The multivariate supOU stochastic volatility model

Ole Eiler Barndorff-Nielsen* Robert Stelzer†

Using positive semidefinite supOU (superposition of Ornstein-Uhlenbeck type) processes to describe the volatility, we introduce a multivariate stochastic volatility model for financial data which is capable of modelling long range dependence effects.

The finiteness of moments and the second order structure of the volatility, the log returns, as well as their "squares" are discussed in detail. Moreover, we give several examples in which long memory effects occur and study how the model as well as the simple Ornstein-Uhlenbeck type stochastic volatility model behave under linear transformations. In particular, the models are shown to be preserved under invertible linear transformations. Finally, we discuss how (sup)OU stochastic volatility models can be combined with a factor modelling approach.

AMS Subject Classification 2000: Primary: 62M10, 91B28
Secondary: 60G51, 91B84

Keywords: factor modelling, Lévy bases, linear transformations, long memory, Ornstein-Uhlenbeck type process, second order moment structure, stochastic volatility

1 Introduction

The well-known Ornstein-Uhlenbeck type stochastic volatility (OU type SV) model introduced in Barndorff-Nielsen and Shephard (2001) has recently been extended to a multivariate set-up in Pigorsch and Stelzer (2009a) using positive semidefinite OU type processes introduced in Barndorff-Nielsen and Stelzer (2007).

In many financial and econometric applications it is important to have adequate models for the joint evolution of the prices of several financial assets. Examples are portfolio optimisation, risk assessment at a portfolio level or pricing of multi-asset derivatives (see Pigorsch and Stelzer (2009a) for a more detailed discussion of the relevance of multivariate asset price models in continuous time).

*Thiele Centre, Department of Mathematical Sciences, Århus University, Ny Munkegade, DK-8000 Århus C, Denmark. oebn@imf.au.dk

†TUM Institute for Advanced Study & Zentrum Mathematik, Technische Universität München, Boltzmannstraße 3, D-85747 Garching, Germany. *Email:* rstelzer@ma.tum.de, www-m4.ma.tum.de

Whereas the OU type model is capable of reproducing most of the so-called stylized facts (stochastic volatility exhibiting jumps, volatility clustering, heavy tails, dependence of the log-returns with zero-autocorrelation, ...) which are usually present in observed financial return series (cf. Cont and Tankov (2004) or Guillaume, Dacorogna, Davé, Müller, Olsen and Pictet (1997)), it is not capable of producing long memory in the volatility or log-returns. In the univariate case one uses superpositions of Ornstein-Uhlenbeck type (supOU) processes (see Barndorff-Nielsen (2001)) to allow for possible long range dependence effects. Using the theory of multivariate supOU processes developed in Barndorff-Nielsen and Stelzer (2009) we introduce and analyse in this paper a multivariate supOU SV model where the volatility or instantaneous covariance matrix (which has to be a positive semidefinite matrix process in a multivariate setting) is modelled via a positive semidefinite supOU process.

In the univariate case positive stationary OU type processes are given by

$$\sigma_t^2 = \int_{-\infty}^{t} e^{-a(t-s)} dL_s$$

where $a > 0$ and L is a Lévy process with non-decreasing paths, i.e. a subordinator, which has a finite logarithmic moment. SupOU processes are intuitively obtained by summing up independent OU type processes with different parameters and independent driving Lévy processes which are identically distributed. This can be extended to integrating up OU type processes with all possible parameters $a > 0$. Formally, this is carried out by using a Lévy basis (or infinitely divisible independently scattered random measure Λ) on $\mathbb{R} \times \mathbb{R}^+ \backslash \{0\}$ and one obtains

$$\sigma_t^2 = \int_0^{\infty} \int_{-\infty}^{t} e^{-a(t-s)} \Lambda(da, ds).$$

Due to the construction, the paths of σ_t^2 exhibit jumps. This is a major difference to fractionally integrated Lévy-driven OU type processes which are continuous and provide another possibility to obtain long range dependence within an OU framework. For a comprehensive overview over the use of supOU type processes to model the stochastic volatility in a financial context we refer to Barndorff-Nielsen and Shephard (2010).

In the multivariate stochastic volatility model of Pigorsch and Stelzer (2009a) the d-dimensional vector of logarithmic stock price processes is given as

$$X_t = \int_0^t (\mu + \Sigma_s \beta) ds + \int_0^t \Sigma_s^{1/2} dW_s$$

where $\mu, \beta \in \mathbb{R}^d$ and the volatility process Σ is given as a stationary positive semidefinite OU type process, i.e.

$$\Sigma_t = \int_{-\infty}^{t} e^{A(t-s)} dL_s e^{A^*(t-s)}$$

with L being a matrix subordinator (i.e. a Lévy process in the positive semidefinite matrices, see Barndorff-Nielsen and Pérez-Abreu (2008)) and A a $d \times d$ matrix with all eigenvalues having strictly negative real part. Note that for notational convenience and to avoid ambiguities we denote the instantaneous covariance matrix by Σ, not its square. Moreover, we call Σ also the volatility process, as this appears most natural in a multivariate setting.

We extend this model by specifying Σ as a positive semidefinite supOU process and furthermore we allow for a leverage effect, a more general drift and the use of more general decompositions of Σ than only the positive semidefinite square root.

The remainder of this paper is structured as follows. Section 2 summarises the notation, gives a precise definition of the OU type stochastic volatility model, provides some background information on Lévy bases and integration with respect to them and summarises the known properties of positive semidefinite supOU processes. Based on this we introduce a multivariate supOU stochastic volatility model in Section 3, derive its second order structure and finally consider examples exhibiting long memory properties. Thereafter, we study the effects of linear transformations and marginalisations for both OU type and supOU stochastic volatility models in Section 4 and finally, address the question how our models can be combined with factor modelling approaches in Section 5.

2 Background and preliminaries

2.1 Notation

We denote the set of real $m \times n$ matrices by $M_{m,n}(\mathbb{R})$. If $m = n$, we simply write $M_n(\mathbb{R})$ and denote the group of invertible $n \times n$ matrices by $GL_n(\mathbb{R})$, the linear subspace of symmetric matrices by $\mathbb{S}_n$, the (closed) positive semidefinite cone by $\mathbb{S}_n^+$ and the open positive definite cone by $\mathbb{S}_n^{++}$ (likewise $\mathbb{S}_n^{--}$ are the strictly negative definite matrices, $\mathbb{R}^{--}$ the strictly negative real numbers, etc.). I_n stands for the $n \times n$ identity matrix. The tensor (Kronecker) product of two matrices A, B is written as $A \otimes B$. vec denotes the well-known vectorisation operator that maps the $n \times n$ matrices to $\mathbb{R}^{n^2}$ by stacking the columns of the matrices below one another. For more information regarding the tensor product and vec operator we refer to Horn and Johnson (1991, Chapter 4). The spectrum of a matrix is denoted by $\sigma(\cdot)$. Finally, A^* is the transpose (adjoint) of a matrix $A \in M_{m,n}(\mathbb{R})$ and A_{ij} stands for the entry of A in the ith row and jth column.

Norms of vectors or matrices are denoted by $\|\cdot\|$. If the norm is not further specified, then it is understood that we take the Euclidean norm or its induced operator norm, respectively. However, due to the equivalence of all norms none of our results really depends on the choice of norms.

For a complex number z we denote by $\Re(z)$ its real part and by $\Im(z)$ its imaginary part. Moreover, the indicator function of a set A is written 1_A.

A mapping $f : V \to W$ is said to be $\mathscr{V}$-$\mathscr{W}$-measurable, if it is measurable when the σ-algebra $\mathscr{V}$ is used on the domain V and the σ-algebra $\mathscr{W}$ is used on the range W. The Borel σ-algebras are denoted by $\mathscr{B}(\cdot)$ and λ typically stands for the Lebesgue measure which in vector or matrix spaces is understood to be defined as the product of the coordinatewise Lebesgue measures.

Throughout we assume that all random variables and processes are defined on a given appropriate filtered probability space $(\Omega, \mathscr{F}, P, \mathbb{F})$ satisfying the usual hypotheses (i.e. complete and right continuous filtration).

Furthermore, we employ an intuitive notation with respect to the (stochastic) integration with matrix-valued integrators referring to any of the standard texts (e.g. Protter (2004)) for a comprehensive treatment of the theory of stochastic integration. Let $(A_t)_{t\in\mathbb{R}^+}$ in $M_{m,n}(\mathbb{R})$, $(B_t)_{t\in\mathbb{R}^+}$ in $M_{r,s}(\mathbb{R})$ be càdlàg and adapted processes and $(L_t)_{t\in\mathbb{R}^+}$ in $M_{n,r}(\mathbb{R})$ be a semimartingale. Then we denote by $\int_0^t A_{s-} dL_s B_{s-}$ the matrix C_t in $M_{m,s}(\mathbb{R})$ which has ij-th element $C_{ij,t} = \sum_{k=1}^n \sum_{l=1}^r \int_0^t A_{ik,s-} B_{lj,s-} dL_{kl,s}$. Equivalently such an integral can be understood in the sense of Métivier and Pellaumail (1980) by identifying it with the integral $\int_0^t \mathbf{A}_{s-} dL_s$ with $\mathbf{A}_t$ being for each fixed t the linear operator $M_{n,r}(\mathbb{R}) \to M_{m,s}(\mathbb{R})$, $X \mapsto A_t X B_t$. Analogous notation is used in the context of integrals with respect to random measures.

