-j} \right\| > 1 \right) < \infty, \end{aligned}$$

and hence $\mathbb{E} \left(\log^+ \left\| \left(\sum_{k=0}^{q} \Phi^{-k}\Theta_k \right) Z_0 \right\| \right) < \infty$. Obviously, this is equivalent to condition (i).

The case $|\lambda| > 1$.

Suppose that $|\lambda| > 1$. Multiplying Equation (2.22) by Φ^{-n} gives for $n \geq q$

$$\begin{aligned}\Phi^{-n}Y_t &= \sum_{j=0}^{q-1}\Phi^{-(n-j)}\left(\sum_{k=0}^{j}\Phi^{-k}\Theta_k\right)Z_{t-j} + \sum_{j=1}^{n-q}\Phi^{-j}\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-n+j}\\ &\quad + \sum_{j=0}^{q-1}\Phi^{j}\left(\sum_{k=j+1}^{q}\Phi^{-k}\Theta_k\right)Z_{t-(n+j)} + Y_{t-n}.\end{aligned}$$

Defining $\tilde{\Phi} := \Phi^{-1}$, and substituting $u = t - n$ yields

$$\begin{aligned}Y_u &= -\sum_{j=0}^{q-1}\tilde{\Phi}^{-j}\left(\sum_{k=j+1}^{q}\Phi^{-k}\Theta_k\right)Z_{u-j} - \sum_{j=1}^{n-q}\tilde{\Phi}^{j}\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{u+j}\\ &\quad - \sum_{j=0}^{q-1}\tilde{\Phi}^{n-j}\left(\sum_{k=0}^{j}\Phi^{-k}\Theta_k\right)Z_{u+n-j} + \tilde{\Phi}^{n}Y_{u+n}.\end{aligned} \tag{2.25}$$

Letting $n \to \infty$ then gives condition (i) with the same arguments as in the case $|\lambda| \in (0,1)$.

The case $|\lambda| = 1$ and symmetric noise (Z_t).

Suppose that Z_0 is symmetric and that $|\lambda| = 1$. Denoting

$$J_1 := \Phi - \lambda\,\mathrm{Id}_m \quad \text{and} \quad J_l := J_1^l \quad \text{for} \quad j \in \mathbb{N}_0,$$

we have

$$\Phi^j = \sum_{l=0}^{m-1}\binom{j}{l}\lambda^{j-l}J_l, \quad j \in \mathbb{N}_0,$$

since $J_l = 0$ for $l \geq m$ and $\binom{j}{l} = 0$ for $l > j$. Further, since for $l \in \{0, \ldots, m-1\}$ we have

$$J_l = (e_{l+1}, e_{l+2}, ..., e_m, 0_m, ..., 0_m) \in \mathbb{C}^{m\times m},$$

with unit vectors $e_{l+1}, ..., e_m$ in $\mathbb{C}^m$, it is easy to see that for $i = 1, ..., m$ the i^{th} row of the matrix Φ^j is given by

$$e_i^T\Phi^j = \sum_{l=0}^{m-1}\binom{j}{l}\lambda^{j-l}e_i^T J_l = \sum_{l=0}^{i-1}\binom{j}{l}\lambda^{j-l}e_{i-l}^T, \quad j \in \mathbb{N}_0. \tag{2.26}$$

It follows from Equations (2.22) and (2.26) that for $n \geq q$ and $t \in \mathbb{Z}$,

$$\begin{aligned} e_i^T Y_t &= \sum_{j=0}^{q-1} \left(\sum_{l=0}^{i-1} \binom{j}{l} \lambda^{j-l} e_{i-l}^T \right) \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ &+ \sum_{j=q}^{n-1} \left(\sum_{l=0}^{i-1} \binom{j}{l} \lambda^{j-l} e_{i-l}^T \right) \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ &+ \sum_{j=0}^{q-1} \left(\sum_{l=0}^{i-1} \binom{n+j}{l} \lambda^{n+j-l} e_{i-l}^T \right) \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \\ &+ \sum_{l=0}^{i-1} \binom{n}{l} \lambda^{n-l} e_{i-l}^T Y_{t-n}. \end{aligned} \tag{2.27}$$

We claim that

$$e_i^T \sum_{k=0}^{q} \Phi^{-k} \Theta_k Z_t = 0 \text{ a.s.} \quad \forall\, i \in \{1, \ldots, m\} \quad \forall\, t \in \mathbb{Z}, \tag{2.28}$$

which clearly gives conditions (ii) and (iii), respectively, with $\alpha = \alpha_1 = 0_m$. Equation (2.28) will be proved by induction on $i = 1, \ldots, m$. We start with $i = 1$. From Equation (2.27) we know that for $n \geq q$

$$\begin{aligned} & e_1^T Y_t - \lambda^n e_1^T Y_{t-n} \\ & - \sum_{j=0}^{q-1} \lambda^j e_1^T \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} - \sum_{j=0}^{q-1} \lambda^{n+j} e_1^T \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \\ = & \sum_{j=q}^{n-1} \lambda^j e_1^T \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j}. \end{aligned} \tag{2.29}$$

Due to the stationarity of $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$, there exists a constant $K_1 > 0$ such that

$$\begin{aligned} \mathbb{P} \Bigg(\Bigg| & e_1^T Y_t - \lambda^n e_1^T Y_{t-n} - \sum_{j=0}^{q-1} \lambda^j e_1^T \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ & - \sum_{j=0}^{q-1} \lambda^{n+j} e_1^T \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \Bigg| < K_1 \Bigg) \geq \frac{1}{2} \quad \forall n \geq q. \end{aligned}$$

By (2.29) this implies

$$\mathbb{P} \left(\left| \sum_{j=q}^{n-1} \lambda^j e_1^T \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} \right| < K_1 \right) \geq \frac{1}{2} \quad \forall n \geq q. \tag{2.30}$$

Therefore $\left| \sum_{j=q}^{n-1} \lambda^j e_1^T \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} \right|$ does not converge in probability to $+\infty$ as $n \to \infty$. Since this is a sum of independent and symmetric terms, this implies that it converges almost surely (see Kallenberg [13], Theorem 4.17), and the Borel-Cantelli lemma then shows that

$$e_1^T \left(\sum_{k=0}^{q} \Phi^{-k}\Theta_k \right) Z_t = 0, \quad t \in \mathbb{Z},$$

which is (2.28) for $i = 1$. With this condition, Equation (2.29) simplifies for $t = 0$ and $n \geq q$ to

$$e_1^T Y_0 - \lambda^n e_1^T Y_{-n} = \sum_{j=0}^{q-1} \lambda^j e_1^T \left(\sum_{k=0}^{j} \Phi^{-k}\Theta_k \right) Z_{-j} + \sum_{j=0}^{q-1} \lambda^{n+j} e_1^T \left(\sum_{k=j+1}^{q} \Phi^{-k}\Theta_k \right) Z_{-(n+j)}.$$

Now setting $t := -n$ in the above equation, multiplying it with $\lambda^t = \lambda^{-n}$ and recalling that $e_1^T \Phi^j = \lambda^j e_1^T$ by (2.26) yields for $t \leq -q$

$$e_1^T Y_t = -\sum_{j=0}^{q-1} e_1^T \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k}\Theta_k \right) Z_{t-j} + \lambda^t e_1^T \left(Y_0 - \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k}\Theta_k \right) Z_{-j} \right).$$

For the induction step let $i \in \{2, \dots, m\}$ and assume that

$$e_r^T \left(\sum_{k=0}^{q} \Phi^{-k}\Theta_k \right) Z_t = 0 \text{ a.s.}, \quad r \in \{1, ..., i-1\}, \ t \in \mathbb{Z}, \tag{2.31}$$

together with

$$e_r^T Y_t = -e_r^T \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k}\Theta_k \right) Z_{t-j} + \begin{cases} 0, & r \in \{1, \dots, i-2\},\ t \leq -rq, \\ \lambda^t e_r^T V_r, & r = i-1,\ t \leq -rq, \end{cases} \tag{2.32}$$

where

$$V_r := \lambda^{(r-1)q} \left(Y_{-(r-1)q} - \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k}\Theta_k \right) Z_{-j-(r-1)q} \right), \quad r \in \{1, \dots, m\}.$$

We are going to show that this implies

$$e_i^T \left(\sum_{k=0}^{q} \Phi^{-k}\Theta_k \right) Z_t = 0 \text{ a.s.}, \quad t \in \mathbb{Z}, \tag{2.33}$$

and

$$e_i^T Y_t = -e_i^T \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k}\Theta_k \right) Z_{t-j} + \lambda^t e_i^T V_i \text{ a.s.}, \quad t \leq -iq, \tag{2.34}$$

together with

$$e_{i-1}^T V_{i-1} = 0. \tag{2.35}$$

This will then imply (2.28). For doing that, in a first step we are going to prove the following:

Lemma 2.4. *Let $i \in \{2, \dots, m\}$ and assume (2.31) and (2.32). Then it holds for $t \le -(i-1)q$ and $n \ge q$,*

$$\begin{aligned} & e_i^T Y_t - \lambda^n e_i^T Y_{t-n} \\ = & \sum_{j=0}^{q-1} e_i^T \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} + \sum_{j=q}^{n-1} \lambda^j e_i^T \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ & + \lambda^n \sum_{j=0}^{q-1} e_i^T \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} + n\lambda^{t-1} e_{i-1}^T V_{i-1}, \end{aligned} \tag{2.36}$$

Proof. Let $t \le -(i-1)q$ and $n \ge q$. Using (2.32) and (2.26), the last summand of (2.27) can be written as

$$\begin{aligned} & \sum_{l=0}^{i-1} \binom{n}{l} \lambda^{n-l} e_{i-l}^T Y_{t-n} \\ = & \lambda^n e_i^T Y_{t-n} + \sum_{r=1}^{i-1} \binom{n}{i-r} \lambda^{n-(i-r)} e_r^T Y_{t-n}, \\ = & \lambda^n e_i^T Y_{t-n} - \sum_{j=0}^{q-1} \left(\sum_{r=1}^{i-1} \sum_{l=0}^{r-1} \binom{j}{l} \binom{n}{i-r} \lambda^{n-(i-r)} \lambda^{j-l} e_{r-l}^T \right) \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \\ & + n\lambda^{t-1} e_{i-1}^T V_{i-1} \\ = & \lambda^n e_i^T Y_{t-n} - \sum_{j=0}^{q-1} \left(\sum_{s=1}^{i-1} \binom{n+j}{s} \lambda^{n+j-s} e_{i-s}^T \right) \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \\ & + \lambda^n \sum_{j=0}^{q-1} \left(\sum_{s=1}^{i-1} \binom{j}{s} \lambda^{j-s} e_{i-s}^T \right) \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} + n\lambda^{t-1} e_{i-1}^T V_{i-1}, \end{aligned}$$

where we substituted $s := i - r + l$ and $p := s - l$ and used Vandermonde's identity $\sum_{p=1}^{s} \binom{j}{s-p} \binom{n}{p} = \binom{n+j}{s} - \binom{j}{s}$ in the last equation. Inserting this back into Equation (2.27) and using (2.31), we get for $t \le -(i-1)q$ and $n \ge q$

$$\begin{aligned} & e_i^T Y_t - \lambda^n e_i^T Y_{t-n} \\ = & \sum_{j=0}^{q-1} \left(\sum_{l=0}^{i-1} \binom{j}{l} \lambda^{j-l} e_{i-l}^T \right) \left(\sum_{k=0}^{j} \Phi^{-k} \Theta_k \right) Z_{t-j} \\ & + \sum_{j=q}^{n-1} \lambda^j e_i^T \left(\sum_{k=0}^{q} \Phi^{-k} \Theta_k \right) Z_{t-j} + \sum_{j=0}^{q-1} \lambda^{n+j} e_i^T \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \\ & + \lambda^n \sum_{j=0}^{q-1} \left(\sum_{s=1}^{i-1} \binom{j}{s} \lambda^{j-s} e_{i-s}^T \right) \left(\sum_{k=j+1}^{q} \Phi^{-k} \Theta_k \right) Z_{t-(n+j)} \\ & + n\lambda^{t-1} e_{i-1}^T V_{i-1}. \end{aligned}$$

An application of (2.26) then shows (2.36), completing the proof of the lemma. □

To continue with the induction step, we first show that (2.35) holds true. Dividing (2.36) by n and letting $n \to \infty$, the strict stationarity of $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ imply that for $t \le -(i-1)q$,

$$n^{-1}\sum_{j=q}^{n-1}\lambda^j e_i^T\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j}$$

converges in probability to $-\lambda^{t-1}e_{i-1}^T V_{i-1}$. On the other hand, this limit in probability must be clearly measurable with respect to the tail-σ-algebra $\cap_{k\in\mathbb{N}}\sigma(\cup_{l\ge k}\sigma(Z_{t-l}))$, which by Kolmogorov's zero-one law is $\mathbb{P}$-trivial. Hence this probability limit must be constant, and because of the assumed symmetry of Z_0 it must be symmetric, hence is equal to 0, i.e.

$$e_{i-1}^T V_{i-1} = 0 \ \text{ a.s.},$$

which is (2.35). Using this, we get from Lemma 2.4 that

$$\begin{aligned}
&e_i^T Y_t - \lambda^n e_i^T Y_{t-n}\\
&-\sum_{j=0}^{q-1} e_i^T\Phi^j\left(\sum_{k=0}^{j}\Phi^{-k}\Theta_k\right)Z_{t-j} - \lambda^n\sum_{j=0}^{q-1}e_i^T\Phi^j\left(\sum_{k=j+1}^{q}\Phi^{-k}\Theta_k\right)Z_{t-(n+j)}\\
&=\sum_{j=q}^{n-1}\lambda^j e_i^T\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j}, \quad t \le -(i-1)q.
\end{aligned} \tag{2.37}$$

Again due to the stationarity of $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ there exists a constant $K_2 > 0$ such that

$$\begin{aligned}
\mathbb{P}\Bigg(\Bigg|&e_i^T Y_t - \lambda^n e_i^T Y_{t-n} - \sum_{j=0}^{q-1}e_i^T\Phi^j\left(\sum_{k=0}^{j}\Phi^{-k}\Theta_k\right)Z_{t-j}\\
&-\lambda^n\sum_{j=0}^{q-1}e_i^T\Phi^j\left(\sum_{k=j+1}^{q}\Phi^{-k}\Theta_k\right)Z_{t-(n+j)}\Bigg| < K_2\Bigg) \ge \frac{1}{2} \quad \forall\, n \ge q,
\end{aligned}$$

so that

$$\mathbb{P}\left(\left|\sum_{j=q}^{n-1}\lambda^j e_i^T\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j}\right| < K_2\right) \ge \frac{1}{2} \quad \forall\, n \ge q, \quad t \le -(i-1)q.$$

Therefore $\left|\sum_{j=q}^{n-1}\lambda^j e_i^T\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_{t-j}\right|$ does not converge in probability to $+\infty$ as $n \to \infty$. Since this is a sum of independent and symmetric terms, this implies that it converges almost surely (see Kallenberg [13], Theorem 4.17), and the Borel-Cantelli lemma then shows that $e_i^T\left(\sum_{k=0}^{q}\Phi^{-k}\Theta_k\right)Z_t = 0$ a.s. for $t \le -(i-1)q$ and hence for all $t \in \mathbb{Z}$, which is (2.33). Equation (2.37) now simplifies for $t = -(i-1)q$ and $n \geq q$ to

$$\begin{aligned} & e_i^T Y_{-(i-1)q} - \lambda^n e_i^T Y_{-(i-1)q-n} \\ &= \sum_{j=0}^{q-1} e_i^T \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k}\Theta_k \right) Z_{-(i-1)q-j} + \lambda^n \sum_{j=0}^{q-1} e_i^T \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k}\Theta_k \right) Z_{-(i-1)q-n-j}. \end{aligned}$$

Multiplying this equation by λ^{-n} and denoting $t := -(i-1)q - n$, it follows that for $t \leq -iq$ it holds

$$\begin{aligned} e_i^T Y_t &= -\sum_{j=0}^{q-1} e_i^T \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k}\Theta_k \right) Z_{t-j} \\ &\quad + \lambda^{t+(i-1)q} e_i^T \left(Y_{-(i-1)q} - \sum_{j=0}^{q-1} \Phi^j \left(\sum_{k=0}^{j} \Phi^{-k}\Theta_k \right) Z_{-j-(i-1)q} \right) \\ &= -\sum_{j=0}^{q-1} e_i^T \Phi^j \left(\sum_{k=j+1}^{q} \Phi^{-k}\Theta_k \right) Z_{t-j} + \lambda^t e_i^T V_i, \end{aligned}$$

which is Equation (2.34). This completes the proof of the induction step and hence of (2.28). It follows that conditions (ii) and (iii), respectively, hold with $\alpha_1 = 0$ if $|\lambda| = 1$ and Z_0 is symmetric.