Finally, integrals of the form $\int_A \int_B f(x,y) m(dx,dy)$ are understood to be over the set A in x and over B in y.

2.2 The multivariate OU type stochastic volatility model

Now, we first give a precise definition of the simple OU type stochastic volatility model. Let $L(\mathbb{S}_d, \mathbb{R}^d)$ denote the set of all linear operators from $\mathbb{S}_d$ to $\mathbb{R}^d$. For details on matrix subordinators, i.e. Lévy processes in the positive semidefinite matrices, we refer to Barndorff-Nielsen and Pérez-Abreu (2008).

Definition 2.1. *Let W be a d-dimensional standard Brownian motion, a be a predictable $\mathbb{R}^d$-valued process, L be a $d \times d$ matrix subordinator with $E(\max(\ln(\|L_1\|), 0)) < \infty$ independent of W, and let $A \in M_d(\mathbb{R})$ be such that $\sigma(A) \subset (-\infty, 0) + i\mathbb{R}$ and finally $\psi \in L(\mathbb{S}_d, \mathbb{R}^d)$. Assume that $X = (X_t)_{t \in \mathbb{R}^+}$ is given by*

$$dX_t = a_t dt + r(\Sigma_{t-}) dW_t + \psi(dL_t), \qquad X_0 = 0, \tag{2.1}$$

for some continuous function $r : \mathbb{S}_d^+ \to M_d(\mathbb{R})$ such that $x = r(x) r(x)^* \, \forall x \in \mathbb{S}_d^+$ and where Σ is the unique stationary solution to

$$d\Sigma_t = (A\Sigma_{t-} + \Sigma_{t-} A^*) dt + dL_t, \tag{2.2}$$

i.e. Σ is a positive semidefinite OU type process. Then we say that X follows a multivariate OU type stochastic volatility (SV) model with leverage*, abbreviated* $\mathrm{SVOU}(a, r, \psi, A, L)$.

If $\psi = 0$, i.e. there is no leverage effect present, we say X follows a multivariate OU type SV model.

Possible choices for r are the positive semidefinite square root and the Cholesky factorisation, for instance. Note that (2.2) can easily be solved explicitely, which gives

$$\Sigma_t = \int_{-\infty}^{t} e^{A(t-s)} dL_s e^{A^*(t-s)}, \, t \in \mathbb{R}^+. \tag{2.3}$$

Moreover, the definition of a multivariate Ornstein-Uhlenbeck type stochastic volatility model given above slightly generalises the one of Pigorsch and Stelzer (2009a) by allowing for a leverage effect and a general drift and by considering general "square-root-like factorisations" r instead of only the positive semi-definite square root.

The following result shows that using different factorisations r of Σ does not make too big a difference.

Proposition 2.2. *Assume $X^{(1)}$ is* $\mathrm{SVOU}(a, r, \psi, A, L)$ *and $X^{(2)}$ is* $\mathrm{SVOU}(a, \tilde{r}, \psi, A, L)$ *and that (a, L) is independent of W. Then $X^{(1)}$ and $X^{(2)}$ are equal in distribution.*

Proof. For the sake of notational simplicity we only prove the case without leverage. The proof of the leverage case is then obvious, since W and (a, L) are independent. It suffices to show that $X^{(1)}|(a, L)$ and $X^{(2)}|(a, L)$ are equal in finite dimensional distributions. Therefore observing that

$$X_t^{(1)}|(a,L) \stackrel{\mathscr{D}}{=} N\left(\int_0^t a_s ds, \int_0^t \Sigma_s ds\right) \stackrel{\mathscr{D}}{=} X_t^{(2)}|(a,L) \quad \forall t \in \mathbb{R}^+$$

$$\left.\begin{pmatrix} X_t^{(1)} \\ X_u^{(1)} \end{pmatrix}\right| (a,L) \stackrel{\mathscr{D}}{=} N\left(\begin{pmatrix} \int_0^t a_s ds \\ \int_0^u a_s ds \end{pmatrix}, \begin{pmatrix} \int_0^t \Sigma_s ds & \int_0^{t\wedge u} \Sigma_s ds \\ \int_0^{t\wedge u} \Sigma_s ds & \int_0^u \Sigma_s ds \end{pmatrix}\right) \stackrel{\mathscr{D}}{=} \left.\begin{pmatrix} X_t^{(2)} \\ X_u^{(2)} \end{pmatrix}\right| (a,L) \qquad \forall t, u \in \mathbb{R}^+$$

and likewise for all higher dimensional marginals concludes the proof. □

Remark 2.3. *The independence of W and (a, L) holds, for instance, if a is adapted to the filtration generated by L.*

2.3 Lévy bases and integration

In this preliminary section we give an introduction to $\mathbb{S}_d$-valued Lévy bases and the relevant integration theory needed in the following (see Rajput and Rosinski (1989) and Pedersen (2003) for more details).

To see the relation between Lévy bases and processes easier, recall that a univariate Lévy process can be understood as a signed random measure on the real numbers. If $L = (L_t)_{t\in\mathbb{R}}$ is a Lévy process, this measure is simply determined by $L((a,b]) = L(b) - L(a)$ for all $a, b \in \mathbb{R}, a < b$.

Define now $M_d^- := \{X \in M_d(\mathbb{R}) : \sigma(X) \subset (-\infty, 0) + i\mathbb{R}\}$ and $\mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$ to be the bounded Borel sets of $M_d^- \times \mathbb{R}$.

Definition 2.4. *A family $\Lambda = \{\Lambda(B) : B \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)\}$ of $\mathbb{S}_d$-valued random variables is called an* $\mathbb{S}_d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ *if:*

(a) *the distribution of $\Lambda(B)$ is infinitely divisible for all $B \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$,*

(b) *for any $n \in \mathbb{N}$ and pairwise disjoint sets $B_1, \dots, B_n \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$ the random variables $\Lambda(B_1), \dots, \Lambda(B_n)$ are independent and*

(c) *for any pairwise disjoint sets $B_i \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$ with $i \in \mathbb{N}$ satisfying $\bigcup_{n\in\mathbb{N}} B_n \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$ the series $\sum_{n=1}^{\infty} \Lambda(B_n)$ converges almost surely and $\Lambda\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sum_{n=1}^{\infty} \Lambda(B_n)$.*

In some literature Lévy bases are also often called infinitely divisible independently scattered random measures (abbreviated i.d.i.s.r.m.) instead.

In the present paper we consider only special $\mathbb{S}_d$-valued Lévy bases having characteristic function of the form

$$E\left(\exp(i\mathrm{tr}(u\Lambda(B)))\right) = \exp\left(\varphi(u)\Pi(B)\right) \tag{2.4}$$

for all $u \in \mathbb{S}_d$ and $B \in \mathscr{B}_b\left(M_d^-(\mathbb{R}) \times \mathbb{R}\right)$, where $\Pi = \pi \times \lambda$ is the product of a probability measure π on $M_d^-(\mathbb{R})$ and the Lebesgue measure λ on $\mathbb{R}$ and

$$\varphi(u) = i\mathrm{tr}(u\gamma_0) + \int_{\mathbb{S}_d} \left(e^{i\mathrm{tr}(ux)} - 1\right) \nu(dx) \tag{2.5}$$

is the cumulant transform of an infinitely divisible distribution on $\mathbb{S}_d$ with Lévy-Khintchine triplet $(\gamma_0, 0, \nu)$ (taken with respect to the truncation function constantly equal to zero), i.e. $\gamma_0 \in \mathbb{S}_d$ and ν is a Lévy measure – a Borel measure on $\mathbb{S}_d$ with $\nu(\{0\}) = 0$ and $\int_{\mathbb{S}_d} (\|x\| \wedge 1)\nu(dx) < \infty$. The triplet (γ_0, ν, π) determines the distribution of the Lévy basis completely and is henceforth referred to as the "generating triplet".

The Lévy process L defined by

$$L_t = \Lambda(M_d^- \times (0,t]) \text{ and } L_{-t} = \Lambda(M_d^- \times (-t, 0)) \text{ for } t \in \mathbb{R}^+$$

has characteristic triplet $(\gamma_0, 0, \nu)$ and is called "the underlying Lévy process".

A Lévy basis has a Lévy-Itô decomposition, which is of the following special form for the $\mathbb{S}_d$-valued Lévy bases we consider. For the necessary background on the integration with respect to Poisson random measures we refer to Jacod and Shiryaev (2003, Section 2.1) and Kallenberg (2002, Lemma 12.13).

Theorem 2.5 (Lévy-Itô decomposition)**.** *Let Λ be an $\mathbb{S}_d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating triplet (γ_0, ν, π). Then there exists a modification $\tilde{\Lambda}$ of Λ which is also a Lévy basis with generating* *triplet (γ_0, ν, π) such that there exists a Poisson random measure μ on $(\mathbb{S}_d \times M_d^- \times \mathbb{R}, \mathscr{B}(\mathbb{S}_d \times M_d^- \times \mathbb{R}))$ with intensity measure $\nu \times \pi \times \lambda$ satisfying*

$$\tilde{\Lambda}(B) = \gamma_0(\pi \times \lambda)(B) + \int_{\mathbb{R}^d} \int_B x\mu(dx, dA, ds) \tag{2.6}$$

for all $B \in \mathscr{B}_b(M_d^- \times \mathbb{R})$. Moreover, the integral with respect to μ exists as a Lebesgue integral for all $\omega \in \Omega$.

Here an $\mathbb{S}_d$-valued Lévy basis $\tilde{\Lambda}$ on $M_d^- \times \mathbb{R}$ is called a *modification* of a Lévy basis Λ if $\tilde{\Lambda}(B) = \Lambda(B)$ a.s. for all $B \in \mathscr{B}_b(M_d^- \times \mathbb{R})$.

In the following we assume without loss of generality that all $\mathbb{S}_d$-valued Lévy bases are such that they have the special Lévy-Itô decomposition (2.6).

As the underlying Lévy process has finite variation, we can always do ω-wise Lebesgue integration with respect to a Lévy basis. Below $L(\mathbb{S}_d)$ denotes the set of all linear operators from $\mathbb{S}_d$ to $\mathbb{S}_d$, which we identify with a linear subspace of $M_{d^2}(\mathbb{R})$.

Proposition 2.6. *Let Λ be an $\mathbb{S}_d$-valued Lévy basis with characteristic triplet (γ_0, ν, π). Furthermore, let $f : M_d^- \times \mathbb{R} \to L(\mathbb{S}_d)$ be a $\mathscr{B}\left(M_d^- \times \mathbb{R}\right) - \mathscr{B}\left(L(\mathbb{S}_d)\right)$ −measurable function satisfying*

$$\int_{M_d^-} \int_{\mathbb{R}} \|f(A,s)\| ds\pi(dA) < \infty, \tag{2.7}$$

$$\int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{S}_d} (1 \wedge \|f(A,s)x\|)\nu(dx) ds\pi(dA) < \infty. \tag{2.8}$$

Then

$$\int_{M_d^-} \int_{\mathbb{R}} f(A,s)\Lambda(dA,ds) = \int_{M_d^-} \int_{\mathbb{R}} f(A,s)\gamma_0 ds\pi(dA) + \int_{\mathbb{S}_d} \int_{M_d^-} \int_{\mathbb{R}} f(A,s)x\mu(dx,dA,ds) \tag{2.9}$$

exists and the right hand side is a Lebesgue integral for every $\omega \in \Omega$

Moreover, the distribution of $\int_{M_d^-} \int_{\mathbb{R}} f(A,s)\Lambda(dA,ds)$ is infinitely divisible with characteristic function

$$E\left(\exp\left(i\mathrm{tr}\left(u\int_{M_d^-} \int_{\mathbb{R}} f(A,s)\Lambda(dA,ds)\right)\right)\right) = \exp\left(i\mathrm{tr}(u\gamma_{\mathrm{int},0}) + \int_{\mathbb{S}_d} \left(e^{i\mathrm{tr}(ux)} - 1\right)\nu_{\mathrm{int}}(dx)\right), u \in \mathbb{S}_d,$$

where

$$\gamma_{\mathrm{int},0} = \int_{M_d^-} \int_{\mathbb{R}} f(A,s)\gamma_0 ds\pi(dA), \tag{2.10}$$

$$\nu_{\mathrm{int}}(B) = \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{S}_d} 1_B(f(A,s)x)\nu(dx)ds\pi(dA) \text{ for all Borel sets } B \subseteq \mathbb{S}_d. \tag{2.11}$$

Proof. Follows from the Lévy-Itô decomposition and the usual integration theory with respect to Poisson random measures (see Kallenberg (2002, Lemma 12.13)). □

2.4 Positive semidefinite supOU processes

Based on the previous section, we recall now the definition (Equation 2.15 below) of positive semidefinite supOU processes from Barndorff-Nielsen and Stelzer (2009) and summarise results relevant later on. Intuitively "superposition" means that we are "adding up" independent positive semidefinite OU type processes as given in (2.3) with all possible mean reversion parameters A in the set $M_d^- := \{A \in M_d(\mathbb{R}) : \max(\Re(\sigma(A))) < 0\}$.

Just as one has to restrict the driving Lévy process to matrix subordinators in the OU type processes to get a positive semidefinite OU type process, one needs to impose a comparable condition on the Lévy basis below. Note that for a $d \times d$ matrix-valued Lévy-basis Λ we denote by $\mathrm{vec}(\Lambda)$ the $\mathbb{R}^{d^2}$-valued Lévy basis given by $\mathrm{vec}(\Lambda)(B) = \mathrm{vec}(\Lambda(B))$ for all Borel sets B. Moreover, observe that $\mathrm{tr}(XY^*)$ (with $X, Y \in M_d(\mathbb{R})$ and tr denoting the usual trace functional) defines a scalar product on $M_d(\mathbb{R})$ and that the vec operator is a Hilbert space isometry between $M_d(\mathbb{R})$ equipped with this scalar product and $\mathbb{R}^{d^2}$ with the usual Euclidean scalar product. For proofs of the results in this section we refer to Barndorff-Nielsen and Stelzer (2009).

Theorem 2.7. *Let Λ be an $\mathbb{S}_d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating triplet (γ_0, ν, π) where ν is a Lévy measure on $\mathbb{S}_d$ satisfying $\nu(\mathbb{S}_d \backslash \mathbb{S}_d^+) = 0$ and*

$$\int_{\|x\|>1} \ln(\|x\|)\nu(dx) < \infty. \tag{2.12}$$

Moreover, assume there exist measurable functions $\rho : M_d^- \to \mathbb{R}^+ \backslash \{0\}$ and $\kappa : M_d^- \to [1,\infty)$ such that:

$$\|e^{As}\| \leq \kappa(A) e^{-\rho(A)s} \forall s \in \mathbb{R}^+, \quad \pi - \textit{almost surely}, \tag{2.13}$$

and

$$\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty. \tag{2.14}$$

Then the process $(\Sigma_t)_{t\in\mathbb{R}}$ given by

$$\begin{aligned} \Sigma_t &= \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \Lambda(dA, ds) e^{A^*(t-s)} \qquad (2.15) \\ &= \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \gamma_0 e^{A^*(t-s)} ds \pi(dA) + \int_{\mathbb{S}_d} \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} x e^{A^*(t-s)} \mu(dx, dA, ds) \end{aligned}$$

is well-defined as a Lebesgue integral for all $t \in \mathbb{R}$ and $\omega \in \Omega$ and Σ is stationary with $\Sigma_t \in \mathbb{S}_d^+$ for all $t \in \mathbb{R}$.

Moreover,

$$\mathrm{vec}(\Sigma_t) = \int_{M_d^-} \int_{-\infty}^t e^{(A \otimes I_d + I_d \otimes A)(t-s)} \mathrm{vec}(\Lambda)(dA, ds) \tag{2.16}$$

and the distribution of Σ_t is infinitely divisible with characteristic function

$$E\left(\exp\left(i\mathrm{tr}(u\Sigma_t)\right)\right) = \exp\left(i\mathrm{tr}(u\gamma_{\Sigma,0}) + \int_{\mathbb{S}_d} \left(e^{i\mathrm{tr}(ux)} - 1\right) \nu_\Sigma(dx)\right), u \in \mathbb{S}_d,$$

where

$$\gamma_{\Sigma,0} = \int_{M_d^-} \int_0^\infty e^{As} \gamma_0 e^{A^* s} ds \pi(dA), \tag{2.17}$$

$$\nu_\Sigma(B) = \int_{M_d^-} \int_0^\infty \int_{\mathbb{S}_d^+} 1_B(e^{As} x e^{A^* s}) \nu(dx) ds \pi(dA) \textit{ for all Borel sets } B \subseteq \mathbb{S}_d. \tag{2.18}$$

Remark 2.8. *Observe that* $\kappa(A)$ *can be replaced by* 1 *and* $\rho(A)$ *by* $-\max(\Re(\sigma(A)))$ *in* (2.14) *(and also in* (2.19) *and* (2.20) *below), provided* π *is concentrated on the normal matrices or finitely many diagonalisable rays (i.e. there are* $k \in \mathbb{N}$ *and diagonalisable* $A_1, \ldots, A_k \in M_d^-$ *such that* $\pi(\{\alpha A_i : i \in \{1,\ldots,k\}, \alpha \in (0,\infty)\}) = 1$*).*

Next we recall the existence of moments and the second order structure.