The case $|\lambda| = 1$ and not necessarily symmetric noise (Z_t).

As in the previous section, assume that $|\lambda| = 1$, but not necessarily that Z_0 is symmetric. Let $(Y_t', Z_t')_{t\in\mathbb{Z}}$ be an independent copy of $(Y_t, Z_t)_{t\in\mathbb{Z}}$ and denote $\widetilde{Y}_t := Y_t - Y_t'$ and $\widetilde{Z}_t := Z_t - Z_t'$. Then $(\widetilde{Y}_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of $\widetilde{Y}_t - \Phi\widetilde{Y}_{t-1} = \sum_{k=0}^{q} \Theta_k \widetilde{Z}_{t-k}$, and $(\widetilde{Z}_t)_{t\in\mathbb{Z}}$ is i.i.d. with $\widetilde{Z}_0$ being symmetric. It hence follows from the previous section that

$$\left(\sum_{k=0}^{q} \Phi^{q-k}\Theta_k \right) Z_0 - \left(\sum_{k=0}^{q} \Phi^{q-k}\Theta_k \right) Z_0' = \left(\sum_{k=0}^{q} \Phi^{q-k}\Theta_k \right) \widetilde{Z}_0 = 0.$$

Since Z_0 and Z_0' are independent, this implies that there is a constant $\alpha \in \mathbb{C}^m$ such that $\sum_{k=0}^{q} \Phi^{q-k}\Theta_k Z_0 = \alpha$ a.s., which is (2.11), hence condition (ii) if $\lambda \neq 1$. To show condition (iii) in the case $\lambda = 1$, recall that the deviation of (2.30) in the previous section did not need the symmetry assumption on Z_0. Hence by (2.30) there is some constant K_1 such that $\mathbb{P}(|\sum_{j=q}^{n-1} 1^j e_1^T \alpha| < K_1) \geq 1/2$ for all $n \geq q$, which clearly implies $e_1^T \alpha = 0$ and hence condition (iii).

2.3.2 The sufficiency of the conditions

Suppose that conditions (i) — (iii) are satisfied, and let $X_t^{(h)}$, $t \in \mathbb{Z}$, $h \in \{1, \ldots, H\}$, be defined by (2.12). The fact that $X_t^{(h)}$ as defined in (2.12) converges a.s. for $|\lambda_h| \in (0,1)$ is in complete analogy to the proof in the one-dimensional case treated in Brockwell and Lindner [4], but we give the short argument for completeness: observe that there are constants $a, b > 0$ such that $\|\Phi_h^j\| \le a e^{-bj}$ for $j \in \mathbb{N}_0$. Hence for $b' \in (0, b)$ we can estimate

$$\begin{aligned}
&\sum_{j=q}^{\infty} \mathbb{P}\left(\left\|\Phi_h^{j-q} \sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k Z_{t-j}\right\| > e^{-b'(j-q)}\right) \\
&\quad \le \sum_{j=q}^{\infty} \mathbb{P}\left(\log^+\left(a \left\|\sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k Z_{t-j}\right\|\right) > (b-b')(j-q)\right) < \infty,
\end{aligned}$$

the last inequality being due to the fact that $\left\|\sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k Z_{t-j}\right\|$ has the same distribution as $\left\|\sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k Z_0\right\|$ and the latter has finite log-moment by Equation (2.10). An application of the Borel–Cantelli lemma then shows that the event $\{\|\Phi_h^{j-q} \sum_{k=0}^{q} \Phi_h^{q-k} I_h S^{-1} \Theta_k Z_{t-j}\| > e^{-b'(j-q)}$ for infinitely many $j\}$ has probability zero, giving the almost sure absolute convergence of the series in (2.12). The almost sure absolute convergence of (2.12) if $|\lambda_h| > 1$ is established similarly.
It is obvious that $((X_t^{(1)T}, \ldots, X_t^{(H)T})^T)_{t \in \mathbb{Z}}$ as defined in (2.12) and hence $(Y_t)_{t \in \mathbb{Z}}$ defined by (2.9) is strictly stationary, so it only remains to show that $(X_t^{(h)})_{t \in \mathbb{Z}}$ solves (2.8) for each $h \in \{1, \ldots, H\}$. For $|\lambda_h| \ne 0, 1$, this is an immediate consequence of (2.12). For $|\lambda_h| = 1$, we have by (2.12) and the definition of f_h that

$$\begin{aligned}
X_t^{(h)} - \Phi_h X_{t-1}^{(h)} &= \alpha_h + \sum_{j=0}^{q-1} \sum_{k=0}^{j} \Phi_h^{j-k} I_h S^{-1} \Theta_k Z_{t-j} - \sum_{j=1}^{q} \sum_{k=0}^{j-1} \Phi_h^{j-k} I_h S^{-1} \Theta_k Z_{t-j} \\
&= \alpha_h + \sum_{j=0}^{q-1} I_h S^{-1} \Theta_j Z_{t-j} - \sum_{k=0}^{q-1} \Phi_h^{q-k} I_h S^{-1} \Theta_k Z_{t-q} \\
&= I_h S^{-1} \sum_{j=0}^{q} \Theta_j Z_{t-j},
\end{aligned}$$

where the last equality follows from (2.11). Finally, if $\lambda_h = 0$, then $\Phi_h^j = 0$ for $j \ge m$, implying that $X_t^{(h)}$ defined by (2.12) solves (2.8) also in this case.

2.3.3 The uniqueness of the solution

Suppose that $|\lambda_h| \ne 1$ for all $h \in \{1, \ldots, H\}$ and let $(Y_t)_{t \in \mathbb{Z}}$ be a strictly stationary solution of (2.3). Then $(X_t^{(h)})_{t \in \mathbb{Z}}$, as defined by (2.6), is a strictly stationary solution

of (2.8) for each $h \in \{1, \dots, H\}$. It then follows as in Section 2.3.1 that by the equation corresponding to (2.24), $X_t^{(h)}$ is uniquely determined if $|\lambda_h| \in (0,1)$. Similarly, $X_t^{(h)}$ is uniquely determined if $|\lambda_h| > 1$. The uniqueness of $X_t^{(h)}$ if $\lambda_h = 0$ follows from the equation corresponding to (2.22) with $n \geq m$, since then $\Phi_h^j = 0$ for $j \geq m$. We conclude that $((X_t^{(1)T}, \dots, X_t^{(H)T})^T)_{t\in\mathbb{Z}}$ is unique and hence so is $(Y_t)_{t\in\mathbb{Z}}$.
Now suppose that there is $h \in \{1, \dots, H\}$ such that $|\lambda_h| = 1$. Let U be a random variable which is uniformly distributed on $[0,1)$ and independent of $(Z_t)_{t\in\mathbb{Z}}$. Then $(R_t)_{t\in\mathbb{Z}}$, defined by $R_t := \lambda_h^t (0, \dots 0, e^{2\pi i U})^T \in \mathbb{C}^{r_{h+1}-r_h}$, is strictly stationary and independent of $(Z_t)_{t\in\mathbb{Z}}$ and satisfies $R_t - \Phi_h R_{t-1} = 0$. Hence, if $(Y_t)_{t\in\mathbb{Z}}$ is the strictly stationary solution of (2.3) specified by (2.12) and (2.9), then

$$Y_t + S(0^T_{r_2-r_1}, \dots, 0^T_{r_h-r_{h-1}}, R_t^T, 0^T_{r_{h+2}-r_{h+1}}, \dots, 0^T_{r_{H+1}-r_H})^T, \quad t \in \mathbb{Z},$$

is another strictly stationary solution of (2.3), violating uniqueness.

2.4 Proof of Theorem 2.3

In this section we shall prove Theorem 2.3. Denote

$$R := U^* \begin{pmatrix} D^{1/2} & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} \quad \text{and} \quad W_t := \begin{pmatrix} D^{-1/2} & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} U(Z_t - \mathbb{E}Z_0), \quad t \in \mathbb{Z},$$

where $D^{1/2}$ is the unique diagonal matrix with strictly positive eigenvalues such that $(D^{1/2})^2 = D$. Then $(W_t)_{t\in\mathbb{Z}}$ is a white noise sequence in $\mathbb{C}^d$ with expectation 0 and covariance matrix $\begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}$. It is further clear that all singularities of $M(z)$ on the unit circle are removable if and only if all singularities of $M'(z) := P^{-1}(z)Q(z)R$ on the unit circle are removable, and in that case, the Laurent expansions of both $M(z)$ and $M'(z)$ converge absolutely in a neighbourhood of the unit circle.
To see the sufficiency of the condition, suppose that (2.19) has a solution g and that $M(z)$ and hence $M'(z)$ have only removable singularities on the unit circle. Define $Y = (Y_t)_{t\in\mathbb{Z}}$ by (2.20), i.e.

$$Y_t = g + \sum_{j=-\infty}^{\infty} M_j \begin{pmatrix} D^{1/2} & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} W_{t-j} = g + M'(B)W_t, \quad t \in \mathbb{Z}.$$

The series converges almost surely absolutely due to the exponential decrease of the entries of M_j as $|j| \to \infty$. Further, Y is clearly weakly stationary, and since the last

$(d-s)$ components of $U(Z_t - \mathbb{E}Z_0)$ vanish, having expectation zero and variance zero, it follows that

$$RW_t = U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} U(Z_t - \mathbb{E}Z_0) = U^*U(Z_t - \mathbb{E}Z_0) = Z_t - \mathbb{E}Z_0, \quad t \in \mathbb{Z}.$$

We conclude that

$$P(B)(Y_t - g) = P(B)M'(B)W_t = P(B)P^{-1}(B)Q(B)RW_t = Q(B)(Z_t - \mathbb{E}Z_0), \quad t \in \mathbb{Z}.$$

Since $P(1)g = Q(1)\mathbb{E}Z_0$, this shows that $(Y_t)_{t\in\mathbb{Z}}$ is a weakly stationary solution of (2.1).
Conversely, suppose that $Y = (Y_t)_{t\in\mathbb{Z}}$ is a weakly stationary solution of (2.1). Taking expectations in (2.1) yields $P(1)\,\mathbb{E}Y_0 = Q(1)\,\mathbb{E}Z_0$, so that (2.19) has a solution. The $\mathbb{C}^{m\times m}$-valued spectral measure μ_Y of Y satisfies

$$P(e^{-i\omega})\, d\mu_Y(\omega)\, P(e^{-i\omega})^* = \frac{1}{2\pi} Q(e^{-i\omega}) \Sigma Q(e^{-i\omega})^*\, d\omega, \quad \omega \in (-\pi, \pi].$$

It follows that, with the finite set $N := \{\omega \in (-\pi,\pi] : P(e^{-i\omega}) = 0\}$,

$$d\mu_Y(\omega) = \frac{1}{2\pi} P^{-1}(e^{-i\omega}) Q(e^{-i\omega}) \Sigma Q(e^{-i\omega})^* P^{-1}(e^{-i\omega})^*\, d\omega \quad \text{on} \quad (-\pi,\pi] \setminus N.$$

Observing that $RR^* = \Sigma$, it follows that the function $\omega \mapsto M'(e^{-i\omega})M'(e^{-i\omega})^*$ must be integrable on $(-\pi,\pi]\setminus N$. Now assume that the matrix rational function M' has a non-removable singularity at z_0 with $|z_0| = 1$ in at least one matrix element. This must then be a pole of order $r \geq 1$. Denoting the spectral norm by $\|\cdot\|_2$ it follows that there are $\varepsilon > 0$ and $K > 0$ such that

$$\|M'(z)^*\|_2 \geq K|z - z_0|^{-1} \quad \forall\, z \in \mathbb{C} : |z| = 1, z \neq z_0, |z - z_0| \leq \varepsilon;$$

this may be seen by considering first the row sum norm of $M'(z)^*$ and then using the equivalence of norms. Since the matrix $M'(z)M'(z)^*$ is hermitian, we conclude that

$$\|M'(z)M'(z)^*\|_2 = \sup_{v\in\mathbb{C}^n : |v|=1} |v^* M'(z)M'(z)^* v| = \sup_{v\in\mathbb{C}^n : |v|=1} |M'(z)^* v|^2 \geq K^2 |z - z_0|^{-2}$$

for all $z \neq z_0$ on the unit circle such that $|z - z_0| \leq \varepsilon$. But this implies that $\omega \mapsto M'(e^{-i\omega})M'(e^{-i\omega})^*$ cannot be integrable on $(-\pi,\pi]\setminus N$, giving the desired contradiction. This finishes the proof of Theorem 2.3.

2.5 Proof of Theorem 2.2

In this section we shall prove Theorem 2.2. For that, we first observe that ARMA(p,q) equations can be embedded into higher dimensional ARMA$(1,q)$ processes, as stated in the following proposition. This is well known and its proof is immediate, hence omitted.

Proposition 2.5. *Let $m,d,p \in \mathbb{N}$, $q \in \mathbb{N}_0$, and let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathbb{C}^d$-valued random vectors. Let $\Psi_1,\ldots,\Psi_p \in \mathbb{C}^{m\times m}$ and $\Theta_0,\ldots,\Theta_q \in \mathbb{C}^{m\times d}$ be complex-valued matrices. Define the matrices $\underline{\Phi} \in \mathbb{C}^{mp\times mp}$ and $\underline{\Theta}_k \in \mathbb{C}^{mp\times d}$, $k \in \{0,\ldots,q\}$, by*

$$\underline{\Phi} := \begin{pmatrix} \Psi_1 & \Psi_2 & \cdots & \Psi_{p-1} & \Psi_p \\ \mathrm{Id}_m & 0_{m,m} & \cdots & 0_{m,m} & 0_{m,m} \\ 0_{m,m} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0_{m,m} & \vdots \\ 0_{m,m} & \cdots & 0_{m,m} & \mathrm{Id}_m & 0_{m,m} \end{pmatrix} \quad \text{and} \quad \underline{\Theta}_k = \begin{pmatrix} \Theta_k \\ 0_{m,d} \\ \vdots \\ 0_{m,d} \end{pmatrix}. \tag{2.38}$$

Then the ARMA(p,q) equation (2.1) *admits a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ of m-dimensional random vectors Y_t if and only if the ARMA$(1,q)$ equation*

$$\underline{Y}_t - \underline{\Phi}\,\underline{Y}_{t-1} = \underline{\Theta}_0 Z_t + \underline{\Theta}_1 Z_{t-1} + \ldots + \underline{\Theta}_q Z_{t-q}, \quad t \in \mathbb{Z}, \tag{2.39}$$

admits a strictly stationary solution $(\underline{Y}_t)_{t\in\mathbb{Z}}$ of mp-dimensional random vectors $\underline{Y}_t$. More precisely, if $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of (2.1)*, then*

$$(\underline{Y}_t)_{t\in\mathbb{Z}} := ((Y_t^T, Y_{t-1}^T, \ldots, Y_{t-(p-1)}^T)^T)_{t\in\mathbb{Z}} \tag{2.40}$$

is a strictly stationary solution of Equation (2.39)*, and conversely, if $(\underline{Y}_t)_{t\in\mathbb{Z}} = ((Y_t^{(1)T},\ldots,Y_t^{(p)T})^T)_{t\in\mathbb{Z}}$ with random components $Y_t^{(i)} \in \mathbb{C}^m$ is a strictly stationary solution of* (2.39)*, then $(Y_t)_{t\in\mathbb{Z}} := (Y_t^{(1)})_{t\in\mathbb{Z}}$ is a strictly stationary solution of* (2.1).