Proposition 2.9. *Let* Σ *be a stationary* $\mathbb{S}_d^+$*-valued supOU process driven by a Lévy basis* Λ *satisfying the conditions of Theorem 2.7.*

(i) If

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty \tag{2.19}$$

for $r \in (0,1]$*, then* Σ *has a finite r-th moment, i.e.* $E(\|\Sigma_t\|^r) < \infty$*.*

(ii) If $r \in (1,\infty)$ *and*

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty, \quad \int_{M_d^-} \frac{\kappa(A)^{2r}}{\rho(A)} \pi(dA) < \infty, \tag{2.20}$$

then Σ *has a finite r-th moment, i.e.* $E(\|\Sigma_t\|^r) < \infty$*.*

(iii) If the conditions given in (ii) are satisfied for $r = 2$*, then the second order structure of* Σ *is given by:*

$$E(\Sigma_0) = -\int_{M_d^-} \mathbf{A}(A)^{-1}\left(\gamma_0 + \int_{\mathbb{S}_d} x\nu(dx)\right)\pi(dA),$$

$$\mathrm{var}(\mathrm{vec}(\Sigma_0)) = -\int_{M_d^-} (\mathscr{A}(A))^{-1}\left(\int_{\mathbb{S}_d} \mathrm{vec}(x)\mathrm{vec}(x)^* \nu(dx)\right)\pi(dA),$$

$$\mathrm{cov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0)) = -\int_{M_d^-} e^{(A\otimes I_d + I_d \otimes A)h}(\mathscr{A}(A))^{-1}\left(\int_{\mathbb{S}_d} \mathrm{vec}(x)\mathrm{vec}(x)^* \nu(dx)\right)\pi(dA) \text{ for } h \in \mathbb{N}$$

with $\mathbf{A}(A) : M_d(\mathbb{R}) \to M_d(\mathbb{R}), X \mapsto AX + XA^*$ *and* $\mathscr{A}(A) : M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R}), X \mapsto (A \otimes I_d + I_d \otimes A)X + X(A^* \otimes I_d + I_d \otimes A^*)$*.*

Moreover, it holds that

$$\lim_{h\to\infty} \mathrm{cov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0)) = 0 \tag{2.21}$$

Most important is the following result which, in particular, ensures that integrals like $\int_0^t \Sigma_s^{1/2} dW_s$, occurring in the definition of the multivariate supOU stochastic volatility models in the next section, do indeed exist. Using (i) below they are defined in the L^2-sense of Øksendal (1998) provided Σ has a finite first moment, which is the case if $\int_{\|x\|>1} \|x\|\nu(dx) < \infty$, and in the general case given in (iii) in the sense of the stochastic integration with respect to semimartingales (see Protter (2004), for instance).

Below $\mathscr{G}$ is the σ-algebra generated by the Lévy basis Λ, i.e. by the set of random variables $\{\Lambda(B) : B \in \mathscr{B}(M_d^- \times \mathbb{R})\}$.

Theorem 2.10. *Let* Σ *be the positive semidefinite supOU process of Theorem 2.7. Then:*

(i) $\Sigma_t(\omega)$ *is* $\mathscr{B}(\mathbb{R}) \times \mathscr{G}$*-*$\mathscr{B}(\mathbb{S}_d)$ *measurable as a function of* $t \in \mathbb{R}$ *and* $\omega \in \Omega$ *and adapted to the filtration* $(\mathscr{G}_t)_{t\in\mathbb{R}}$ *generated by* Λ*, i.e.* $\mathscr{G}_t$ *is the* σ*-algebra generated by the set of random variables* $\{\Lambda(B) : B \in \mathscr{B}(M_d^- \times (-\infty, t])\}$*.*

(ii) If

$$\int_{M_d^-} \kappa(A)^2 \pi(dA) < \infty, \tag{2.22}$$

the paths of Σ are locally uniformly bounded in t for every $\omega \in \Omega$.

(iii) Provided that

$$\int_{M_d^-} \frac{(\|A\| \vee 1)\kappa(A)^2}{\rho(A)} \pi(dA) < \infty \tag{2.23}$$

and

$$\int_{M_d^-} \|A\| \kappa(A)^2 \pi(dA) < \infty \tag{2.24}$$

it holds that

$$\Sigma_t = \Sigma_0 + \int_0^t Z_u du + L_t \tag{2.25}$$

where L is the underlying matrix subordinator and

$$Z_u = \int_{M_d^-} \int_{-\infty}^{u} \left(A e^{A(u-s)} \Lambda(dA, ds) e^{A^*(u-s)} + e^{A(u-s)} \Lambda(dA, ds) e^{A^*(u-s)} A^* \right) \tag{2.26}$$

for all $u \in \mathbb{R}$, with the integral existing ω-wise.
Moreover, the paths of Σ are càdlàg and of finite variation on compacts.

Remark 2.11. *(i) Of course, our given filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in \mathbb{R}}$ is such that $\mathscr{G}_t \subset \mathscr{F}_t$ for all $t \in \mathbb{R}$ and hence the measurability properties in (i) are also true with $\mathscr{F}_t$ in place of $\mathscr{G}_t$*

(ii) Intuitively (2.24) *means that π does not place too much mass on the elements of M_d^- with high norm and thus very fast exponential decay rates.*

If π is concentrated on the normal matrices or finitely many diagonalisable rays (as in Remark 2.8), then (2.23) *and* (2.24) *become*

$$-\int_{M_d^-} \frac{(\|A\| \vee 1)}{\max \Re(\sigma(A))} \pi(dA) < \infty \text{ and } \int_{M_d^-} \|A\| \pi(dA) < \infty. \tag{2.27}$$

In particular, the second condition simply means that π has a finite first moment.

If π is concentrated on $\mathbb{S}_d^{--}$, then we have $\|A\| = -\min(\sigma(A))$ and (2.23) *becomes*

$$\int_{\mathbb{S}_d^-} \frac{(\min(\sigma(A)) \wedge -1)}{\max(\sigma(A))} \pi(dA) < \infty, \tag{2.28}$$

which can be seen as a condition on the spread between the different exponential decay rates as measured by the eigenvalues.

3 The supOU stochastic volatility model

3.1 Definition

Now we can define a stochastic volatility model based on positive semidefinite supOU processes.

Definition 3.1. *Let W be a d-dimensional standard Brownian motion, a be a predictable $\mathbb{R}^d$-valued process, Λ be an $\mathbb{S}_d^+$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating triplet (γ_0, ν, π), independent of W and satisfying the conditions of Theorem 2.7. Let, moreover L be the underlying Lévy process of Λ and finally $\psi \in L(\mathbb{S}_d, \mathbb{R}^d)$. Assume that $X = (X_t)_{t\in\mathbb{R}^+}$ is given by*

$$dX_t = a_t dt + r(\Sigma_{t-})dW_t + \psi(dL_t), \qquad X_0 = 0, \tag{3.1}$$

for some continuous function $r : \mathbb{S}_d^+ \to M_d(\mathbb{R})$ such that $x = r(x)r(x)^*$ and where

$$\Sigma_t = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)} \Lambda(dA, ds) e^{A^*(t-s)} \forall t \in \mathbb{R}^+.$$

Then we say that X follows a multivariate supOU type stochastic volatility (SV) model with leverage*, abbreviated* $\mathrm{SVsupOU}(a, r, \psi, \gamma_0, \nu, \pi)$.

If $\psi = 0$, i.e. there is no leverage effect present, we say X follows a multivariate supOU type SV model.

Obviously, an $\mathrm{SVOU}(a, r, \psi, A, L)$ model is an $\mathrm{SVsupOU}(a, r, \psi, \gamma_{0,L}, \nu_L, \delta_A)$ model with $(\gamma_{0,L}, 0, \nu_L)$ being the Lévy-Khintchine triplet of L and δ_A denoting the Dirac delta distribution with unit mass at A.

The following is obtained along the same lines as Proposition 2.2.

Proposition 3.2. *Assume $X^{(1)}$ is $\mathrm{SVsupOU}(a, r, \psi, \gamma_0, \nu, \pi)$ and $X^{(2)}$ is $\mathrm{SVsupOU}(a, \tilde{r}, \psi, \gamma_0, \nu, \pi)$ and that (a, Λ) is independent of W. Then $X^{(1)}$ and $X^{(2)}$ are equal in distribution.*

Remark 3.3. *The independence of W and (a, Λ) holds, for instance, if a is adapted to the filtration $(\mathscr{G}_t)_{t\in\mathbb{R}}$ generated by Λ, i.e. $\mathscr{G}_t$ is the σ-algebra generated by the set of random variables $\{\Lambda(B) : B \subseteq \mathscr{B}(M_d^- \times (-\infty, t])\}$ for $t \in \mathbb{R}$.*

Since in distribution the choice of r does not matter, our supOU stochastic volatility model falls into the general class of stochastic volatility models analysed in Pigorsch and Stelzer (2009a) regarding distributional properties like moments.

3.2 Second order properties

When thinking about X as the log-price process of d financial assets, it is clear that one typically will observe neither X continuously nor the volatility process Σ, but only X at a discrete set of times. In the following we assume that we observe X at an equally spaced time grid with given grid size $\Delta > 0$. Then one is typically interested in the log-returns Y over the grid intervals as well as the integrated volatility $\mathbf{V}$ over them. (For more background we refer to Pigorsch and Stelzer (2009a).) Thus we assume given an $\mathrm{SVsupOU}(0, r, 0, \gamma_0, \nu, \pi)$ model with the volatility process Σ having finite second moments.

Note that we restrict ourselves to the non-leverage case and $a = 0$. The reason is that we want to calculate the second order properties as explicitly as possible which can only be done under these restrictions. If a_t is an affine function of Σ_t, some explicit results can still be obtained, see Pigorsch and Stelzer (2009a).

The subsequent log returns over time intervals of length $\Delta \in \mathbb{R}^{++}$ are denoted by $Y = (Y_n)_{n\in\mathbb{N}}$. In many financial applications the time intervals, i.e. $[(n-1)\Delta, n\Delta]$ with $n \in \mathbb{N}$, will represent trading days, for example. The logarithmic price increments and the integrated volatilities are defined by

$$Y_n := X_{n\Delta} - X_{(n-1)\Delta} = \int_{(n-1)\Delta}^{n\Delta} r(\Sigma_t) dW_t, \; \mathbf{V}_n := \int_{(n-1)\Delta}^{n\Delta} \Sigma_t dt, \; n \in \mathbb{N}.$$

It is clear that $Y_n | \mathbf{V}_n \sim N_d(0, \mathbf{V}_n)$.

As the twice integrated autocovariance function of the stationary volatility process Σ will be of particular importance, we define

$$\begin{aligned} r^{++}(t) &:= \int_0^t \int_0^u \mathrm{cov}(\Sigma_h, \Sigma_0) dh du \\ &= -\int_0^t \int_0^u \int_{M_d^-} e^{(A\otimes I_d + I_d \otimes A)h} (\mathscr{A}(A))^{-1} \left(\int_{\mathbb{S}^d} \mathrm{vec}(x)\mathrm{vec}(x)^* \nu(dx) \right) \pi(dA) dh du. \end{aligned} \tag{3.2}$$

Using Fubini and

$$\int_0^t \int_0^u e^{(A\otimes I_d+I_d\otimes A)h} dh du = (A\otimes I_d + I_d \otimes A)^{-2} e^{(A\otimes I_d+I_d\otimes A)t} - (A\otimes I_d + I_d\otimes A)^{-2} - (A\otimes I_d + I_d \otimes A)^{-1} t$$

one can immediately integrate with respect to u and h.

Theorem 3.4. *Let X, Σ follow an SVsupOU$(0, r, 0, \gamma_0, \nu, \pi)$ model with the volatility process Σ having finite second moments. Then $(\mathbf{V}_n)_{n\in\mathbb{N}}$ is stationary and square-integrable with*

$$E(\mathbf{V}_1) = -\Delta \int_{M_d^-} \mathbf{A}(A)^{-1} \left(\gamma_0 + \int_{\mathbb{S}_d} x \nu(dx) \right) \pi(dA), \tag{3.3}$$

$$\mathrm{var}(\mathrm{vec}(\mathbf{V}_1)) = r^{++}(\Delta) + r^{++}(\Delta)^*, \tag{3.4}$$

$$\begin{aligned} \mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1)) &= r^{++}(h\Delta + \Delta) - 2r^{++}(h\Delta) + r^{++}(h\Delta - \Delta) \\ &= -\int_{M_d^-} g(A,h) (\mathscr{A}(A))^{-1} \left(\int_{\mathbb{S}_d} \mathrm{vec}(x)\mathrm{vec}(x)^* \nu(dx) \right) \pi(dA),\ h \in \mathbb{N}, \end{aligned} \tag{3.5}$$

with $g(A,h) = (A\otimes I_d + I_d\otimes A)^{-2} \left(e^{(A\otimes I_d+I_d\otimes A)(h\Delta+\Delta)} - 2e^{(A\otimes I_d+I_d\otimes A)h\Delta} + e^{(A\otimes I_d+I_d\otimes A)(h\Delta-\Delta)} \right)$. *It holds that* $\lim_{h\to\infty} \mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1)) = 0$.

Likewise the log-price increments $(Y_n)_{n\in\mathbb{N}}$ as well as their "squares" $(Y_n Y_n^*)_{n\in\mathbb{N}}$ are stationary and square-integrable with

$$E(Y_1) = 0, \quad \mathrm{var}(Y_1) = E(\mathbf{V}_1), \quad \mathrm{cov}(Y_{h+1}, Y_1) = 0 \,\forall\, h \in \mathbb{N}, \tag{3.6}$$

$$E(Y_1 Y_1^*) = E(\mathbf{V}_1), \tag{3.7}$$

$$\mathrm{var}(\mathrm{vec}(Y_1 Y_1^*)) = (I_{d^2} + \mathbf{Q} + \mathbf{PQ}) \left(r^{++}(\Delta) + r^{++}(\Delta)^* \right) + (I_{d^2} + \mathbf{P}) \left(E(\mathbf{V}_1) \otimes E(\mathbf{V}_1) \right) \tag{3.8}$$

$$\mathrm{cov}\left(\mathrm{vec}(Y_{h+1} Y_{h+1}^*),\ \mathrm{vec}(Y_1 Y_1^*) \right) = \mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1)) \ \textit{for}\ h \in \mathbb{N} \tag{3.9}$$

where

$$\mathbf{P} : M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R}),\ (\mathbf{P}X)_{i,(p-1)d+q} = X_{i,(q-1)d+p} \ \textit{for all}\ i = \{1,2,\ldots,d^2\},\ p,q = \{1,2,\ldots d\}$$

$$\mathbf{Q} : M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R}),\ (\mathbf{Q}X)_{(k-1)d+l,(p-1)d+q} = X_{(k-1)d+p,(l-1)d+q} \ \textit{for all}\ k,l,p,q = \{1,2,\ldots d\}$$

are linear operators. Obviously, $\mathbf{P}^{-1} = \mathbf{P}$, $\mathbf{Q}^{-1} = \mathbf{Q}$ and $\mathbf{P}$ is representable as $X \mapsto XP$ with $P \in M_{d^2}(\mathbb{R})$ being a permutation matrix. Moreover, $\mathbf{Q}(\mathrm{vec}(X)\mathrm{vec}(Z)^*) = X \otimes Z$ for all $X, Z \in \mathbb{S}_d$.

Proof. Follows from Pigorsch and Stelzer (2009a, Theorems 3.2, 3.3), Proposition 2.9 and straightforward calculations. That $\lim_{h\to\infty} \mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1)) = 0$ is implied by $\lim_{h\to\infty} \mathrm{cov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0)) = 0$ and the identity

$$\mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1)) = \int_{h\Delta}^{(h+1)\Delta} \int_0^\Delta \mathrm{cov}(\mathrm{vec}(\Sigma_{s-u}), \mathrm{vec}(\Sigma_0)) du ds, \tag{3.10}$$

see Pigorsch and Stelzer (2009a, Proof of Theorem 3.2). □

The relevance of these results is that they provide the basis for (general) method of moments based estimation of the model as in Pigorsch and Stelzer (2009a) for the multivariate SVOU model and Barndorff-Nielsen and Shephard (2010) for the univariate SVsupOU model.

3.2.1 Long memory

Before presenting examples of SVsupOU models which exhibit long memory in the log returns (polynomially decaying autocovariance function of the squared returns), we show that in the exponential and polynomially decaying case, the asymptotic behaviour of the autocovariance function of the integrated volatility (and thus the squared returns) is of the same type as the one of the volatility process (not strictly, though, in the exponential case). The result below is stated for the one-dimensional case, but it immediately extends to the eigenvalues of the matrices dependent on h involved in the general case, as will be illustrated in the examples. Likewise, the result can be applied to the individual components of the autocovariance matrix in a multivariate model.