For the proof of Theorem 2.2 we need some notation: define $\underline{\Phi}$ and $\underline{\Theta}_k$ as in (2.38). Choose an invertible $\mathbb{C}^{mp\times mp}$ matrix $\underline{S}$ such that $\underline{S}^{-1}\underline{\Phi}\,\underline{S}$ is in Jordan canonical form, with H Jordan blocks $\underline{\Phi}_1,\ldots,\underline{\Phi}_H$, say, the h^{th} Jordan block $\underline{\Phi}_h$ starting in row $\underline{r}_h$, with $\underline{r}_1 := 1 < \underline{r}_2 < \cdots < \underline{r}_H < mp+1 =: \underline{r}_{H+1}$. Let $\underline{\lambda}_h$ be the eigenvalue associated with $\underline{\Phi}_h$, and, similarly to (2.7), denote by $\underline{I}_h$ the $(\underline{r}_{h+1}-\underline{r}_h)\times mp$-matrix with components $\underline{I}_h(i,j) = 1$ if $j = i + \underline{r}_h - 1$ and $\underline{I}_h(i,j) = 0$ otherwise. For

$h \in \{1, \ldots, H\}$ and $j \in \mathbb{Z}$ let

$$N_{j,h} := \begin{cases} \mathbf{1}_{j \geq 0} \underline{\Phi}_h^{j-q} \sum_{k=0}^{j \wedge q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\Theta}_k, & |\underline{\lambda}_h| \in (0,1), \\ -\mathbf{1}_{j \leq q-1} \underline{\Phi}_h^{j-q} \sum_{k=(1+j)\vee 0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\Theta}_k, & |\underline{\lambda}_h| > 1, \\ \mathbf{1}_{j \in \{0,\ldots,mp+q-1\}} \sum_{k=0}^{j \wedge q} \underline{\Phi}_h^{j-k} \underline{I}_h \underline{S}^{-1} \underline{\Theta}_k, & \underline{\lambda}_h = 0, \\ \mathbf{1}_{j \in \{0,\ldots,q-1\}} \sum_{k=0}^{j} \underline{\Phi}_h^{j-k} \underline{I}_h \underline{S}^{-1} \underline{\Theta}_k, & |\underline{\lambda}_h| = 1, \end{cases}$$

and

$$\underline{N}_j := \underline{S}^{-1} (N_{j,1}^T, \ldots, N_{j,H}^T)^T \in \mathbb{C}^{mp \times d}. \tag{2.41}$$

Further, let U and K be defined as in the statement of the theorem, and denote

$$W_t := U Z_t, \quad t \in \mathbb{Z}.$$

Then $(W_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence. Equation (2.15) is then an easy consequence of the fact that for $a \in \mathbb{C}^d$ the distribution of $a^* W_0 = (U^* a)^* Z_0$ is degenerate to a Dirac measure if and only if $U^* a \in K$, i.e. if $a \in UK = \{0_s\} \times \mathbb{C}^{d-s}$: taking for a the i^{th} unit vector in $\mathbb{C}^d$ for $i \in \{s+1, \ldots, d\}$, we see that W_t must be of the form $(w_t^T, u^T)^T$ for some $u \in \mathbb{C}^{d-s}$, and taking $a = (b^T, 0_{d-s}^T)^T$ for $b \in \mathbb{C}^s$ we see that $b^* w_0$ is not degenerate to a Dirac measure for $b \neq 0_s$. The remaining proof of the necessity of the conditions, the sufficiency of the conditions and the stated uniqueness will be given in the next subsections.

2.5.1 The necessity of the conditions

Suppose that $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of (2.1). Define $\underline{Y}_t$ by (2.40). Then $(\underline{Y}_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of (2.39) by Proposition 2.5. Hence, by Theorem 2.1, there is $\underline{f}' \in \mathbb{C}^{mp}$, such that $(\underline{Y}'_t)_{t\in\mathbb{Z}}$, defined by

$$\underline{Y}'_t = \underline{f}' + \sum_{j=-\infty}^{\infty} \underline{N}_j Z_{t-j}, \quad t \in \mathbb{Z}, \tag{2.42}$$

is (possibly another) strictly stationary solution of

$$\underline{Y}'_t - \underline{\Phi}\, \underline{Y}'_{t-1} = \sum_{k=0}^{q} \underline{\Theta}_k Z_{t-k} = \sum_{k=0}^{q} \underline{\widetilde{\Theta}}_k W_{t-k}, \quad t \in \mathbb{Z},$$

where $\underline{\widetilde{\Theta}}_k := \underline{\Theta}_k U^*$. The sum in (2.42) converges almost surely absolutely. Now define $A_h \in \mathbb{C}^{(\underline{r}_{h+1} - \underline{r}_h) \times s}$ and $C_h \in \mathbb{C}^{(\underline{r}_{h+1} - \underline{r}_h) \times (d-s)}$ for $h \in \{1, \ldots, H\}$ such that $|\underline{\lambda}_h| = 1$ by

$$(A_h, C_h) := \sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\widetilde{\Theta}}_k. \tag{2.43}$$

By conditions (ii) and (iii) of Theorem 2.1, for every such h with $|\underline{\lambda}_h| = 1$ there exists a vector $\underline{\alpha}_h = (\alpha_{h,1}, \ldots, \alpha_{h,\underline{r}_{h+1}-\underline{r}_h})^T \in \mathbb{C}^{\underline{r}_{h+1}-\underline{r}_h}$ such that

$$(A_h, C_h) W_0 = \underline{\alpha}_h \quad \text{a.s.}$$

with $\alpha_{h,1} = 0$ if $\underline{\lambda}_h = 1$. Since $W_0 = (w_0^T, u^T)^T$, this implies $A_h w_0 = \underline{\alpha}_h - C_h u$, but since $b^* w_0$ is not degenerate to a Dirac measure for any $b \in \mathbb{C}^s \setminus \{0_s\}$, this gives $A_h = 0$ and hence $C_h u = \underline{\alpha}_h$ for $h \in \{1, \ldots, H\}$ such that $|\underline{\lambda}_h| = 1$. Now let $v \in \mathbb{C}^s$ and $(W_t'')_{t\in\mathbb{Z}}$ be an i.i.d. $N(\begin{pmatrix} v \\ u \end{pmatrix}, \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix})$-distributed sequence, and let $Z_t'' := U^* W_t''$. Then

$$(A_h, C_h) W_0'' = C_h u = \underline{\alpha}_h \quad \text{a.s.} \quad \text{for} \quad h \in \{1, \ldots, H\} : |\underline{\lambda}_h| = 1$$

and

$$\mathbb{E} \log^+ \left\| \sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\widetilde{\Theta}}_k W_0'' \right\| < \infty \quad \text{for} \quad h \in \{1, \ldots, H\} : |\underline{\lambda}_h| \neq 0, 1.$$

It then follows from Theorem 2.1 that there is a strictly stationary solution $\underline{Y}_t''$ of the ARMA$(1, q)$ equation $\underline{Y}_t'' - \underline{\Phi}\, \underline{Y}_{t-1}'' = \sum_{k=0}^{q} \underline{\widetilde{\Theta}}_k W_{t-k}'' = \sum_{k=0}^{q} \underline{\Theta}_k Z_{t-k}''$, which can be written in the form $\underline{Y}_t'' = \underline{f}'' + \sum_{j=-\infty}^{\infty} \underline{N}_j Z_{t-j}''$ for some $\underline{f}'' \in \mathbb{C}^{mp}$. In particular, $(\underline{Y}_t'')_{t\in\mathbb{Z}}$ is a Gaussian process. Again from Proposition 2.5 it follows that there is a Gaussian process $(Y_t'')_{t\in\mathbb{Z}}$ which is a strictly stationary solution of

$$Y_t'' - \sum_{k=1}^{p} \Psi_k Y_{t-k}'' = \sum_{k=0}^{q} \widetilde{\Theta}_k W_{t-k}'' = \sum_{k=0}^{q} \Theta_k Z_{t-k}'', \quad t \in \mathbb{Z}.$$

In particular, this solution is also weakly stationary. Hence it follows from Theorem 2.3 that $z \mapsto M(z)$ has only removable singularities on the unit circle and that (2.17) has a solution $g \in \mathbb{C}^m$, since $\mathbb{E} Z_0'' = U^*(v^T, u^T)^T$. Hence we have established that (i) and (iii'), and hence (iii), of Theorem 2.2 are necessary conditions for a strictly stationary solution to exist.
To see the necessity of conditions (ii) and (ii'), we need the following lemma, which is interesting in itself since it expresses the Laurent coefficients of $M(z)$ in terms of the Jordan canonical decomposition of $\underline{\Phi}$.

Lemma 2.6. *With the notations of Theorem 2.2 and those introduced after Proposition 2.5, suppose that condition (i) of Theorem 2.2 holds, i.e. that $M(z)$ has only* *removable singularities on the unit circle. Denote by* $M(z) = \sum_{j=-\infty}^{\infty} M_j z^j$ *the Laurent expansion of* $M(z)$ *in a neighborhood of the unit circle. Then*

$$\underline{M}_j := (M_j^T, M_{j-1}^T, \ldots, M_{j-p+1}^T)^T = \underline{N}_j U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} \quad \forall\, j \in \mathbb{Z}. \tag{2.44}$$

In particular,

$$\underline{M}_j U Z_{t-j} = \underline{N}_j Z_{t-j} - \underline{N}_j U^* (0_s^T, u^T)^T \quad \forall\, j, t \in \mathbb{Z}. \tag{2.45}$$

Proof. Define $\Lambda := \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}$ and let $(Z_t')_{t\in\mathbb{Z}}$ be an i.i.d. $N(0_d, U^*\Lambda U)$-distri-buted noise sequence and define $Y_t' := \sum_{j=-\infty}^{\infty} M_j U Z_{t-j}'$. Then $(Y_t')_{t\in\mathbb{Z}}$ is a weakly and strictly stationary solution of $P(B)Y_t' = Q(B)Z_t'$ by Theorem 2.3, and the entries of M_j decrease geometrically as $|j| \to \infty$. By Proposition 2.5, the process $(\underline{Y}_t')_{t\in\mathbb{Z}}$ defined by $\underline{Y}_t' = ({Y_t'}^T, {Y_{t-1}'}^T, \ldots, {Y_{t-p+1}'}^T) = \sum_{j=-\infty}^{\infty} \underline{M}_j U Z_{t-j}'$ is a strictly stationary solution of

$$\underline{Y}_t' - \underline{\Phi}\, \underline{Y}_{t-1}' = \sum_{j=0}^{q} \underline{\Theta}_j Z_{t-j}', \quad t \in \mathbb{Z}. \tag{2.46}$$

Denoting $\underline{\Theta}_j = 0_{mp,d}$ for $j \in \mathbb{Z} \setminus \{0, \ldots, q\}$, it follows that

$$\sum_{k=-\infty}^{\infty} (\underline{M}_k - \underline{\Phi}\, \underline{M}_{k-1}) U Z_{t-k}' = \sum_{k=-\infty}^{\infty} \underline{\Theta}_k Z_{t-k}',$$

and multiplying this equation from the right by ${Z'}_{t-j}^T$, taking expectations and observing that $M(z)\Lambda = M(z)$ we conclude that

$$(\underline{M}_j - \underline{\Phi}\, \underline{M}_{j-1}) U = (\underline{M}_j - \underline{\Phi}\, \underline{M}_{j-1}) \Lambda U = \underline{\Theta}_j U^* \Lambda U \quad \forall\, j \in \mathbb{Z}. \tag{2.47}$$

Next observe that since $(\underline{Y}_t')_{t\in\mathbb{Z}}$ is a strictly stationary solution of (2.46), it follows from Theorem 2.1 that $(\underline{Y}_t'')_{t\in\mathbb{Z}}$, defined by $\underline{Y}_t'' = \sum_{j=-\infty}^{\infty} \underline{N}_j Z_{t-j}'$, is also a strictly stationary solution of (2.46). With precisely the same argument as above it follows that

$$(\underline{N}_j - \underline{\Phi}\, \underline{N}_{j-1}) U^* \Lambda U = \underline{\Theta}_j U^* \Lambda U \quad \forall\, j \in \mathbb{Z}. \tag{2.48}$$

Now let $L_j := \underline{M}_j - \underline{N}_j U^* \Lambda$, $j \in \mathbb{Z}$. Then $L_j - \underline{\Phi} L_{j-1} = 0_{mp,d}$ from (2.47) and (2.48), and the entries of L_j decrease exponentially as $|j| \to \infty$ since so do the entries of $\underline{M}_j$ and $\underline{N}_j$. It follows that for $h \in \{1, \ldots, H\}$ and $j \in \mathbb{Z}$ we have

$$\underline{I}_h \underline{S}^{-1} L_j - \underline{\Phi}_h \underline{I}_h \underline{S}^{-1} L_{j-1} = \underline{I}_h \left(\underline{S}^{-1} L_j - \begin{pmatrix} \underline{\Phi}_1 & & \\ & \ddots & \\ & & \underline{\Phi}_H \end{pmatrix} \underline{S}^{-1} L_{j-1} \right) = 0_{\underline{r}_{h+1} - \underline{r}_h, d}. \tag{2.49}$$

Since $\underline{\Phi}_h$ is invertible for $h \in \{1, \ldots, H\}$ such that $\underline{\lambda}_h \neq 0$, this gives $\underline{I}_h \underline{S}^{-1} L_0 = \underline{\Phi}_h^{-j} \underline{I}_h \underline{S}^{-1} L_j$ for all $j \in \mathbb{Z}$ and $\underline{\lambda}_h \neq 0$. Since for $|\underline{\lambda}_h| \geq 1$, $\|\underline{\Phi}_h^{-j}\| \leq \kappa j^{mp}$ for all $j \in \mathbb{N}_0$ for some constant κ, it follows that $\|\underline{I}_h \underline{S}^{-1} L_0\| \leq \kappa j^{mp} \|\underline{I}_h \underline{S}^{-1} L_j\|$, which converges to 0 as $j \to \infty$ by the geometric decrease of the coefficients of L_j as $j \to \infty$, so that $\underline{I}_h \underline{S}^{-1} L_k = 0$ for $|\underline{\lambda}_h| \geq 1$ and $k = 0$ and hence for all $k \in \mathbb{Z}$. Similarly, letting $j \to -\infty$, it follows that $\underline{I}_h \underline{S}^{-1} L_k = 0$ for $|\underline{\lambda}_h| \in (0, 1)$ and $k = 0$ and hence for all $k \in \mathbb{Z}$. Finally, for $h \in \{1, \ldots, H\}$ such that $\underline{\lambda}_h = 0$ observe that $\underline{I}_h \underline{S}^{-1} L_k = \underline{\Phi}_h^{mp} \underline{I}_h \underline{S}^{-1} L_{k-mp}$ for $k \in \mathbb{Z}$ by (2.49), and since $\underline{\Phi}_h^{mp} = 0$, this shows that $\underline{I}_h \underline{S}^{-1} L_k = 0$ for $k \in \mathbb{Z}$. Summing up, we have $\underline{S}^{-1} L_k = 0$ and hence $\underline{M}_k = \underline{N}_k U^* \Lambda$ for $k \in \mathbb{Z}$, which is (2.44). Equation (2.45) then follows from (2.15), since

$$\underline{M}_j U Z_{t-j} = \underline{M}_j \begin{pmatrix} w_{t-j} \\ u \end{pmatrix} = \underline{N}_j U^* \begin{pmatrix} w_{t-j} \\ 0_{d-s} \end{pmatrix} = \underline{N}_j U^* \left(U Z_{t-j} - \begin{pmatrix} 0 \\ u \end{pmatrix} \right).$$

□

Returning to the proof of the necessity of conditions (ii) and (ii') for a strictly stationary solution to exist, observe that $\sum_{j=-\infty}^{\infty} \underline{N}_j Z_{t-j}$ converges almost surely absolutely by (2.42), and since the entries of $\underline{N}_j$ decrease geometrically as $|j| \to \infty$, this together with (2.45) implies that $\sum_{j=-\infty}^{\infty} \underline{M}_j U Z_{t-j}$ converges almost surely absolutely, which shows that (ii') must hold. To see (ii), observe that for $j \geq mp + q$ we have

$$N_{j,h} = \begin{cases} \underline{\Phi}_h^{j-q} \sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\Theta}_k, & |\underline{\lambda}_h| \in (0,1), \\ 0, & |\underline{\lambda}_h| \notin (0,1), \end{cases}$$

while

$$N_{-1,h} = \begin{cases} \underline{\Phi}_h^{-1-q} \sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\Theta}_k, & |\underline{\lambda}_h| > 1, \\ 0, & |\underline{\lambda}_h| \leq 1. \end{cases}$$

Since a strictly stationary solution of (2.39) exists, it follows from Theorem 2.1 that $\mathbb{E} \log^+ \|\underline{N}_j Z_0\| < \infty$ for $j \geq mp + q$ and $\mathbb{E} \log^+ \|\underline{N}_{-1} Z_0\| < \infty$. Together with (2.45) this shows that condition (ii) of Theorem 2.2 is necessary.