Proposition 3.5. *Assume given an SVsupOU*$(0,r,0,\gamma_0,\nu,\pi)$ *model in dimension* $d=1$.
(i) If $\mathrm{cov}(\mathrm{vec}(\Sigma_h),\mathrm{vec}(\Sigma_0)) \sim Ch^{-\alpha}$ *for* $h\to\infty$ *with* $\alpha>0$ *and* $C\in\mathbb{R}\backslash\{0\}$, *then*

$$\mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}),\mathrm{vec}(\mathbf{V}_1)) \sim C\Delta^{2-\alpha}h^{-\alpha} \text{ for } h\to\infty. \tag{3.11}$$

(ii) If $\mathrm{cov}(\mathrm{vec}(\Sigma_h),\mathrm{vec}(\Sigma_0)) \sim Ce^{-\alpha h}$ *with* $\alpha>0$ *and* $C\in\mathbb{R}\backslash\{0\}$, *then*

$$\liminf_{h\to\infty}\left|\frac{\mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}),\mathrm{vec}(\mathbf{V}_1))}{C\Delta^2 e^{-\alpha(h\Delta+\Delta)}}\right|\geq 1, \quad \limsup_{h\to\infty}\left|\frac{\mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}),\mathrm{vec}(\mathbf{V}_1))}{C\Delta^2 e^{-\alpha(h\Delta-\Delta)}}\right|\leq 1.$$

Proof. (i): Without loss of generality we assume $C>0$. For any $\varepsilon>0$ there is an $h^*\in\mathbb{R}^+$ such that $(1-\varepsilon)Ch^{-\alpha}\leq \mathrm{cov}(\mathrm{vec}(\Sigma_h),\mathrm{vec}(\Sigma_0))\leq(1+\varepsilon)Ch^{-\alpha}$. Using (3.10) we obtain for all $h\geq(h^*/\Delta)+1$

$$\mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}),\mathrm{vec}(\mathbf{V}_1)) \leq (1+\varepsilon)C\int_{h\Delta}^{(h+1)\Delta}\int_0^{\Delta}(s-u)^{-\alpha}duds \leq (1+\varepsilon)C\Delta^2(h\Delta-\Delta)^{-\alpha}.$$

Hence,

$$\limsup_{h\to\infty}\frac{\mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}),\mathrm{vec}(\mathbf{V}_1))}{C\Delta^{2-\alpha}h^{-\alpha}}\leq 1+\varepsilon$$

and likewise one has

$$\liminf_{h\to\infty}\frac{\mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}),\mathrm{vec}(\mathbf{V}_1))}{C\Delta^{2-\alpha}h^{-\alpha}}\geq 1-\varepsilon.$$

Since ε was arbitrary, this gives (3.11).

(ii): The proof is similar to that of (i) and therefore left out. □

Remark 3.6. *As the proof shows, Proposition 3.5 is valid not only for SVsupOU models, but for the general stochastic volatility model as defined in Pigorsch and Stelzer (2009a).*

This implies that if the spot volatility has long memory (in the sense that it decays like $h^{-\alpha}$ with $\alpha\in(0,1)$), then the integrated volatility increments $\mathbf{V}$ and the "squared returns" YY^* have also long memory.

Unlike in the univariate case, so far no detailed theory for long memory exists for multivariate stochastic processes/observations. Below, we speak of long memory whenever at least one of the components of the autocovariance functions decays asymptotically like $h^{-\alpha}$ for some $\alpha\in(0,1)$ and with the lag h going to infinity. Clearly, this is a case when one may adequately speak of long memory.

Now we consider several examples of SVsupOU$(0,r,0,\gamma_0,\nu,\pi)$ models.

Example 3.1. Let π be given as the distribution of RB with a diagonalisable $B \in M_d^-$ and R being a real $\Gamma(\alpha,\beta)$-distributed random variable with $\alpha > 1, \beta \in \mathbb{R}^+ \backslash \{0\}$. Like in Barndorff-Nielsen and Stelzer (2009, Example 3.1) one shows that the positive semidefinite OU type process Σ exists, is stationary and has finite second moments.

Using spectral matrix calculus, one obtains for the autocovariance function at positive lags h:

$$\mathrm{cov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0)) = -\frac{\beta^\alpha}{\alpha-1}\left(\beta I_{d^2} - (B\otimes I_d + I_d \otimes B)h\right)^{1-\alpha} \mathscr{B}^{-1}\left(\int_{\mathbb{S}^d} \mathrm{vec}(x)\mathrm{vec}(x)^* \nu(dx)\right)$$

with $\mathscr{B} : M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R}), X \mapsto (B\otimes I_d + I_d\otimes B)X + X(B^*\otimes I_d + I_d \otimes B^*)$ and thus we have a polynomially decaying autocovariance function. For $\alpha \in (1,2)$ we obviously get long memory. To see this easily, it should be noted that, if λ is an eigenvalue of $B\otimes I_d + I_d \otimes B$, then $(\beta - \lambda h)^{1-\alpha}$ is an eigenvalue of $\left(\beta I_{d^2} - (B\otimes I_d + I_d\otimes B)h\right)^{1-\alpha}$. Hence, the eigenvalues of this matrix decay polynomially, which implies that the elements of $\left(\beta I_{d^2} - (B\otimes I_d + I_d\otimes B)h\right)^{1-\alpha}$ and in turn the elements of $\mathrm{cov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0))$ decay polynomially at rate $1-\alpha$ unless they are constantly zero in h.

To calculate the autocovariance function of V and YY^* one could use the equation (3.5). However, when one tries to explicitly calculate the integral over M_d^- using spectral calculus (similar to Barndorff-Nielsen and Stelzer (2009, Example 3.1)) one can only do this (without using additional analytic tricks) for $\alpha > 3$. Instead, we note that by (3.10) we have

$$\begin{aligned}
&\mathrm{cov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1)) = \mathrm{cov}(\mathrm{vec}(Y_{h+1}Y_{h+1}^*), \mathrm{vec}(Y_1Y_1^*)) \\
&\quad = -\frac{\beta^\alpha}{\alpha-1}\int_{h\Delta}^{(h+1)\Delta}\int_0^\Delta \left(\beta I_{d^2} - (B\otimes I_d + I_d\otimes B)(s-u)\right)^{1-\alpha} du ds \mathscr{B}^{-1}\left(\int_{\mathbb{S}^d}\mathrm{vec}(x)\mathrm{vec}(x)^*\nu(dx)\right).
\end{aligned}$$

If $\lambda_1, \ldots, \lambda_{d^2}$ are the eigenvalues of $(B\otimes I_d + I_d \otimes B)$, then using spectral calculus and Proposition 3.5 the eigenvalues of $\Gamma_h := \int_{h\Delta}^{(h+1)\Delta}\int_0^\Delta \left(\beta I_{d^2} - (B\otimes I_d + I_d\otimes B)(s-u)\right)^{1-\alpha} du ds$ are asymptotically equal to $(-\lambda_i)^{1-\alpha}\Delta^{3-\alpha}h^{1-\alpha}$ for $h\to\infty$, which shows that we again have a polynomial decay (and long memory for $\alpha \in (1,2)$).

Actually, we can also calculate the integral explicitly via spectral calculus. However, from the results below it appears to be rather non-trivial to see the asymptotic decay. Setting $\mathfrak{B} = (B\otimes I_d + I_d\otimes B)$ we have

$$\begin{aligned}
\Gamma_h =& \frac{\mathfrak{B}^{-2}\left((\beta I_{d^2} - \mathfrak{B}(h\Delta+\Delta))^{3-\alpha} - 2(\beta I_{d^2} - \mathfrak{B}h\Delta)^{3-\alpha} + (\beta I_{d^2} - \mathfrak{B}(h\Delta-\Delta))^{3-\alpha}\right)}{(2-\alpha)(3-\alpha)}, \qquad \alpha \neq 2,3 \\
\Gamma_h =& \mathfrak{B}^{-2}\big((\beta I_{d^2} - \mathfrak{B}(h\Delta+\Delta))\mathrm{Log}(\beta I_{d^2} - \mathfrak{B}(h\Delta+\Delta)) \\
&- 2(\beta I_{d^2} - \mathfrak{B}h\Delta)\mathrm{Log}(\beta I_{d^2} - \mathfrak{B}h\Delta) + (\beta I_{d^2} - \mathfrak{B}(h\Delta-\Delta))\mathrm{Log}(\beta I_{d^2} - \mathfrak{B}(h\Delta-\Delta))\big), \qquad \alpha = 2 \\
\Gamma_h =& \frac{\mathfrak{B}^{-2}\left(\mathrm{Log}(\beta I_{d^2} - \mathfrak{B}(h\Delta+\Delta)) - 2\mathrm{Log}(\beta I_{d^2} - \mathfrak{B}h\Delta) + \mathrm{Log}(\beta I_{d^2} - \mathfrak{B}(h\Delta-\Delta))\right)}{(2-\alpha)}, \qquad \alpha = 3.
\end{aligned}$$

Log denotes the main branch of the complex logarithm. Observe that $\sigma(\mathfrak{B}) = \sigma(B) + \sigma(B)$ ensures that all eigenvalues of $\beta I_{d^2} - \mathfrak{B}s$ are in the right half plane for all $s \in \mathbb{R}^+$.

At a first sight, the above formulae suggest different decay rates for Γ_h than a polynomial decay with rate $1-\alpha$. Straightforward calculations give, however, that, for instance, for $\alpha \neq 2,3$ we have $\lim_{h\to\infty}\Gamma_h/h^{3-\alpha} = 0$.