2.5.2 The sufficiency of the conditions and uniqueness of the solution

In this subsection we shall show that (i), (ii), (iii) as well as (i), (ii'), (iii) of Theorem 2.2 are sufficient conditions for a strictly stationary solution of (2.1) to exist, and prove the uniqueness assertion.

(a) Assume that conditions (i), (ii) and (iii) hold for some $v \in \mathbb{C}^s$ and $g \in \mathbb{C}^m$. Then $\mathbb{E}\log^+ \|\underline{N}_{-1} Z_0\| < \infty$ and $\mathbb{E}\log^+ \|\underline{N}_{mp+q} Z_0\| < \infty$ by (ii) and (2.45). In particular, since $\underline{S}$ is invertible, $\mathbb{E}\log^+ \|N_{-1,h} Z_0\| < \infty$ for $|\underline{\lambda}_h| > 1$ and $\mathbb{E}\log^+ \|N_{mp+q,h} Z_0\| < \infty$ for $|\underline{\lambda}_h| \in (0,1)$. The invertibility of $\underline{\Phi}_h$ for $\underline{\lambda}_h \neq 0$ then shows that

$$\mathbb{E}\log^+ \left\| \sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \underline{\Theta}_k Z_0 \right\| < \infty \quad \forall\, h \in \{1,\ldots,H\} : |\underline{\lambda}_h| \in (0,1) \cup (1,\infty). \tag{2.50}$$

Now let $(W_t''')_{t\in\mathbb{Z}}$ be an i.i.d. $N(\begin{pmatrix} v \\ u \end{pmatrix}, \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix})$ distributed sequence and define $Z_t''' := U^* W_t'''$. Then $\mathbb{E} Z_t''' = U^*(v^T, u^T)^T$. By conditions (i) and (iii) and Theorem 2.3, $(Y_t''')_{t\in\mathbb{Z}}$, defined by $Y_t''' := P(1)^{-1} Q(1) \mathbb{E} Z_0''' + \sum_{j=-\infty}^{\infty} M_j (W_{t-j}''' - (v^T, u^T)^T)$, is a weakly stationary solution of $Y_t''' - \sum_{k=1}^{p} \Psi_k Y_{t-k}''' = \sum_{k=0}^{q} \Theta_k Z_{t-k}'''$, and obviously, it is also strictly stationary. It now follows in complete analogy to the necessity proof presented in Section 2.5.1 that $A_h = 0$ and $C_h u = (\alpha_{h,1}, \ldots, \alpha_{h,\underline{r}_{h+1}-\underline{r}_h})^T$ for $|\underline{\lambda}_h| = 1$, where (A_h, C_h) is defined as in (2.43) and $\alpha_{h,1} = 0$ if $\lambda_h = 1$. Hence $\sum_{k=0}^{q} \underline{\Phi}_h^{q-k} \underline{I}_h \underline{S}^{-1} \widetilde{\underline{\Theta}}_k W_0 = (\alpha_{h,1}, \ldots, \alpha_{h,\underline{r}_{h+1}-\underline{r}_h})^T$ for $|\underline{\lambda}_h| = 1$. By Theorem 2.1, this together with (2.50) implies the existence of a strictly stationary solution of (2.39), so that a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ of (2.1) exists by Proposition 2.5.

(b) Now assume that conditions (i), (ii') and (iii) hold for some $v \in \mathbb{C}^s$ and $g \in \mathbb{C}^m$ and define $Y = (Y_t)_{t\in\mathbb{Z}}$ by (2.18). Then Y is clearly strictly stationary. Since $U Z_t = (w_t^T, u^T)$, we further have, using (iii), that

$$\begin{aligned} P(B) Y_t &= P(1) g - P(1) M(1) \begin{pmatrix} v \\ u \end{pmatrix} + Q(B) U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix} \begin{pmatrix} w_t \\ u \end{pmatrix} \\ &= Q(1) U^* \begin{pmatrix} v \\ u \end{pmatrix} - Q(1) U^* \begin{pmatrix} v \\ 0_{d-s} \end{pmatrix} + Q(B) U^* \begin{pmatrix} w_t \\ 0_{d-s} \end{pmatrix} \\ &= Q(B) U^* \begin{pmatrix} w_t \\ u \end{pmatrix} = Q(B) Z_t \end{aligned}$$

for $t \in \mathbb{Z}$, so that $(Y_t)_{t\in\mathbb{Z}}$ is a solution of (2.1).

(c) Finally, the uniqueness assertion follows from the fact that by Proposition 2.5, (2.1) has a unique strictly stationary solution if and only if (2.39) has a unique strictly stationary solution. By Theorem 2.1, the latter is equivalent to the fact that $\underline{\Phi}$ does not have an eigenvalue on the unit circle, which in turn is equivalent to $\det P(z) \neq 0$ for z on the unit circle, since $\det P(z) = \det(\mathrm{Id}_{mp} - \underline{\Phi} z)$ (e.g. Gohberg et al. [8], p. 14). This finishes the proof of Theorem 2.2.

2.6 Discussion and consequences of main results

In this section we shall discuss the main results and consider special cases. Some consequences of the results are also listed. We start with some comments on Theorem 2.1. If Ψ_1 has only eigenvalues of absolute value in $(0,1)\cup(1,\infty)$, then a much simpler condition for stationarity of (2.3) can be given:

Corollary 2.7. *Let the assumptions of Theorem 2.1 be satisfied and suppose that* Ψ_1 *has only eigenvalues of absolute value in* $(0,1)\cup(1,\infty)$. *Then a strictly stationary solution of* (2.3) *exists if and only if*

$$\mathbb{E}\log^+\left\|\left(\sum_{k=0}^{q}\Psi_1^{q-k}\Theta_k\right)Z_0\right\|<\infty. \tag{2.51}$$

Proof. It follows from Theorem 2.1 that there exists a strictly stationary solution if and only if (2.10) holds for every $h\in\{1,\dots,H\}$. But this is equivalent to

$$\mathbb{E}\log^+\|(\sum_{k=0}^{q}(S^{-1}\Psi_1 S)^{q-k}\mathrm{Id}_m S^{-1}\Theta_k)Z_0\|<\infty,$$

which in turn is equivalent to (2.51), since S is invertible and hence for a random vector $R\in\mathbb{C}^m$ we have $\mathbb{E}\log^+\|SR\|<\infty$ if and only if $\mathbb{E}\log^+\|R\|<\infty$. □

Remark 2.8. *Suppose that* Ψ_1 *has only eigenvalues of absolute value in* $(0,1)\cup(1,\infty)$. *Then* $\mathbb{E}\log^+\|Z_0\|$ *is a sufficient condition for* (2.3) *to have a strictly stationary solution, since it implies* (2.51). *But it is not necessary. For example, let* $q=1$, $m=d=2$ *and*

$$\Psi_1=\begin{pmatrix}2&0\\0&3\end{pmatrix},\quad \Theta_0=\mathrm{Id}_2,\quad \Theta_1=\begin{pmatrix}-1&-1\\1&-4\end{pmatrix},\quad \textit{so that}\ \sum_{k=0}^{1}\Psi_1^{q-k}\Theta_k=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}.$$

By (2.51), *a strictly stationary solution exists for example if the i.i.d. noise* $(Z_t)_{t\in\mathbb{Z}}$ *satisfies* $Z_0=(R_0,R_0+R_0')^T$, *where* R_0' *is a random variable with finite log-moment and* R_0 *a random variable with infinite log-moment. In particular,* $\mathbb{E}\log^+\|Z_0\|=\infty$ *is possible.*

An example like in the remark above cannot occur if the matrix $\sum_{k=0}^{q}\Psi_1^{q-k}\Theta_k$ is invertible if $m=d$. More generally, we have the following result:

Corollary 2.9. *Let the assumptions of Theorem 2.1 be satisfied and suppose that* Ψ_1 *has only eigenvalues of absolute value in* $(0,1)\cup(1,\infty)$. *Suppose further that* $d\le m$ *and that* $\sum_{k=0}^{q}\Psi_1^{q-k}\Theta_k$ *has full rank* d. *Then a strictly stationary solution of* (2.3) *exists if and only if* $\mathbb{E}\log^+\|Z_0\|<\infty$.

Proof. The sufficiency of the condition has been observed in Remark 2.8, and for the necessity, observe that with $A := \sum_{k=0}^{q} \Psi_1^{q-k}\Theta_k$ and $U := AZ_0$ we must have $\mathbb{E}\log^+\|U\| < \infty$ by (2.51). Since A has rank d, the matrix $A^T A \in \mathbb{C}^{d\times d}$ is invertible and we have $Z_0 = (A^T A)^{-1}A^T U$, i.e. the components of Z_0 are linear combinations of those of U. It follows that $\mathbb{E}\log^+\|Z_0\| < \infty$. □

Next, we shall discuss the conditions of Theorem 2.2 in more detail. The following remark is obvious from Theorem 2.2. It implies in particular the well known fact that $\mathbb{E}\log^+\|Z_0\| < \infty$ together with $\det P(z) \neq 0$ for all z on the unit circle is sufficient for the existence of a strictly stationary solution.

Remark 2.10. *(a)* $\mathbb{E}\log^+\|Z_0\| < \infty$ *is a sufficient condition for (ii) of Theorem 2.2.*
(b) $\det P(1) \neq 0$ *is a sufficient condition for (iii) of Theorem 2.2.*
(c) $\det P(z) \neq 0$ *for all z on the unit circle is a sufficient condition for (i) and (iii) of Theorem 2.2.*

With the notations of Theorem 2.2, denote

$$\widetilde{Q}(z) := Q(z)U^* \begin{pmatrix} \mathrm{Id}_s & 0_{s,d-s} \\ 0_{d-s,s} & 0_{d-s,d-s} \end{pmatrix}, \tag{2.52}$$

so that $M(z) = P^{-1}(z)\widetilde{Q}(z)$. It is natural to ask if conditions (i) and (iii) of Theorem 2.2 can be replaced by a removability condition on the singularities on the unit circle of $(\det P(z))^{-1}\det(\widetilde{Q}(z))$ if $d = m$. The following corollary shows that this condition is indeed necessary, but it is not sufficient as pointed out in Remark 2.12.

Corollary 2.11. *Under the assumptions of Theorem 2.1, with $\widetilde{Q}(z)$ as defined in* (2.52)*, a necessary condition for a strictly stationary solution of the ARMA(p,q) equation* (2.1) *to exist is that the function $z \mapsto |\det P(z)|^{-2}\det(\widetilde{Q}(z)\widetilde{Q}(z)^*)$ has only removable singularities on the unit circle. If additionally $d = m$, then a necessary condition for a strictly stationary solution to exist is that the matrix rational function $z \mapsto (\det P(z))^{-1}\det(\widetilde{Q}(z))$ has only removable singularities on the unit circle.*

Proof. The second assertion is immediate from Theorem 2.2, and the first assertion follows from the fact that if $M(z)$ as defined in Theorem 2.2 has only removable singularities on the unit circle, then so does $M(z)M(z)^*$ and hence $\det(M(z)M(z)^*)$. □

Remark 2.12. *In the case* $d = m$ *and* $\mathbb{E}\log^+\|Z_0\| < \infty$*, the condition that the matrix rational function* $z \mapsto (\det P(z))^{-1}\det \widetilde{Q}(z)$ *has only removable singularities on the unit circle is not sufficient for the existence of a strictly stationary solution of* (2.3)*. For example, let* $p = q = 1$*,* $m = d = 2$ *and* $\Psi_1 = \Theta_0 = \mathrm{Id}_2$*,* $\Theta_1 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$*,* $(Z_t)_{t\in\mathbb{Z}}$ *be i.i.d. standard normally distributed and* $U = \mathrm{Id}_2$*. Then* $\det P(z) = \det \widetilde{Q}(z) = (1-z)^2$*, but it does not hold that* $\Psi_1\Theta_0 + \Theta_1 = 0$*, so that condition (iii) of Theorem 2.1 is violated and no strictly stationary solution can exist.*

Next, we shall discuss condition (i) of Theorem 2.2 in more detail. Recall (e.g. Kailath [12]) that a $\mathbb{C}^{m\times m}$ matrix polynomial $R(z)$ is a *left-divisor* of $P(z)$, if there is a matrix polynomial $P_1(z)$ such that $P(z) = R(z)P_1(z)$. The matrix polynomials $P(z)$ and $\widetilde{Q}(z)$ are *left-coprime*, if every common left-divisor $R(z)$ of $P(z)$ and $\widetilde{Q}(z)$ is *unimodular*, i.e. the determinant of $R(z)$ is constant in z. In that case, the matrix rational function $P^{-1}(z)\widetilde{Q}(z)$ is also called *irreducible*. With $\widetilde{Q}$ as defined in (2.52), it is then easy to see that condition (i) of Theorem 2.2 is equivalent to

(i') There exist $\mathbb{C}^{m\times m}$*-valued matrix polynomials* $P_1(z)$ *and* $R(z)$ *and a* $\mathbb{C}^{m\times d}$*-valued matrix polynomial* $Q_1(z)$ *such that* $P(z) = R(z)P_1(z)$*,* $\widetilde{Q}(z) = R(z)Q_1(z)$ *for all* $z \in \mathbb{C}$ *and* $\det P_1(z) \neq 0$ *for all* z *on the unit circle.*

That (i') implies (i) is obvious, and that (i) implies (i') follows by taking $R(z)$ as the greatest common left-divisor (cf. [12], p. 377) of $P(z)$ and $\widetilde{Q}(z)$. The thus remaining right-factors $P_1(z)$ and $Q_1(z)$ are then left-coprime, and since the matrix rational function $M(z) = P^{-1}(z)\widetilde{Q}(z) = P_1^{-1}(z)Q_1(z)$ has no poles on the unit circle, it follows from page 447 in Kailath [12] that $\det P_1(z) \neq 0$ for all z on the unit circle, which establishes (i'). As an immediated consequence, we have:

Remark 2.13. *With the notation of the Theorem 2.2 and* (2.52)*, assume additionally that* $P(z)$ *and* $\widetilde{Q}(z)$ *are left-coprime. Then condition (i) of Theorem 2.2 is equivalent to* $\det P(z) \neq 0$ *for all* z *on the unit circle.*

Next we show how a slight extension of Theorem 4.1 of Bougerol and Picard [2], which characterized the existence of a strictly stationary non-anticipative solution of the ARMA(p, q) equation (2.1), can be deduced from Theorem 2.2. By a *non-anticipative* strictly stationary solution we mean a strictly stationary solution $Y = (Y_t)_{t\in\mathbb{Z}}$ such that for every $t \in \mathbb{Z}$, Y_t is independent of the sigma algebra generated by $(Z_s)_{s>t}$, and by a *causal* strictly stationary solution we mean a strictly stationary solution $Y = (Y_t)_{t\in\mathbb{Z}}$ such that for every $t \in \mathbb{Z}$, Y_t is measurable with respect to the sigma algebra generated by $(Z_s)_{s\le t}$. Clearly, since $(Z_t)_{t\in\mathbb{Z}}$ is assumed to be i.i.d., every causal solution is also non-anticipative. The equivalence of (i) and (iii) in the theorem below was already obtained by Bougerol and Picard [2] under the additional assumption that $\mathbb{E}\log^+\|Z_0\| < \infty$.

Theorem 2.14. *In addition to the assumptions and notations of Theorem 2.2, assume that the matrix polynomials $P(z)$ and $\widetilde{Q}(z)$ are left-coprime, with $\widetilde{Q}(z)$ as defined in* (2.52)*. Then the following are equivalent:*

(i) *There exists a non-anticipative strictly stationary solution of* (2.1)*.*
(ii) *There exists a causal strictly stationary solution of* (2.1)*.*
(iii) $\det P(z) \neq 0$ *for all* $z \in \mathbb{C}$ *such that* $|z| \le 1$ *and if* $M(z) = \sum_{j=0}^{\infty} M_j z^j$ *denotes the Taylor expansion of* $M(z) = P^{-1}(z)\widetilde{Q}(z)$*, then*

$$\mathbb{E}\log^+\|M_j U Z_0\| < \infty \quad \forall\, j \in \{mp+q-p+1,\ldots,mp+q\}. \tag{2.53}$$

Proof. The implication "(iii) ⇒ (ii)" is immediate from Theorem 2.2 and equation (2.18), and "(ii) ⇒ (i)" is obvious since $(Z_t)_{t\in\mathbb{Z}}$ is i.i.d. Let us show that "(i) ⇒ (iii)": since a strictly stationary solution exists, the function $M(z)$ has only removable singularities on the unit circle by Theorem 2.2. Since $P(z)$ and $\widetilde{Q}(z)$ are left-coprime, this implies by Remark 2.13 that $\det P(z) \neq 0$ for all $z \in \mathbb{C}$ such that $|z| = 1$. In particular, by Theorem 2.2, the strictly stationary solution is unique and given by (2.18). By assumption, this solution must then be non-anticipative, so that we conclude that the distribution of $M_j U Z_{t-j}$ must be degenerate to a constant for all $j \in \{-1,-2,\ldots\}$. But since $UZ_0 = (w_0^T, u^T)^T$ and $M_j = (M_j', 0_{m,d-s})$ with certain matrices $M_j' \in \mathbb{C}^{m,s}$, it follows for $j \le -1$ that $M_j U Z_0 = M_j' w_0$, so that $M_j' = 0$ since no non-trivial linear combination of the components of w_0 is constant a.s. It follows that $M_j = 0$ for $j \le -1$, i.e. $M(z)$ has only removable singularities for $|z| \le 1$. Since $P(z)$ and $\widetilde{Q}(z)$ are assumed to be left-coprime, it follows from page 447 in Kailath [12] that $\det P(z) \neq 0$ for all $|z| \le 1$. Equation (2.53) is an immediate consequence of Theorem 2.2. □

It may be possible to extend Theorem 2.14 to situations without assuming that $P(z)$ and $\widetilde{Q}(z)$ are left-coprime, but we did not investigate this question.
The last result is on the interplay of the existence of strictly and of weakly stationary solutions of (2.1) when the noise is i.i.d. with finite second moments:

Theorem 2.15. *Let* $m, d, p \in \mathbb{N}$, $q \in \mathbb{N}_0$, *and let* $(Z_t)_{t\in\mathbb{Z}}$ *be an i.i.d. sequence of* $\mathbb{C}^d$*-valued random vectors with finite second moment. Let* $\Psi_1, \ldots, \Psi_p \in \mathbb{C}^{m\times m}$ *and* $\Theta_0, \ldots, \Theta_q \in \mathbb{C}^{m\times d}$. *Then the ARMA*$(p, q)$ *equation* (2.1) *admits a strictly stationary solution if and only if it admits a weakly stationary solution, and in that case, the solution given by* (2.20) *is both a strictly stationary and weakly stationary solution of* (2.1).

Proof. It follows from Theorem 2.3 that if a weakly stationary solution exists, then one choice of such a solution is given by (2.20), which is clearly also strictly stationary. On the other hand, if a strictly stationary solution exists, then by Theorem 2.2, one such solution is given by (2.18), which is clearly weakly stationary. □

Finally, we remark that most of the results presented in this chapter can be applied also to the case when $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence of $\mathbb{C}^{d\times d'}$ random matrices and $(Y_t)_{t\in\mathbb{Z}}$ is $\mathbb{C}^{m\times d'}$-valued. This can be seen by stacking the columns of Z_t into a $\mathbb{C}^{dd'}$-variate random vector Z_t', those of Y_t into a $\mathbb{C}^{md'}$-variate random vector Y_t', and considering the matrices

$$\Psi_k' := \begin{pmatrix} \Psi_k & & \\ & \ddots & \\ & & \Psi_k \end{pmatrix} \in \mathbb{C}^{md'\times md'} \quad \text{and} \quad \Theta_k' := \begin{pmatrix} \Theta_k & & \\ & \ddots & \\ & & \Theta_k \end{pmatrix} \in \mathbb{C}^{md'\times dd'}.$$

The question of existence of a strictly stationary solution of (2.1) with matrix-valued Z_t and Y_t is then equivalent to the existence of a strictly stationary solution of $Y_t' - \sum_{k=1}^{p} \Psi_k' Y_{t-k}' = \sum_{k=0}^{q} \Theta_k' Z_{t-k}'$.

3 Strictly stationary solutions of ARMA equations with fractional noise and ARIMA processes

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Abstract. We obtain necessary and sufficient conditions for the existence of strictly stationary solutions of ARIMA equations with independent and identically distributed noise. No a priori assumptions are made on the driving noise sequence. We interpret ARIMA equations as ARMA equations with fractional noise, and characterize for which i.i.d. noise sequences the series defining fractional noise converges almost surely.

3.1 Introduction

Let $(Z_t)_{t\in\mathbb{Z}}$ be a real-valued noise sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define polynomials

$$\Phi(z) := 1 - \sum_{k=1}^{p} \varphi_k z^k, \quad \text{and} \quad \Theta(z) := 1 + \sum_{k=1}^{q} \theta_k z^k, \quad z \in \mathbb{C}, \tag{3.1}$$

with $p, q \in \mathbb{N}_0$, $\varphi_1, ..., \varphi_p, \theta_1, ..., \theta_q \in \mathbb{C}$, $\varphi_p \neq 0$ and $\theta_q \neq 0$, where $\varphi_0 := \theta_0 := 1$. For any $D \in \mathbb{R} \setminus \{1, 2, ...\}$ and B the backwards shift operator, define the difference operator $\nabla^D = (1-B)^D$ by means of the binomial expansion,

$$\nabla^D = (1-B)^D = \sum_{j=0}^{\infty} (-1)^j \binom{D}{j} B^j.$$

For $(Z_t)_{t\in\mathbb{Z}}$ weak white noise, i.e. uncorrelated and with zero mean and finite second moment, Granger and Joyeux [10] and Hosking [11] introduced weak ARIMA(p,D,q) processes as weakly stationary solutions of the equation

$$\Phi(B)\nabla^D Y_t = \Theta(B)Z_t, \quad t \in \mathbb{Z}. \tag{3.2}$$

It is shown in [11] that a sufficient condition for a weak ARIMA process to exist is $D < \frac{1}{2}$ and $\Phi(z)$ having no zeros on the unit circle. Furthermore, they found out that a sufficient condition for a solution of (3.2) to be invertible is $D > -\frac{1}{2}$ and $\Theta(z)$ having no zeros on the unit circle.
A couple of years later, Kokoszka and Taqqu [14] and Kokoszka [15] developed the theory of infinite variance stable fractional ARIMA(p,D,q) time series defined by the equation

$$\Phi(B)Y_t = [\Theta(B)\nabla^{-D}]Z_t, \quad t \in \mathbb{Z}, \tag{3.3}$$

where the noise sequence $(Z_t)_{t\in\mathbb{Z}}$ is i.i.d. symmetric α-stable (in [14]) or belongs to the domain of attraction of an α-stable law (in [15]), respectively, with $0 < \alpha < 2$ and fractional D such that the right hand side of (3.3) converges. Among other results, they obtained a unique strictly stationary solution of (3.3) with this specific noise in terms of the Laurent series of $\Theta(z)(1-z)^{-D}/\Phi(z)$, provided $\Phi(z) \neq 0$ for all $|z| \leq 1$, and $\Phi(z)$ and $\Theta(z)$ having no roots in common.
In this chapter, we study a slightly different approach by interpreting Equation (3.3) as an ARMA(p,q) equation with fractional noise $\nabla^{-D}Z_t$, i.e.

$$\Phi(B)Y_t = \Theta(B)[\nabla^{-D}Z_t], \quad t \in \mathbb{Z}, \tag{3.4}$$

where $D \in \mathbb{R}\backslash\{-1, -2, ...\}$, and $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence of real random variables, not necessarily with finite variance. Here, the fractional noise has a representation $\nabla^{-D}Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ with coefficients

$$\psi_j = (-1)^j \binom{-D}{j} = \prod_{0<k\leq j} \frac{k-1+D}{k} = \frac{\Gamma(j+D)}{\Gamma(j+1)\Gamma(D)}, \quad j = 0, 1, 2, ... \tag{3.5}$$

Note that an application of Stirling's formula, according to which $\Gamma(x) \sim \sqrt{2\pi}e^{-x+1}$ $(x-1)^{x-1/2}$ as $x \to \infty$, yields

$$\psi_j \sim \frac{j^{D-1}}{\Gamma(D)} \quad \text{as } j \to \infty, \tag{3.6}$$

and this implies that there are constants $0 < C_1 \le C_2$ such that

$$C_1 j^{D-1} \le |\psi_j| \le C_2 j^{D-1}, \quad j \in \mathbb{N}. \tag{3.7}$$

Now, we call a complex-valued process $Y := (Y_t)_{t\in\mathbb{Z}}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a strict ARIMA(p,D,q) process (or more precisely a strict ARMA(p,q) process with fractional noise) if the series $\nabla^{-D} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ converges almost surely and Y is a strictly stationary solution of (3.4). Our aim is to give necessary and sufficient conditions for such a process to exist when the noise sequence $(Z_t)_{t\in\mathbb{Z}}$ is i.i.d. Such conditions are given in Theorem 3.5, where also an explicit solution to (3.4), given its existence, is derived and the question of uniqueness of this solution is solved. In contrast to the results in [14] and [15], we make no a priori assumptions on the roots of Φ and Θ, and allow for more general noise distributions.
As the definition of a strict ARIMA(p,D,q) process requires the series $\nabla^{-D} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ to converge almost surely, questions of convergence of this series need to be addressed before being able to give equivalent conditions for the existence of a strict ARIMA(p,D,q) process. These questions are solved in Section 3.2, where Theorem 3.1 gives a necessary and sufficient condition for $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ to converge almost surely in terms of moment conditions on Z_0.
Throughout this chapter we restrict ourselves to the cases $D \in (-\infty, 0)\backslash\{-1, -2, ...\}$ and $0 < D < \frac{1}{2}$. This is because for $D \ge \frac{1}{2}$, the series $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ can only converge for $Z_t \equiv 0$, $t \in \mathbb{Z}$, because in this case the series $\sum_{j=0}^{\infty} \psi_j^2$ does not converge due to the asymptotic behaviour (3.6) of the coefficients ψ_j. But the convergence of this series is necessary for $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ to converge (see Chow and Teicher [6], Theorem 5.1.4), i.e. fractional noise $\nabla^{-D} Z_t$ cannot exist for $D \ge \frac{1}{2}$ unless $Z_t \equiv 0$. In the case $D \in \{0, -1, -2, ...\}$, Equation (3.4) reduces to an ARMA equation with i.i.d. noise sequence and the question of existence and uniqueness of strictly stationary solutions to this equation is addressed in Brockwell and Lindner [4] (cf. Theorem 1.4 in this thesis).

3.2 Fractional noise

In this section we characterize for which i.i.d. noise sequences $(Z_t)_{t\in\mathbb{Z}}$ the series $\nabla^{-D} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ defining fractional noise converges almost surely. Observe that almost sure convergence of this sum is equivalent to its convergence in distribution, being a sum of independent random variables (see Kallenberg [13], Theorem 4.18).

Theorem 3.1. *Let* $(Z_t)_{t\in\mathbb{Z}}$ *be an i.i.d. sequence of real random variables and* ψ_j *defined as in* (3.5)*. Then for* $D \in (-\infty, 0) \setminus \{-1, -2, ...\}$, $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ *converges almost surely if and only if* $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$*. For* $D \in (0, \frac{1}{2})$, $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ *converges almost surely if and only if* $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$ *and* $\mathbb{E}Z_0 = 0$.

Proof. (a) Let $D \in (-\infty, 0) \setminus \{-1, -2, ...\}$. To show the sufficiency of the condition let $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$. We apply Kolmogorov's three series criterion (see Kallenberg [13], Theorem 4.18). According to this criterion, $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ converges almost surely if and only if the following conditions hold: $\sum_{j=1}^{\infty} \mathbb{P}(|\psi_j Z_{t-j}| > 1) < \infty$, $\sum_{j=1}^{\infty} \mathbb{E}(\psi_j Z_{t-j} \mathbb{1}_{\{|\psi_j Z_{t-j}|\le 1\}})$ converges, and $\sum_{j=1}^{\infty} \mathbb{V}(\psi_j Z_{t-j} \mathbb{1}_{\{|\psi_j Z_{t-j}|\le 1\}}) < \infty$.
To show the convergence of the first series, observe that

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{P}(|\psi_j Z_{t-j}| > 1) &= \sum_{j=1}^{\infty} \mathbb{P}(|Z_0| > |\psi_j^{-1}|) \\ &\le \sum_{j=1}^{\infty} \mathbb{P}(|Z_0| > C_2^{-1} j^{1-D}) \\ &= \sum_{j=1}^{\infty} \mathbb{P}(|Z_0|^{1/(1-D)} > C_2^{1/(D-1)} j) < \infty,\end{aligned}$$

the second inequality following from (3.7) and the last inequality from the condition $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$.
To show the convergence of the second series, observe that

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{E}\left|\psi_j Z_{t-j} \mathbb{1}_{\{|\psi_j Z_{t-j}|\le 1\}}\right| &\le \mathbb{E}\left(|Z_0| \sum_{j=1}^{\infty} \psi_j \mathbb{1}_{\{|\psi_j| \le |Z_0|^{-1}\}}\right) \\ &\le \mathbb{E}\left(|Z_0| \sum_{j=1}^{\infty} C_2 j^{D-1} \mathbb{1}_{\{C_1 j^{D-1} \le |Z_0|^{-1}\}}\right) \\ &= C_2\, \mathbb{E}\left(|Z_0| \sum_{j=\lceil (C_1|Z_0|)^{1/(1-D)} \rceil}^{\infty} j^{D-1}\right). \end{aligned} \tag{3.8}$$

From the integral criterion for convergence we know that there is a constant $C_3 \in (0, \infty)$ such that for $C_1|v| > 2$

$$\sum_{j=\lceil (C_1|v|)^{1/(1-D)} \rceil}^{\infty} j^{D-1} \le \int_{\lceil (C_1|v|)^{1/(1-D)} \rceil - 1}^{\infty} y^{D-1} dy \le C_3 (C_1|v|)^{D/(1-D)}, \tag{3.9}$$

and (3.9) obviously also holds true for $C_1|v| \leq 2$ by choosing C_3 large enough. Thus it follows that

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{E}\left|\psi_j Z_{t-j} \mathbb{1}_{\{|\psi_j Z_{t-j}| \leq 1\}}\right| &\leq C_2\, C_3\, \mathbb{E}\left(|Z_0|(C_1|Z_0|)^{D/(1-D)}\right) \\ &= C_1^{D/(1-D)} C_2 C_3\, \mathbb{E}|Z_0|^{1/(1-D)} < \infty.\end{aligned}$$

To show the convergence of the third series, observe that it follows analogue to (3.8) that

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{V}\left(\psi_j Z_{t-j} \mathbb{1}_{\{|\psi_j Z_{t-j}| \leq 1\}}\right) &\leq \mathbb{E}\left(Z_0^2 \sum_{j=1}^{\infty} \psi_j^2 \mathbb{1}_{\{|\psi_j| \leq |Z_0|^{-1}\}}\right) \\ &\leq C_2^2 \cdot \mathbb{E}\left(Z_0^2 \sum_{j=\lceil (C_1|Z_0|)^{1/(1-D)}\rceil}^{\infty} j^{2D-2}\right), \qquad (3.10)\end{aligned}$$

and then analogue to Equation (3.9) it follows with the integral criterion for convergence that there is a constant $C_3 \in (0, \infty)$ such that

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{V}\left(\psi_j Z_{t-j} \mathbb{1}_{\{|\psi_j Z_{t-j}| \leq 1\}}\right) &\leq C_2^2 \cdot \mathbb{E}\left(Z_0^2 C_3 (C_1|Z_0|)^{(2D-1)/(1-D)}\right) \\ &= C_1^{(2D-1)/(1-D)} C_2^2 C_3 \cdot \mathbb{E}\left(|Z_0|^{1/(1-D)}\right) < \infty. \qquad (3.11)\end{aligned}$$

Altogether now, the three series criterion of Kolmogorov yields the claimed almost sure convergence of $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$.