The above results can be easily extended to the case where π is concentrated on finitely many diagonalisable rays or concentrated on the negative definite matrices and specified in terms of a measure on the unit sphere and a Γ-distributed kernel for the radial part (similar to Barndorff-Nielsen and Stelzer (2009, Examples 3.2, 3.3)). However, as in these cases everything can be calculated, by straightforward combinations of the above Example 3.1 and arguments from Barndorff-Nielsen and Stelzer (2009, Examples 3.2, 3.3), and as the resulting formulae for the autocovariances are simply sums or integrals over terms of the form obtained in the above Example 3.1, we refrain from giving details.

Example 3.2. Let Λ be now an $\mathbb{S}_2^+$-valued Lévy basis with π concentrated on $\mathbb{D}_2^{--}$, the 2 x 2 diagonal matrices with strictly negative entries on the diagonal. We identify $\mathbb{D}_2^{--}$ with $(\mathbb{R}^{--})^2$ and we assume that π has Lebesgue density

$$\pi(da_1, da_2) = \frac{\beta_1^{\alpha_1}\beta_2^{\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}(-a_1)^{\alpha_1-1}(-a_2)^{\alpha_2-1}e^{\beta_1 a_1+\beta_2 a_2}1_{(\mathbb{R}^{--})^2}(a_1,a_2)da_1da_2$$

with $\alpha_1, \alpha_2 > 1$ and $\beta_1, \beta_2 > 0$. So the diagonal elements are independent and their absolute values follow Gamma distributions. Using the arguments from Barndorff-Nielsen and Stelzer (2009, Example 3.4), one can show that the positive semidefinite supOU process Σ exists, is stationary and has finite second moments.

Let us now consider the individual components $\Sigma_{11,t}$, $\Sigma_{22,t}$ and $\Sigma_{12,t}$ of Σ_t. Denote by

$$P_{ij} : \mathbb{S}_2 \to \mathbb{R}, \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \mapsto x_{ij},$$

the projection onto the ij-th coordinate with $i, j \in \{1,2\}$, $i \le j$, and define $\mathbb{R}^+$-valued Lévy bases Λ_{ii} on $\mathbb{R}^{--} \times \mathbb{R}$ via $\Lambda_{ii}(da_i, ds) = P_{ii}(\Lambda(P_{ii}^{-1}(da_i), ds)$ and a Lévy measure ν_{ii} on $\mathbb{R}$ via $\nu_{ii}(dx_{ii}) = \nu(P_{ii}^{-1}(dx_{ii}))$. Then Λ_{ii} has characteristic triplet $(\gamma_{ii}, \nu_{ii}, \pi_i)$ with π_i having Lebesgue density

$$\pi_i(da_i) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i))}(-a_i)^{\alpha_i - i}e^{\beta_i a_i}1_{(\mathbb{R}^{--})}(a_i)da_i$$

and

$$\Sigma_{ii,t} = \int_{\mathbb{R}^{--}}\int_{-\infty}^{t} e^{2a_i(t-s)}\Lambda_{ii}(da_i, ds), \tag{3.12}$$

$$\Sigma_{12,t} = \int_{\mathbb{R}^{--}}\int_{\mathbb{R}^{--}}\int_{-\infty}^{t} e^{(a_1+a_2)(t-s)}(P_{12}(\Lambda))(da_1, da_2, ds). \tag{3.13}$$

It is now also straightforward to see by substituting $2a_i$ with $\tilde{a}_i$ that Σ_{ii} is an $\mathbb{R}^+$-valued supOU process of the form

$$\Sigma_{ii,t} = \int_{\mathbb{R}^{--}}\int_{-\infty}^{t} e^{a_i(t-s)}\tilde{\Lambda}_{ii}(da_i, ds)$$

with $\tilde{\Lambda}$ having characteristic triplet $(\gamma_{ii}, \nu_{ii}, -\Gamma(\alpha_i, \beta_i/2))$. Here, $-\Gamma(\alpha, \beta)$ denotes the probability distribution of the random variable $-X$ when X has a $\Gamma(\alpha, \beta)$-distribution.

For the autocovariance function of the variance components we thus get

$$\operatorname{cov}(\Sigma_{ii,h}, \Sigma_{ii,0}) = \frac{(\beta_i/2)^{\alpha_i}}{2(\alpha_i - 1)}((\beta_i/2) + h)^{1-\alpha_i}\left(\int_{\mathbb{R}} x_1^2 \nu_1(dx_1)\right), \; h \in \mathbb{R}^+.$$

In particular, we have long memory in the i-th variance component provided $\alpha_i \in (1,2)$. Using Proposition 3.5 and the upcoming results of Section 4.2 which give that X_i is in distribution equal to a one-dimensional SVsupOU$(0,\sqrt{\cdot},0,\gamma_{0,ii},\nu_{ii},-\Gamma(\alpha_i,\beta_i/2))$-model, we get for the integrated variances and squared returns

$$\operatorname{cov}(\mathbf{V}_{ii,h+1},\mathbf{V}_{ii,1})=\operatorname{cov}(Y_{i,h+1}^2,Y_{i,1}^2)\sim\frac{(\beta_i/2)^{\alpha_i}\Delta^{3-\alpha_i}\int_{\mathbb{R}}x_1^2\nu_1(dx_1)}{2(\alpha_i-1)}h^{1-\alpha_i}\text{ as }h\to\infty.$$

These covariances can also easily be calculated explicitly using the above formulae, but we refrain from stating them.

The importance of this example is that it allows one to choose ν_{ii} such that the stationary distribution of the variance Σ_{ii} is a certain prescribed selfdecomposable distribution concentrated on $\mathbb{R}^+$ (using Barndorff-Nielsen (2001, Theorem 3.1, Corollary 3.1), Fasen and Klüppelberg (2007, Remark 2.2) or Pigorsch and Stelzer (2009b, Theorem 4.9)). Then one only needs to combine the margins ν_{ii} in a suitable way to a Lévy measure ν on $\mathbb{S}_d^+$ in order to construct an SVsupOU model where the univariate margins of the stationary distribution of Σ are the prescribed ones.

Obviously this example has a straightforward extension to general dimension d.

4 Linear transformations and marginalisations

Now we study the effects of linear transformations and marginalisations on multivariate (sup)OU stochastic volatility models. Note that these models fall into the framework of so-called CGPII-processes studied in Barndorff-Nielsen and Pedersen (2009). In that paper the question of stability under linear transformations was also raised and answered positively. However, in our special set-up we can show more refined results by rather elementary calculations. In particular, we show below that under invertible linear transformations (sup)OU models are again (sup)OU with different parameters, but the same Wiener process and the driving Lévy process or Lévy basis being defined ω-wise in terms of the original one. Moreover, we are able to treat the case with a general drift and leverage term.

4.1 Ornstein-Uhlenbeck-type stochastic volatility models

We first analyse OU type models (i.e. without superposition) separately in this section, as this emphasises the gist of our results.

Theorem 4.1. *The classes of multivariate OU type SV models with and without leverage are each preserved under invertible linear transformations, i.e. if X is* SVOU(a,r,ψ,A,L) *and $B\in GL_d(\mathbb{R})$, then $Z=BX$ is* SVOU$(\tilde{a},\tilde{r},\tilde{\psi},\tilde{A},\tilde{L})$*) with:*

$$\tilde{a}_t=Ba_t\,\forall t\in\mathbb{R}^+,\quad \tilde{r}:\mathbb{S}_d^+\to M_d(\mathbb{R}),X\mapsto Br(B^{-1}XB^{-*}),\qquad \tilde{\psi}\in L(\mathbb{S}_d,\mathbb{R}^d):X\mapsto B\psi(B^{-1}XB^{-*})$$
$$\tilde{A}=BAB^{-1},\quad \tilde{L}_t=BL_tB^*\,\forall t\in\mathbb{R}^+.$$

Proof. Follows immediately from

$$\begin{aligned}dZ_t&=BdX_t=Ba_tdt+\tilde{V}_{t-}dW_t+B\psi\left(B^{-1}d(BL_tB^*)B^{-*}\right),\\ \tilde{V}_t&:=Br(\Sigma_t),\quad \tilde{\Sigma}_t:=\tilde{V}_t\tilde{V}_t^*=Br(\Sigma_t)r(\Sigma_t)^*B^*,\\ d\tilde{\Sigma}_t&=Bd\Sigma_tB^*=(BAB^{-1}\tilde{\Sigma}_{t-}+\tilde{\Sigma}_{t-}B^{-*}A^*B^*)dt+d(BL_tB^*)\end{aligned}$$

and the fact that the function given by $X\mapsto B\psi(B^{-1}XB^{-*})$ is in $L(\mathbb{S}_d,\mathbb{R}^d)$ and that $(BL_tB^*)_{t\in\mathbb{R}^+}$ is again a matrix subordinator. Furthermore, note that $\tilde{V}_t=Br(B^{-1}\tilde{\Sigma}_tB^{-*})$. □

Remark 4.2. *(i) The above result means that when one takes a different set of securities to linearly span the same market one remains within the same class of models. This is clearly a very desirable property.*

(ii) A case of some particular interest for modelling is given by $a_t = \mu t + \beta(\Sigma_{t-})$ with $\beta \in L(\mathbb{S}_d, \mathbb{R}^d)$ and $\mu \in \mathbb{R}^d$. A straightforward variant of the above proof shows that the class of multivariate OU type SV models (with leverage) with such a drift term is preserved under invertible linear transformations. This remains true when μ or β or both are fixed to the value zero.