For the necessity of the condition suppose that $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ converges a.s. Applying Equation (3.7), it follows that

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{P}(|Z_0|^{1/(1-D)} > C_1^{1/(D-1)} j) &\leq \sum_{j=1}^{\infty} \mathbb{P}(|Z_0| > |\psi_j^{-1}|) \\ &= \sum_{j=1}^{\infty} \mathbb{P}(|\psi_j Z_{t-j}| > 1) < \infty, \qquad (3.12)\end{aligned}$$

so that $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$ as claimed. Here we applied the Borel-Cantelli lemma in the last inequality.

(b) Let $D \in (0, \frac{1}{2})$. For the sufficiency let $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$ and $\mathbb{E}Z_0 = 0$. Denote $S_n := \sum_{j=1}^{n} \psi_j Z_{t-j}$. Since $\mathbb{E}Z_0 = 0$ and since $(Z_t)_{t\in\mathbb{Z}}$ is i.i.d., $(S_n)_{n\in\mathbb{N}}$ is a martingale with respect to its natural filtration.

By Theorem 1 of Manstavičius [17], there exists a constant c such that

$$\mathbb{E}\,|S_n| \leq c \left(\sum_{j=1}^{n} \int_{\{|u|<1\}} u^2 \, d\mathbb{P}_{\psi_j \tilde{Z}_0}(u)\right)^{1/2} + c \sum_{j=1}^{n} \int_{\{|u|\geq 1\}} |u| \, d\mathbb{P}_{\psi_j \tilde{Z}_0}(u),$$

where $(\tilde{Z}_t)_{t\in\mathbb{Z}}$ is the corresponding symmetrized version of $(Z_t)_{t\in\mathbb{Z}}$.
In the following technical Lemma 3.2 below we show that these both summands are uniformly bounded in n, so that we get $\sup_{n\in\mathbb{N}} \mathbb{E}|S_n| < \infty$. Hence $(S_n)_{n\in\mathbb{N}}$ is an L^1-bounded martingale and so it converges a.s. (see e.g. Kallenberg [13], Theorem 7.18).

For the necessity of the condition suppose that $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ converges almost surely. Similarly to above in part (a), it follows that $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$. Denote $W_t := Z_t - \mathbb{E}Z_0$. Then $\mathbb{E}|W_t|^{1/(1-D)} < \infty$ and $\mathbb{E}W_t = 0$. From the sufficiency part it follows that $\sum_{j=0}^{\infty} \psi_j W_{t-j}$ converges almost surely and hence $\sum_{j=0}^{\infty} \psi_j \mathbb{E}Z_0 = \sum_{j=0}^{\infty} \psi_j (Z_{t-j} - W_{t-j})$ converges a.s. But this implies $\mathbb{E}Z_0 = 0$, since $\sum_{j=0}^{\infty} \psi_j$ does not converge due to (3.6) and $D \in (0, \frac{1}{2})$. □

Lemma 3.2. *With the notations and assumptions of the previous Theorem 3.1, let* $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$*, and assume* Z_0 *to be symmetric. Then*

$$\sum_{j=1}^{\infty} \int_{\{|u|<1\}} u^2 d\mathbb{P}_{\psi_j Z_0}(u) < \infty, \tag{3.13}$$

and for $0 < p < \frac{1}{1-D}$

$$\sum_{j=1}^{\infty} \int_{\{|u|\geq 1\}} |u|^p d\mathbb{P}_{\psi_j Z_0}(u) < \infty. \tag{3.14}$$

Proof. To establish the finiteness of (3.13), observe that analogue to Equations (3.10) and (3.11) in the proof of Theorem 3.1, we can find a constant $C_3 \in (0,\infty)$ such that

$$\begin{aligned}\sum_{j=1}^{\infty} \int_{\{|u|<1\}} u^2 d\mathbb{P}_{\psi_j Z_0}(u) &\leq \sum_{j=1}^{\infty} \int_{\{|v|<|\psi_j|^{-1}\}} v^2 \psi_j^2 d\mathbb{P}_{Z_0}(v) \\ &\leq C_1^{(2D-1)/(1-D)} C_2^2 C_3 \int_{-\infty}^{\infty} |v|^{1/(1-D)} d\mathbb{P}_{Z_0}(v) < \infty,\end{aligned}$$

the last inequality following from $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$.
Now, let $0 < p < \frac{1}{1-D}$. Then

$$\begin{aligned}\sum_{j=1}^{\infty} \int_{\{|u|\geq 1\}} |u|^p d\mathbb{P}_{\psi_j Z_0}(u) &= \sum_{j=1}^{\infty} \int_{\{|v|\geq |\psi_j^{-1}|\}} |v|^p |\psi_j|^p d\mathbb{P}_{Z_0}(v) \\ &\leq C_2^p \sum_{j=1}^{\infty} \int_{\{|v|\geq C_2^{-1} j^{1-D}\}} |v|^p (j^{D-1})^p d\mathbb{P}_{Z_0}(v) \\ &\leq C_2^p \int_{-\infty}^{\infty} |v|^p \sum_{\substack{j\in\mathbb{N}: \\ j\leq (C_2|v|)^{1/(1-D)}}} j^{p(D-1)} d\mathbb{P}_{Z_0}(v).\end{aligned}$$

Again it follows with the integral criterion for convergence analogue to (3.9) that there is a constant $C_4 \in (0,\infty)$ such that

$$\sum_{\substack{j\in\mathbb{N}:\\ j\le (C_2|v|)^{1/(1-D)}}} j^{p(D-1)} \le C_4 \left((C_2|v|)^{1/(1-D)} \right)^{p(D-1)+1} \quad \text{for every } v \in \mathbb{R}.$$

So it follows that

$$\sum_{j=1}^{\infty} \int_{\{|u|\ge 1\}} |u|^p d\mathbb{P}_{\psi_j Z_0}(u) \le C_2^{1/(1-D)} C_4 \int_{-\infty}^{\infty} |v|^{1/(1-D)} d\mathbb{P}_{Z_0}(v) < \infty,$$

the last inequality following from $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$. This shows (3.14) and hence the claim. □

Remark 3.3. *Note that the proof of Theorem 3.1 does not rely on the special representation of the ψ_j as the coefficients of the binomial series, but rather only on their asymptotic behaviour. So the assertion of the theorem is true for any real sequence $(\psi_j)_{j\in\mathbb{Z}}$ with $\psi_j \sim Cj^{D-1}$ as $j\to\infty$, $C \ne 0$.*

Since $\mathbb{E}|Z_0|^{1/(1-D)}$ is necessary for $\nabla^{-D} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ to converge almost surely, it is natural to ask if $\nabla^{-D} Z_t$ does also have finite $1/(1-D)$-moment. The next proposition clearifies this question.

Proposition 3.4. *Let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of random variables and $V_t := \nabla^{-D} Z_t = (1-B)^{-D} Z_t$ for $D \in (-\infty, \frac{1}{2}) \setminus \{0,-1,-2,...\}$, $t \in \mathbb{Z}$, such that the series defining V_t converges almost surely. Then $\mathbb{E}|V_t|^{1/(1-D)-\varepsilon} < \infty$ for all $\varepsilon \in (0, 1/(1-D))$. Moreover, $\mathbb{E}|V_t|^{1/(1-D)} < \infty$ if and only if $\mathbb{E}(|Z_0|^{1/(1-D)} \log^+ |Z_0|) < \infty$.*

Proof. Recall that $V_t = \nabla^{-D} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, where

$$\psi_j = \frac{\Gamma(j+D)}{\Gamma(j+1)\Gamma(D)} \sim \frac{j^{D-1}}{\Gamma(D)} \quad \text{as } j \to \infty.$$

By Theorem 3.1 we know that $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$ for $D \in (-\infty, \frac{1}{2}) \setminus \{0,-1,-2,...\}$. Let $\varepsilon \in (0, 1/(1-D))$. Applying Fatou's lemma, it follows that

$$\mathbb{E}|V_t - Z_t|^{1/(1-D)-\varepsilon} \le \liminf_{m\to\infty} \mathbb{E} \left| \sum_{j=1}^{m} \psi_j Z_{t-j} \right|^{1/(1-D)-\varepsilon}$$

Via symmetrizing, we may assume without loss of generality that Z_0 is symmetric. Then by Theorems 1 and 2 of Manstavičius [17], there exists a constant $c_\varepsilon = c_\varepsilon(D)$

such that

$$\mathbb{E}\left|\sum_{j=1}^{m}\psi_j Z_{t-j}\right|^{1/(1-D)-\varepsilon} \leq c_\varepsilon\left(\sum_{j=1}^{m}\int_{\{|u|<1\}} u^2 d\mathbb{P}_{\psi_j Z_0}(u)\right)^{(1/(1-D)-\varepsilon)/2} + c_\varepsilon\sum_{j=1}^{m}\int_{\{|u|\geq 1\}} |u|^{1/(1-D)-\varepsilon} d\mathbb{P}_{\psi_j Z_0}(u).$$

Hence, for showing that $\mathbb{E}|V_t|^{1/(1-D)-\varepsilon} < \infty$, it is sufficient to observe that

$$\sum_{j=1}^{\infty}\int_{\{|u|<1\}} u^2 d\mathbb{P}_{\psi_j Z_0}(u) < \infty$$

and

$$\sum_{j=1}^{\infty}\int_{\{|u|\geq 1\}} |u|^{1/(1-D)-\varepsilon} d\mathbb{P}_{\psi_j Z_0}(u) < \infty,$$

which follows immediately with Lemma 3.2.
For the sufficiency of the asserted equivalence let $\mathbb{E}(|Z_0|^{1/(1-D)}\log^+|Z_0|) < \infty$. The assertion then follows analogue to above when observing that there is a constant $C_3 \in (0,\infty)$ such that

$$\begin{aligned} &\sum_{j=1}^{\infty}\int_{\{|u|\geq 1\}} |u|^{1/(1-D)} d\mathbb{P}_{\psi_j Z_0}(u) \\ = &\sum_{j=1}^{\infty}\int_{\{|v|\geq|\psi_j^{-1}|\}} |v|^{1/(1-D)}|\psi_j|^{1/(1-D)} d\mathbb{P}_{Z_0}(v) \\ \leq &C_2^{1/(1-D)}\sum_{j=1}^{\infty}\int_{\{|v|\geq C_2^{-1}j^{1-D}\}} |v|^{1/(1-D)}(j^{D-1})^{1/(1-D)} d\mathbb{P}_{Z_0}(v) \\ = &C_2^{1/(1-D)}\int_{-\infty}^{\infty} |v|^{1/(1-D)}\sum_{\substack{j\in\mathbb{N}:\\ j\leq(C_2|v|)^{1/(1-D)}}} j^{-1} d\mathbb{P}_{Z_0}(v) \\ \leq &C_2^{1/(1-D)}C_3\int_{-\infty}^{\infty} |v|^{1/(1-D)}\log^+(C_2|v|+1) d\mathbb{P}_{Z_0}(v) < \infty. \end{aligned}$$

For the necessity let $\mathbb{E}|V_t|^{1/(1-D)} < \infty$. We show that $\mathbb{E}(|Z_0|^{1/(1-D)}\log^+|Z_0|) < \infty$ and do this indirectly by assuming that this moment is not finite. Observe that it follows from Theorems 1 and 2 in [17] that there is a universal constant $c(D)$ such that

$$\mathbb{E}|X|^{1/(1-D)} \leq c(D)\cdot\mathbb{E}|X+Y|^{1/(1-D)},$$

whenever X, Y are symmetric and independent and $\mathbb{E}|X+Y|^{1/(1-D)} < \infty$. Applying this to $\nabla^{-D} Z_t = Z_0 + \sum_{j=1}^m \psi_j Z_{-j} + \sum_{j=m+1}^\infty \psi_j Z_{-j}$, we get that

$$\sup_{m\in\mathbb{N}} \mathbb{E}\left|\sum_{j=1}^m \psi_j Z_{t-j}\right|^{1/(1-D)} < \infty. \tag{3.15}$$

Now, for all $m \in \mathbb{N}$, $t \geq 0$, let

$$A_{t,m} := \left(\sum_{j=1}^m \int_{\{|u|<t\}} u^2 d\mathbb{P}_{\psi_j Z_0}(u)\right)^{1/(2-2D)}, \quad B_{t,m} := \sum_{j=1}^m \int_{\{|u|\geq t\}} |u|^{1/(1-D)} d\mathbb{P}_{\psi_j Z_0}(u).$$

Then it follows with (3.15) and again Theorems 1 and 2 in [17] that

$$\sup_{m\in\mathbb{N}} \inf_{t\geq 0}(A_{t,m} + B_{t,m}) < \infty.$$

Thus there is a constant $C < \infty$ and a sequence $(t_m)_{m\in\mathbb{N}}$ with $t_m \geq 0$ such that

$$A_{t_m,m} + B_{t_m,m} \leq C \quad \text{for all } m \in \mathbb{N}. \tag{3.16}$$

Then we shall see that

$$t_m \to \infty \quad \text{as } m \to \infty. \tag{3.17}$$

Otherwise we could find a subsequence $(t_{m_k})_{k\in\mathbb{N}}$ and $T > 0$ such that $t_{m_k} \leq T$ for all $k \in \mathbb{N}$, and thus

$$\begin{aligned} B_{t_{m_k},m_k} &= \sum_{j=1}^{m_k} \int_{\{|u|\geq t_{m_k}\}} |u|^{1/(1-D)} d\mathbb{P}_{\psi_j Z_0}(u) \\ &\geq \sum_{j=1}^{m_k} \int_{\{|u|\geq T\}} |u|^{1/(1-D)} d\mathbb{P}_{\psi_j Z_0}(u) \quad \text{for all } k \in \mathbb{N}. \end{aligned}$$

So it follows that there are constants $C_4, C_5 \in (0, \infty)$ such that

$$\begin{aligned} \sup_{k\in\mathbb{N}} B_{t_{m_k},m_k} &\geq \sum_{j=1}^{\infty} \int_{\{|u|\geq T\}} |u|^{1/(1-D)} d\mathbb{P}_{\psi_j Z_0}(u) \\ &= \sum_{j=1}^{\infty} \int_{\{|v|\geq |\psi_j|^{-1}T\}} |v|^{1/(1-D)} |\psi_j|^{1/(1-D)} d\mathbb{P}_{Z_0}(v) \\ &\geq C_1^{1/(1-D)} \sum_{j=1}^{\infty} \int_{\{|v|\geq C_1^{-1} j^{1-D} T\}} |v|^{1/(1-D)} j^{-1} d\mathbb{P}_{Z_0}(v) \\ &= C_1^{1/(1-D)} \int_{-\infty}^{\infty} |v|^{1/(1-D)} \sum_{\substack{j\in\mathbb{N}: \\ j\leq (C_1|v|T^{-1})^{1/(1-D)}}} j^{-1} d\mathbb{P}_{Z_0}(v) \\ &\geq C_1^{1/(1-D)} C_4 \int_{-\infty}^{\infty} |v|^{1/(1-D)} \log^+(C_1|v|T^{-1}) d\mathbb{P}_{Z_0}(v) \\ &\geq C_1^{1/(1-D)} C_5 \int_{-\infty}^{\infty} |v|^{1/(1-D)} \log^+ |v| d\mathbb{P}_{Z_0}(v). \end{aligned}$$

So, by the assumption that $\mathbb{E}(|Z_0|^{1/(1-D)}\log^+|Z_0|)$ is not finite, it then follows that $\sup_{k\in\mathbb{N}} B_{t_{m_k},m_k}$ is not finite, which is a contradiction to (3.16) because $A_{t_m,m}\geq 0$ for all $m\in\mathbb{N}$. Thus indeed (3.17) holds.
Finally, this gives

$$\begin{aligned}\sup_{m\in\mathbb{N}} A_{t_m,m} &= \sup_{m\in\mathbb{N}}\left(\sum_{j=1}^{m}\int_{\{|u|<t_m\}} u^2 d\mathbb{P}_{\psi_j Z_0}(u)\right)^{1/(2-2D)}\\ &\geq \sup_{m\in\mathbb{N}}\left(\int_{\{|u|<t_m\}} u^2 d\mathbb{P}_{\psi_1 Z_0}(u)\right)^{1/(2-2D)}\\ &= \left(\int_{-\infty}^{\infty} u^2 d\mathbb{P}_{\psi_1 Z_0}(u)\right)^{1/(2-2D)},\end{aligned}$$

but since $1/(1-D)<2$ and $\mathbb{E}(|Z_0|^{1/(1-D)}\log^+|Z_0|)$ is not finite, this is not finite, which is a contradiction to (3.16). Thus it follows that $\mathbb{E}(|Z_0|^{1/(1-D)}\log^+|Z_0|)<\infty$ as asserted. □

3.3 Strict ARMA processes with fractional noise

After having resolved the question of convergence of $\nabla^{-D}Z_t$, we are able to state a complete characterization of the existence and uniqueness of strict ARIMA(p,D,q) processes. Keep in mind that the definition of such processes requires the series $\nabla^{-D}Z_t=\sum_{j=0}^{\infty}\psi_j Z_{t-j}$, with $(\psi_j)_{j=0,1,2,...}$ as in (3.5), to converge almost surely.

Theorem 3.5. *Let $(Z_t)_{t\in\mathbb{Z}}$ be a nondeterministic i.i.d. sequence of real random variables and let Φ and Θ be defined as in* (3.1).
If $D\in(0,\frac{1}{2})$, the ARIMA equation

$$\Phi(B)Y_t=\Theta(B)[\nabla^{-D}Z_t],\quad t\in\mathbb{Z}, \tag{3.18}$$

admits a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ if and only if all singularities of $\Theta(z)/\Phi(z)$ on the unit circle are removable, $\mathbb{E}|Z_0|^{1/(1-D)}<\infty$ and $\mathbb{E}Z_0=0$.
If $D\in[-\frac{1}{2},0)$, the ARIMA equation (3.18) *admits a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ if and only if all singularities of $\Theta(z)/\Phi(z)$ on the unit circle are removable and $\mathbb{E}|Z_0|^{1/(1-D)}<\infty$.*
If $D\in(-\infty,-\frac{1}{2})\setminus\{-1,-2,...\}$, let $\Phi(1)\neq 0$. Then the ARIMA equation (3.18) *admits a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ if and only if all singularities of $\Theta(z)/\Phi(z)$ on the unit circle are removable and $\mathbb{E}|Z_0|^{1/(1-D)}<\infty$.*

In all three cases, a strictly stationary solution of (3.18) *is given by*

$$Y_t = \sum_{j=-\infty}^{\infty} \xi_j \left(\nabla^{-D} Z_{t-j}\right), \quad t \in \mathbb{Z}, \tag{3.19}$$

where

$$\sum_{j=-\infty}^{\infty} \xi_j z^j = \frac{\Theta(z)}{\Phi(z)}, \quad 1-c < |z| < 1+c \text{ for some } c \in (0,1),$$

is the Laurent expansion of $\Theta(z)/\Phi(z)$ *around zero. The sum in* (3.19) *converges absolutely almost surely, in the sense that* $\sum_{j=-\infty}^{\infty} |\xi_j||\nabla^{-D} Z_{t-j}| < \infty$ *a.s. If* $\Phi(z) \neq 0$ *for all* $|z| = 1$*, then* (3.19) *is the unique strictly stationary solution of* (3.18).

Before proving Theorem 3.5 we need to establish conditions under which common factors of $\Phi(z)$ and $\Theta(z)$ can be cancelled. This is done in the following lemma, whose proof is in wide parts analogue to the proof of Lemma 1 in Brockwell and Lindner [4], but which we give here for completeness.

Lemma 3.6. *Suppose that* $Y = (Y_t)_{t\in\mathbb{Z}}$ *is a strict ARIMA(p,D,q) process satisfying (3.18). Suppose that* $\lambda_1 \in \mathbb{C}$ *is such that* $\Phi(\lambda_1) = \Theta(\lambda_1) = 0$*, and define*

$$\Phi_1(z) := \frac{\Phi(z)}{1-\lambda_1^{-1}z}, \quad \Theta_1(z) := \frac{\Theta(z)}{1-\lambda_1^{-1}z}, \quad z \in \mathbb{C}.$$

If $|\lambda_1| = 1$ *suppose further that* Z_0 *is symmetric and that* $\Phi_1(\lambda_1) = 0$*, i.e. the multiplicity of the zero* λ_1 *of* Φ *is at least* 2*. Then* Y *is an ARIMA(p* $-$ *1,D,q* $-$ *1) process with autoregressive polynomial* Φ_1 *and moving average polynomial* Θ_1*, i.e.*

$$\Phi_1(B)Y_t = \Theta_1(B)[\nabla^{-D} Z_t], \quad t \in \mathbb{Z}. \tag{3.20}$$

Proof of the lemma. Define

$$W_t := \Phi_1(B)Y_t, \quad t \in \mathbb{Z}.$$

Then $(W_t)_{t\in\mathbb{Z}}$ is strictly stationary, and since $\Phi(z) = (1-\lambda_1^{-1}z)\Phi_1(z)$ we have

$$W_t - \lambda_1^{-1} W_{t-1} = \Theta(B)\nabla^{-D} Z_t, \quad t \in \mathbb{Z}.$$

But as we know that

$$\begin{aligned} \Theta(B)\nabla^{-D} Z_t &= (1-\lambda_1^{-1}B)\Theta_1(B)\nabla^{-D} Z_t \\ &= \Theta_1(B)\nabla^{-D} Z_t - \lambda_1^{-1}\Theta_1(B)\nabla^{-D} Z_{t-1}, \quad t \in \mathbb{Z}, \end{aligned}$$

we get

$$W_t - \lambda_1^{-1} W_{t-1} = \Theta_1(B)\nabla^{-D} Z_t - \lambda_1^{-1}\Theta_1(B)\nabla^{-D} Z_{t-1}, \quad t \in \mathbb{Z}.$$

Iterating gives for $n \in \mathbb{N}_0$

$$W_t - \lambda_1^{-n} W_{t-n} = \Theta_1(B)\nabla^{-D} Z_t - \lambda_1^{-n}\Theta_1(B)\nabla^{-D} Z_{t-n}, \quad t \in \mathbb{Z}. \tag{3.21}$$

Now if $|\lambda_1| > 1$, then the strict stationarity of $(W_t)_{t\in\mathbb{Z}}$ and an application of Slutsky's lemma show that $\lambda_1^{-n} W_{t-n}$ converges in probability to 0 as $n \to \infty$. And because $(Z_t)_{t\in\mathbb{Z}}$ is i.i.d. and hence $(\Theta_1(B)\nabla^{-D} Z_{t-n})_{n\in\mathbb{Z}}$ strictly stationary, an application of Slutsky's lemma shows that $\lambda_1^{-n}\Theta_1(B)\nabla^{-D} Z_{t-n}$ converges in probability to 0 as $n \to \infty$. So we have $\Phi_1(B)Y_t = W_t = \Theta_1(B)\nabla^{-D} Z_t$, $t \in \mathbb{Z}$, which is (3.20). If $|\lambda_1| < 1$, the same argument as in Brockwell and Lindner [4] applies.
Now let $|\lambda_1| = 1$ and assume $\Phi_1(\lambda_1) = 0$ as well as the symmetry of Z_0. Define

$$\Phi_2(z) := \frac{\Phi_1(z)}{1 - \lambda_1^{-1} z} = \frac{\Phi(z)}{(1 - \lambda_1^{-1} z)^2}, \quad z \in \mathbb{C},$$

and $X_t := \Phi_2(B)Y_t$, $t \in \mathbb{Z}$. Then $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary, and

$$X_t - \lambda_1^{-1} X_{t-1} = \Phi_1(B)Y_t = W_t. \tag{3.22}$$

Defining

$$C_t := W_t - \Theta_1(B)\nabla^{-D} Z_t, \quad t \in \mathbb{Z}, \tag{3.23}$$

it follows from (3.21) that

$$C_t = \lambda_1^{-n} C_{t-n} = \lambda_1^{-n} W_{t-n} - \lambda_1^{-n}\Theta_1(B)\nabla^{-D} Z_{t-n}, \quad n \in \mathbb{N}_0, \quad t \in \mathbb{Z}.$$

Summing this over n from 0 to $N \geq 0$ and inserting (3.22) gives

$$(N+1)C_t = \sum_{n=0}^{N} \lambda_1^{-n}(X_{t-n} - \lambda_1^{-1} X_{t-n-1}) - \sum_{n=0}^{N} \lambda_1^{-n}\Theta_1(B)\nabla^{-D} Z_{t-n},$$

so that for $t \in \mathbb{Z}$ and $N \geq 0$ we get

$$C_t - (N+1)^{-1}\Big(X_t - \lambda_1^{-(N+1)} X_{t-N-1}\Big) = -(N+1)^{-1}\sum_{n=0}^{N} \lambda_1^{-n}\Theta_1(B)\nabla^{-D} Z_{t-n}. \tag{3.24}$$

But since $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary and since $|\lambda_1| = 1$, we get with the same argument as above that the left hand-side of (3.24) converges in probability to C_t

as $N \to \infty$. But now the probability limit as $N \to \infty$ of the right-hand side must be measurable with respect to the tail-σ-algebra $\bigcap_{M\in\mathbb{N}} \sigma\{\bigcup_{k\geq M} \sigma(Z_{t-k})\}$. So, by Kolmogorov's zero-one law the right-hand side of (3.24) is $\mathbb{P}$-trivial. Hence C_t is independent of itself, so that C_t must be deterministic. Since Z_0 is symmetric, so is the right-hand side of (3.24) and hence also C_t is symmetric, which implies $C_t = 0$. Equation (3.23) then shows that $\Phi_1(B)Y_t = W_t = \Theta_1(B)\nabla^{-D}Z_t$, $t \in \mathbb{Z}$, which is (3.20). □

Proof of Theorem 3.5. The definition of a strictly stationary solution of (3.18) requires the series $V_t := \nabla^{-D}Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ to converge a.s. By Theorem 3.1, this is equivalent to $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$ in the case $D \in (-\infty, 0) \setminus \{-1, -2, ...\}$ and to $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$ and $\mathbb{E}Z_0 = 0$ in the case $D \in (0, \frac{1}{2})$. Thus it remains to show the sufficiency and necessity of the condition that $\Theta(z)/\Phi(z)$ has only removable singularities on the unit circle, as well as the uniqueness assertion. The sufficiency of this condition and the uniqueness assertion follow in complete analogy to the sufficiency and uniqueness proof of Theorem 1 in Brockwell and Lindner [4], since V_t has finite log-moment by Proposition 3.4. To show that the condition is necessary, let $(Y_t)_{t\in\mathbb{Z}}$ be a strictly stationary solution of (3.18). Let λ_1 be a zero of Φ on the unit circle of multiplicity $m_\Phi(\lambda_1) \geq 1$. We assume that the singularity of Θ/Φ at λ_1 is not removable, i.e. $m_\Phi(\lambda_1) > m_\Theta(\lambda_1)$, and derive a contradiction. By the same argument as in Brockwell and Lindner [4], we may assume without loss of generality that Z_0 is symmetric, with $Z_0 \neq 0$ due to the assumption that Z_0 is not deterministic. By Lemma 3.6 we further may assume without loss of generality that $m_\Theta(\lambda_1) = 0$. Define

$$W_t := \Phi_1(B)Y_t, \quad t \in \mathbb{Z},$$

with $\Phi_1(z) = \Phi(z)/(1 - \lambda_1^{-1}z)$. Then we know that

$$W_t - \lambda_1^{-1}W_{t-1} = \Theta(B)\nabla^{-D}Z_t, \quad t \in \mathbb{Z},$$

Iterating gives for $n \in \mathbb{N}_0$

$$W_t = \lambda_1^{-n}W_{t-n} + \sum_{j=0}^{n-1} \lambda_1^{-j}\Theta(B)\nabla^{-D}Z_{t-j}.$$

Now, we can write

$$\Theta(B)\nabla^{-D}Z_t = \Theta(B)\sum_{k=0}^{\infty} \psi_k Z_{t-k} = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k\wedge q} \theta_m \psi_{k-m} \right) Z_{t-k}. \tag{3.25}$$

So it follows with $\psi'_k := \sum_{m=0}^{k\wedge q} \theta_m \psi_{k-m}$ that

$$\begin{aligned}
& W_t - \lambda_1^{-n} W_{t-n} \\
= \; & \sum_{j=0}^{n-1} \lambda_1^{-j} \sum_{k=0}^{\infty} \psi'_k Z_{t-k-j} \\
= \; & \sum_{j=0}^{\infty} \sum_{k=0}^{j\wedge(n-1)} \lambda_1^{-k} \psi'_{j-k} Z_{t-j} \\
= \; & \sum_{j=0}^{n-2} \lambda_1^{-j} \left(\sum_{k=0}^{j} \psi'_{j-k} \lambda_1^{j-k} \right) Z_{t-j} + \sum_{j=n-1}^{\infty} \lambda_1^{-j} \left(\sum_{k=0}^{n-1} \psi'_{j-k} \lambda_1^{j-k} \right) Z_{t-j} \\
= \; & \sum_{j=0}^{n-2} \lambda_1^{-j} \left(\sum_{k=0}^{j} \psi'_{k} \lambda_1^{k} \right) Z_{t-j} + \sum_{j=n-1}^{\infty} \lambda_1^{-j} \left(\sum_{k=0}^{n-1} \psi'_{j-k} \lambda_1^{j-k} \right) Z_{t-j} \\
=: \; & A_t^n + B_t^n. \qquad (3.26)
\end{aligned}$$

In the following, we shall see that there are constants $K, c > 0$ such that

$$\mathbb{P}\left(|A_t^n| < K\right) \geq c \quad \text{for all } n \in \mathbb{N}. \qquad (3.27)$$

This is because of the stationarity of $(W_t)_{t\in\mathbb{Z}}$ and because of $|\lambda_1| = 1$ we can find constants $0 < \varepsilon < \frac{1}{4}$, $K_1 > 0$ such that

$$\mathbb{P}\left(\left|W_t - \lambda_1^{-n} W_{t-n}\right| < K_1\right) \geq 1 - \varepsilon \quad \text{for all } n \in \mathbb{N},$$

and hence, by (3.26)

$$\mathbb{P}\left(|A_t^n + B_t^n| < K_1\right) \geq 1 - \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Obviously, it follows that for all $n \in \mathbb{N}$

$$\mathbb{P}\left(|\mathrm{Re}\,(A_t^n + B_t^n)| < K_1\right) \geq 1 - \varepsilon, \quad \text{and} \quad \mathbb{P}\left(|\mathrm{Im}\,(A_t^n + B_t^n)| < K_1\right) \geq 1 - \varepsilon. \qquad (3.28)$$

Then for $\frac{1}{2} < c_1 < 1 - 2\varepsilon$ the following holds

$$\mathbb{P}\left(|\mathrm{Re}\,A_t^n| < K_1\right) \geq c_1, \quad \text{and} \quad \mathbb{P}\left(|\mathrm{Im}\,A_t^n| < K_1\right) \geq c_1 \quad \text{for all } n \in \mathbb{N}. \qquad (3.29)$$

Because otherwise we could find $n_0, n_1 \in \mathbb{N}$ such that

$$\begin{aligned}
\mathbb{P}\left(|\mathrm{Re}\,A_t^{n_0}| \geq K_1\right) & \geq 1 - c_1, \quad \text{or} \\
\mathbb{P}\left(|\mathrm{Im}\,A_t^{n_1}| \geq K_1\right) & \geq 1 - c_1
\end{aligned}$$

which means by symmetry of Z_0 that

$$\mathbb{P}\left(\operatorname{Re} A_t^{n_0} \geq K_1\right) \geq \frac{1-c_1}{2}, \quad \mathbb{P}\left(\operatorname{Re} A_t^{n_0} \leq -K_1\right) \geq \frac{1-c_1}{2}, \quad \text{or}$$
$$\mathbb{P}\left(\operatorname{Im} A_t^{n_1} \geq K_1\right) \geq \frac{1-c_1}{2}, \quad \mathbb{P}\left(\operatorname{Im} A_t^{n_1} \leq -K_1\right) \geq \frac{1-c_1}{2}.$$

But also by symmetry of Z_0 we know that

$$\mathbb{P}\left(\operatorname{Re} B_t^{n_0} \leq 0\right) \geq \frac{1}{2}, \quad \mathbb{P}\left(\operatorname{Im} B_t^{n_1} \leq 0\right) \geq \frac{1}{2}.$$

So, because of the independence of $A_t^{n_0}$ and $B_t^{n_0}$ and because of $1-c_1 > 2\varepsilon$ we get

$$\mathbb{P}\left(\operatorname{Re} A_t^{n_0} + \operatorname{Re} B_t^{n_0} \geq K_1\right) = \mathbb{P}\left(\operatorname{Re} A_t^{n_0} + \operatorname{Re} B_t^{n_0} \leq -K_1\right) \geq \frac{1}{4}(1-c_1) > \frac{\varepsilon}{2}, \quad \text{or}$$
$$\mathbb{P}\left(\operatorname{Im} A_t^{n_1} + \operatorname{Im} B_t^{n_1} \geq K_1\right) = \mathbb{P}\left(\operatorname{Im} A_t^{n_1} + \operatorname{Im} B_t^{n_1} \leq -K_1\right) \geq \frac{1}{4}(1-c_1) > \frac{\varepsilon}{2},$$

which is a contradiction to (3.28), so that (3.29) holds. And then it easily follows that

$$\mathbb{P}\left(|A_t^n| < \sqrt{2}K_1\right) \geq 2c_1 - 1 \quad \text{for all } n \in \mathbb{N},$$

which is (3.27) with $K := \sqrt{2}K_1$ and $c := 2c_1 - 1$.
By definition of A_t^n, it follows that $|\sum_{j=0}^{n-2} \lambda_1^{-j} \left(\sum_{k=0}^{j} \psi'_k \lambda_1^k\right) Z_{t-j}|$ does not converge in probability to $+\infty$ as $n \to \infty$. Since $\sum_{j=0}^{n-2} \lambda_1^{-j} \left(\sum_{k=0}^{j} \psi'_k \lambda_1^k\right) Z_{-j}$ is a sum of independent symmetric terms, this implies that $\sum_{j=0}^{\infty} \lambda_1^{-j} \left(\sum_{k=0}^{j} \psi'_k \lambda_1^k\right) Z_{-j}$ converges almost surely (see Kallenberg [13], Theorem 4.17). Now we derive a contradiction in each of the cases (a) $|\lambda_1| = 1$, $\lambda_1 \neq 1$; (b) $\lambda_1 = 1$ and $D \in [-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$.
(a) Obviously, $(1-z)^{-D}$ is continuous and nonzero on $\{|z| \leq 1\} \setminus \{z = 1\}$. So, for $D \in (-\infty, 0) \setminus \{-1, -2, ...\}$ it is clear that

$$(1-\lambda_1)^{-D} = \sum_{j=0}^{\infty} \psi_j \lambda_1^j \tag{3.30}$$

converges absolutely. For $D \in (0, \frac{1}{2})$ the convergence of the series on the right-hand side of (3.30) follows by an application of the Leibniz criterion when keeping in mind that the coefficients ψ_j are monotone decreasing, because $\psi_{j+1}/\psi_j = (D+j)/(j+1) < 1$. So (3.30) holds for $D \in (0, \frac{1}{2})$ as well.
Thus we know that $0 \neq \Theta(\lambda_1)(1-\lambda_1)^{-D} = \sum_{k=0}^{\infty} \psi'_k \lambda_1^k$, and it follows that

$$\left|\sum_{k=0}^{j} \psi'_k \lambda_1^k\right| \geq \left|\frac{\Theta(\lambda_1)(1-\lambda_1)^{-D}}{2}\right|, \quad \text{for } j \geq j_0, \text{ with } j_0 \text{ sufficiently large.}$$

From the obtained a.s. convergence of $\sum_{j=0}^{\infty} \lambda_1^{-j} \left(\sum_{k=0}^{j} \psi_k' \lambda_1^k\right) Z_{-j}$ it follows that for every $r > 0$

$$\mathbb{P}\left(\limsup_{j\to\infty}\left\{\left|\sum_{k=0}^{j} \psi_k' \lambda_1^k\right| \cdot |Z_{-j}| > \frac{|\Theta(\lambda_1)(1-\lambda_1)^{-D}|}{2} \cdot r\right\}\right) = 0.$$

Hence $\mathbb{P}(\limsup_{j\to\infty}\{|Z_{-j}| > r\}) = 0$, so that $\sum_{j=0}^{\infty} \mathbb{P}(|Z_{-j}| > r) < \infty$ by the Borel-Cantelli lemma. So it follows that $\mathbb{P}(|Z_0| \geq r) = 0$ for each $r > 0$, so that Z_0 is almost surely zero, which is impossible since Z_0, and hence its symmetrization, was assumed to be nondeterministic.
(b) Now, let $\lambda_1 = 1$ and $D \in [-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$. We know that $\sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \psi_k'\right) Z_{-j}$ converges almost surely. Then it follows by an application of Theorem 5.1.4 in Chow/Teicher [6] that

$$\sum_{j=0}^{\infty} \left|\sum_{k=0}^{j} \psi_k'\right|^2 < \infty. \tag{3.31}$$

(Observe that the theorem there is only stated for real coefficients. If $\theta_1, ..., \theta_q$ are real, the coefficents ψ_k' are as well. And otherwise split in real and imaginary part and then apply the theorem.)
Now, from Equation (3.25) we know that

$$\Theta(z)(1-z)^{-D} = \sum_{j=0}^{\infty} \psi_j' z^j, \quad |z| < 1,$$

with $\psi_j' = \sum_{k=0}^{j\wedge q} \theta_k \psi_{j-k}$. Multiplying both sides of this equation with $(1-z)^{-1}$ it follows that

$$\Theta(z)(1-z)^{-D-1} = \left(\sum_{j=0}^{\infty} \psi_j' z^j\right)\left(\sum_{m=0}^{\infty} z^m\right) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} \psi_k'\right) z^j, \quad |z| < 1. \tag{3.32}$$

But by means of the binomial expansion we also know that $(1-z)^{-D-1} = \sum_{j=0}^{\infty} \pi_j z^j$, where

$$\pi_j \sim \frac{j^D}{\Gamma(D+1)} \quad \text{as } j \to \infty. \tag{3.33}$$

And then it follows that

$$\Theta(z)(1-z)^{-D-1} = \left(\sum_{k=0}^{q} \theta_k z^k\right)\left(\sum_{j=0}^{\infty} \pi_j z^j\right) = \sum_{j=0}^{\infty} \pi_j' z^j, \quad |z| < 1,$$

with

$$\pi'_j := \sum_{k=0}^{j\wedge q} \theta_k \pi_{j-k} = \pi_j \left(1 + \sum_{k=1}^{j\wedge q} \theta_k \frac{\pi_{j-k}}{\pi_j}\right).$$

Because of (3.33) it is clear that

$$1 + \sum_{k=1}^{j\wedge q} \theta_k \frac{\pi_{j-k}}{\pi_j} \to 1 + \sum_{k=1}^{q} \theta_k = \Theta(1) \neq 0 \quad \text{as } j \to \infty,$$

so that $\pi'_j \sim \frac{j^D}{\Gamma(D+1)}\Theta(1)$ as $j \to \infty$. But as we know from (3.32) that $\pi'_j = \sum_{k=0}^{j} \psi'_k$, $j \in \mathbb{N}$, it follows that

$$\sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \psi'_k \right|^2 = \sum_{j=0}^{\infty} |\pi'_j|^2$$

is not finite, which is a contradiction to (3.31). □

Remark 3.7. *If $\Phi(1) = 0$ and $D < -\frac{1}{2}$, it is still necessary for a strictly stationary solution to exist that all singularities λ on the unit circle with $\lambda \neq 1$ are removable and that $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$, as follows from the proof of Theorem 3.5. That $m_\Phi(1) \leq m_\Theta(1)$, where m_Φ and m_Θ denote the multiplicity of the zero 1 of Φ or Θ, respectively, is however not necessary in that case. For example, let $D < -\frac{1}{2}$, $\Phi(z) = 1 - z$, $\Theta(z) = 1$, and Z_t such that $\mathbb{E}|Z_0|^{1/(1-(D+1))} = \mathbb{E}|Z_0|^{1/(-D)} < \infty$ and $\mathbb{E}Z_0 = 0$ if $D \in [-1, -\frac{1}{2})$. Then $Y_t := \nabla^{-(D+1)} Z_t$ converges almost surely, as does $V_t := \nabla^{-D} Z_t$, and $(1-B)Y_t = \nabla^{-D} Z_t$, so that $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of $\Phi(B)Y_t = \nabla^{-D} Z_t$ in that case. It seems feasible to obtain a full characterization of when (3.18) admits a strictly stationary solution in the case when $D < -\frac{1}{2}$ and 1 is a singularity of Θ/Φ which is not removable, but we have not investigated this issue.*

Corollary 3.8. *The solution Y_t as defined in Theorem 3.5 has finite $(1/(1-D)-\varepsilon)$-moment for every $\varepsilon \in (0, 1/(1-D))$, and finite $1/(1-D)$-moment if $\mathbb{E}(|Z_0|^{1/(1-D)} \log^+ |Z_0|) < \infty$.*

Proof. Let $\varepsilon \in (0, 1/(1-D))$ and $V_t := \nabla^{-D} Z_t$. From Proposition 3.4 it follows that $\mathbb{E}|V_0|^r < \infty$, with $r = 1/(1-D) - \varepsilon$ in the case $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$ but $\mathbb{E}|Z_0|^{1/(1-D)} \log^+ |Z_0|$ not finite, and $r = 1/(1-D)$ in the case $\mathbb{E}|Z_0|^{1/(1-D)} \log^+ |Z_0| < \infty$. For $r \geq 1$ we thus get with the Minkowski inequality

$$\begin{aligned}(\mathbb{E}|Y_t|^r)^{1/r} = \left(\mathbb{E}\left|\sum_{j=-\infty}^{\infty} \xi_j V_{t-j}\right|^r\right)^{1/r} &\leq \sum_{j=-\infty}^{\infty} (\mathbb{E}|\xi_j V_{t-j}|^r)^{1/r} \\ &= (\mathbb{E}|V_0|^r)^{1/r} \sum_{j=-\infty}^{\infty} |\xi_j| < \infty,\end{aligned}$$

the last inequality following because the ξ_j are the coefficients of the Laurent expansion of $\Theta(z)/\Phi(z)$ around zero, which is convergent in a neighbourhood of the unit circle.
For $0 < r < 1$ we have

$$\mathbb{E}|Y_t|^r = \mathbb{E}\left|\sum_{j=-\infty}^{\infty} \xi_j V_{t-j}\right|^r \leq \sum_{j=-\infty}^{\infty} \mathbb{E}|\xi_j V_{t-j}|^r = \mathbb{E}|V_0|^r \sum_{j=-\infty}^{\infty} |\xi_j|^r < \infty.$$

□

Finally, we shall discuss the connection of our results to the results of Kokoszka and Taqqu [14]. In their paper, they study fractional ARIMA processes defined by the equations

$$\Phi(B)Y_t = R(B)Z_t, \tag{3.34}$$

with

$$R(z) := \Theta(z)(1-z)^{-D} = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j\wedge q} \theta_k \psi_{j-k}\right) z^j = \sum_{j=0}^{\infty} \psi'_j z^j,$$

and i.i.d. symmetric α-stable noise $(Z_t)_{t\in\mathbb{Z}}$. Among other results, they obtain a unique strictly stationary solution of (3.34) with this specific noise in terms of the Laurent series of $R(z)/\Phi(z)$, provided $\Phi(z) \neq 0$ for all $|z| \leq 1$, and $\Phi(z)$ and $\Theta(z)$ having no roots in common.
In contrast, we characterized all strictly stationary solutions of the equations

$$\Phi(B)Y_t = \Theta(B)[\nabla^{-D} Z_t], \tag{3.35}$$

that are, strictly speaking, ARMA equations with fractional noise.
However, it is not immediately clear that these both approaches are equivalent. But the following theorem puts things right.

Theorem 3.9. *Let $\Theta(1) \neq 0$. Then, with the notations of above, $\sum_{j=0}^{\infty} \psi'_j Z_{t-j}$ converges almost surely if and only if $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ converges almost surely. In this case, $R(B)Z_t = \Theta(B)[\nabla^{-D} Z_t]$.*
Furthermore, (3.34) admits a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ (in the sense that $\sum_{j=0}^{\infty} \psi'_j Z_{t-j}$ converges almost surely and (Y_t) satisfies (3.34) and is strictly stationary) if and only if (3.35) admits a strictly stationary solution. Any strictly stationary solution of (3.34) is a solution of (3.35) and vice versa.

Proof. Let $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ converge almost surely. Then it is clear by definition of $R(z) = \Theta(z) \sum_{j=0}^{\infty} \psi_j z^j$ that $\sum_{j=0}^{\infty} \psi'_j Z_{t-j}$ must converge almost surely as well, and that in this case $R(B)Z_t = \Theta(B)[\nabla^{-D} Z_t]$.
Conversely, let $\sum_{j=0}^{\infty} \psi'_j Z_{t-j}$ converge almost surely. Then

$$\psi'_j = \sum_{k=0}^{j \wedge q} \theta_k \psi_{j-k} = \psi_j \left(1 + \sum_{k=1}^{j \wedge q} \theta_k \frac{\psi_{j-k}}{\psi_j} \right).$$

Because of (3.7) it is clear that

$$1 + \sum_{k=1}^{j \wedge q} \theta_k \frac{\psi_{j-k}}{\psi_j} \to 1 + \sum_{k=1}^{q} \theta_k = \Theta(1) \neq 0 \quad \text{as } j \to \infty,$$

so that

$$\psi'_j \sim \frac{j^{D-1}}{\Gamma(D)} \Theta(1) \quad \text{as } j \to \infty.$$

But then it follows with Theorem 3.1 and Remark 3.3 that $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$ in the case $D \in (-\infty, 0) \setminus \{-1, -2, ...\}$, and $\mathbb{E}|Z_0|^{1/(1-D)} < \infty$ together with $\mathbb{E}Z_0 = 0$ in the case $D \in (0, \frac{1}{2})$. (Observe that Remark 3.3 is only stated for real coefficients. But as can be seen from the proof of (3.12), the necessity assertion carries over to complex coefficients with this asymptotic behaviour without difficulty.) An application of Theorem 3.1 then yields the almost sure convergence of $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$.
The assertion regarding the equivalence of the solutions to (3.34) and (3.35) is clear from $R(B)Z_t = \Theta(B)[\nabla^{-D} Z_t]$. □

Remark 3.10. *The case* $\Theta(1) = 0$ *is more complicated as the coefficients* ψ'_j, $j \in \mathbb{N}$, *show a different asymptotic behaviour. For example, let* $\Theta(z) = 1 - z$ *and* $\Phi(z) = 1$. *Then*

$$\psi'_j \;=\; \psi_j - \psi_{j-1} = (-1)^j \binom{-D}{j} + (-1)^j \binom{-D}{j-1} = (-1)^j \binom{-D+1}{j}.$$

Applying Stirling's formula, we obtain

$$\psi'_j \sim \frac{j^{D-2}}{\Gamma(D-1)} \quad \text{as } j \to \infty.$$

From Remark 3.3 we then obtain that $\sum_{j=0}^{\infty} \psi'_j Z_{t-j}$ *converges almost surely if and only if* $\mathbb{E}|Z_0|^{1/(2-D)} < \infty$. *Hence, whenever* $\mathbb{E}|Z_0|^{1/(2-D)} < \infty$, *but* $\mathbb{E}|Z_0|^{1/(1-D)}$ *is not finite,* $\sum_{j=0}^{\infty} \psi'_j Z_{t-j}$ *is a solution of* (3.34) *(with* $\Theta(z) = 1 - z$, $\Phi(z) = 1$*), but* (3.35) *does not have a strictly stationary solution.*

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Curriculum Vitae

Personal Data

Name	Bernd Karl Vollenbröker
born on	07/13/1982
in	Ochtrup

Professional experience and education

03/2008 - current	Research and teaching assistant at the Institute of Mathematical Stochastics, Technical University of Braunschweig
12/2007	Diplom-Mathematiker (diploma in mathematics)
10/2001 - 12/2007	Studies at University of Münster, major subject: mathematics, minor subject: economics
Juni 2001	Abitur (general university admission)
1988 - 2001	School

Lebenslauf

Persönliche Daten

Name	Bernd Karl Vollenbröker
geboren am	13. Juli 1982
in	Ochtrup

Werdegang

März 2008 - dato	Wissenschaftlicher Mitarbeiter am Institut für Mathematische Stochastik der Technischen Universität Braunschweig
Dezember 2007	Diplom-Mathematiker
Oktober 2001 - Dezember 2007	Studium der Mathematik mit Nebenfach Volkswirtschaftslehre an der Universität Münster
Juni 2001	Abitur am Arnold-Janssen-Gymnasium, St. Arnold
1988 - 2001	Schulausbildung