Next we turn our focus on studying the effects of marginalisation in OU type SV models. Assuming $a = 0$ and $\psi = 0$ for the sake of simplicity, note that the stochastic variance process of the first component X_1, for instance, is given by Σ_{11}, but $X_{1,t} = \int_0^t \sum_{i=1}^d (\Sigma_s^{1/2})_{1,i} dW_{i,s} \neq \int_0^t (\Sigma_{11,s})^{1/2} dW_{1,s}$ (as random variables defined on the common probability space). However, this is different when we do not demand equality of the processes in a strong sense on the same probability space, but only equality of the distributions.

Proposition 4.3. *Assume that X is $\mathrm{SVOU}(a,r,\psi,A,L)$ and that (a,L) is independent of W. Then it holds for the first one-dimensional component X_1 that*

$$(X_{1,t})_{t\in\mathbb{R}^+} \overset{\mathscr{D}}{=} \left(\int_0^t a_{1,s} ds + \int_0^t (\Sigma_{11,s})^{1/2} dW_{1,s} + \int_0^t \psi_1(dL_t) \right)_{t\in\mathbb{R}^+}.$$

with $\psi_1 := P\psi$ where $P : \mathbb{R}^d \to \mathbb{R}, (x_1, x_2, \ldots, x_p)^* \mapsto x_1$.

Proof. Due to the independence of W and (a,L), it suffices to show that

$$(X_{1,t})_{t\in\mathbb{R}^+} | (a,L) \overset{fidi}{=} \left(\int_0^t a_{1,s} ds + \int_0^t (\Sigma_{11,s})^{1/2} dW_{1,s} \right)_{t\in\mathbb{R}^+} \Bigg| (a,L)$$

with $\overset{fidi}{=}$ denoting equality of all finite dimensional distributions. The latter is immediate from

$$\begin{aligned} X_{1,t}|(a,L) &= \int_0^t a_{1,s} ds + \sum_{i=1}^d \int_0^t r(\Sigma_s)_{1,i} dW_{i,s} \Bigg| (a,L) \overset{\mathscr{D}}{=} N\left(\int_0^t a_{1,s} ds, \int_0^t \Sigma_{11,s} ds \right) \\ &\overset{\mathscr{D}}{=} \int_0^t a_{1,s} ds + \int_0^t (\Sigma_{11,s})^{1/2} dW_{1,s} \Bigg| (a,L) \quad \forall t \in \mathbb{R}, \\ \begin{pmatrix} X_{1,t} \\ X_{1,\tilde{t}} \end{pmatrix} \Bigg| (a,L) &\overset{\mathscr{D}}{=} N\left(\begin{pmatrix} \int_0^t a_{1,s} ds \\ \int_0^{\tilde{t}} a_{1,s} ds \end{pmatrix}, \int_0^\infty \begin{pmatrix} \Sigma_{11,s} 1_{\{s\le t\}} & \Sigma_{11,s} 1_{\{s\le \min\{t,\tilde{t}\}\}} \\ \Sigma_{11,s} 1_{\{s\le \min\{t,\tilde{t}\}\}} & \Sigma_{11,s} 1_{\{s\le \tilde{t}\}} \end{pmatrix} ds \right) \\ &\overset{\mathscr{D}}{=} \begin{pmatrix} \int_0^t a_{1,s} ds + \int_0^t (\Sigma_{11,s})^{1/2} dW_{1,s} \\ \int_0^{\tilde{t}} a_{1,s} ds + \int_0^{\tilde{t}} (\Sigma_{11,s})^{1/2} dW_{1,s} \end{pmatrix} \Bigg| (a,L) \quad \forall t, \tilde{t} \in \mathbb{R}^+ \end{aligned}$$

and likewise for all higher dimensional distributions. □

Remark 4.4. *(i) Combining Proposition 4.3 with Pigorsch and Stelzer (2009a, Section 3.3) shows that, if X is $\mathrm{SVOU}(0,r,0,A,L)$ with the matrix A being (real) diagonalisable, then the first component X_1 equals in distribution a driftless stochastic volatility model with the volatility process being a superposition of possibly dependent (real) univariate OU type processes. Likewise, if a is not zero but $a_t = (\mu_1 + \beta_1 \Sigma_{11,t}, \mu_2 + \beta_2 \Sigma_{22,t}, \ldots, \mu_d + \beta_d \Sigma_{dd,t})^*$, then the first component X_1 equals in distribution a stochastic volatility model with the volatility process Σ_{11} being a superposition of possibly dependent (real) univariate OU type processes and the drift being $\mu_1 + \beta_1 \Sigma_{11}$.*

(ii) The same effect arises in the Hubalek-Nicolato model (see Hubalek and Nicolato (2009)). The equality of the marginal processes as supOU BNS models, stated in Section 3.2.2 of that paper, is to be understood in distribution.

The above results apply, of course, not only to the first but also to all other components. Moreover, Proposition 4.3 can easily be generalised to the joint behaviour of several components as follows. If $x \in \mathbb{R}^d$, $z \in M_d(\mathbb{R})$ and $\mathfrak{I} \subseteq \{1,2,\ldots,d\}$, then we define $x_{\mathfrak{I}} := (x_i)_{i\in\mathfrak{I}} \in \mathbb{R}^{|\mathfrak{I}|}$ and $z_{\mathfrak{I}} := (z_{ij})_{i,j\in\mathfrak{I}} \in M_{|\mathfrak{I}|}(\mathbb{R})$.

Proposition 4.5. *Assume that X is $\mathrm{SVOU}(a,r,\psi,A,L)$ and that (a,L) is independent of W. Then it holds for any $\mathfrak{I} \subset \{1,2,\ldots,d\}$ that*

$$(X_{\mathfrak{I},t})_{t\in\mathbb{R}^+} \stackrel{\mathscr{D}}{=} \left(\int_0^t a_{\mathfrak{I},s}ds + \int_0^t (\Sigma_{\mathfrak{I},s})^{1/2} dW_{\mathfrak{I},s} + \int_0^t \psi_{\mathfrak{I}}(dL_t)\right)_{t\in\mathbb{R}^+}$$

with $\psi_{\mathfrak{I}} := P_{\mathfrak{I}}\psi$ where $P_{\mathfrak{I}} : \mathbb{R}^d \to \mathbb{R}^{|\mathfrak{I}|}, (x_1,x_2,\ldots,x_p)^* \mapsto x_{\mathfrak{I}}$.

Remark 4.6. *Theorem 4.1 has an obvious extension to injective linear transformations from $\mathbb{R}^d \to \mathbb{R}^m$ with $m > d$ which correspond to $m\times d$ matrices B of full rank, because these functions as well as the maps $\mathbb{S}_d^+ \to \mathbb{S}_m^+, X \mapsto BXB^*$ are invertible on the image sets.*

A surjective linear transformation from $\mathbb{R}^d \to \mathbb{R}^m$ with $m < d$, which corresponds to an $m\times d$ matrix B with full rank, can be written as an invertible linear transformation from $\mathbb{R}^d$ to $\mathbb{R}^d$ followed by a projection on the first m coordinates. Thus, a combination of Theorem 4.1 and Proposition 4.5 shows what happens to our stochastic volatility model under such a transformation.

The only remaining type of linear transformations are those being neither injective nor surjective (thus corresponding to a matrix having non-full rank r). However, such linear transformations B can always be represented as $B = B_2PB_1$ with $B_1 : \mathbb{R}^d \to \mathbb{R}^d$ invertible, P being the projection on the first r coordinates and $B_2 : \mathbb{R}^r \to \mathbb{R}^m$ being injective. Theorem 4.1 and Proposition 4.5 show the effects of B_1 and P. Moreover, if

$$(X_{\mathfrak{I},t})_{t\in\mathbb{R}^+} \stackrel{\mathscr{D}}{=} \left(\int_0^t a_{\mathfrak{I},s}ds + \int_0^t (\Sigma_{\mathfrak{I},s})^{1/2} dW_{\mathfrak{I},s} + \int_0^t \psi_{\mathfrak{I}}(dL_t)\right)_{t\in\mathbb{R}^+}.$$

then

$$B_2(X_{\mathfrak{I},t})_{t\in\mathbb{R}^+} \stackrel{\mathscr{D}}{=} \left(\int_0^t B_2 a_{\mathfrak{I},s}ds + \int_0^t B_2(\Sigma_{\mathfrak{I},s})^{1/2} dW_{\mathfrak{I},s} + \int_0^t B_2\psi_{\mathfrak{I}}(dL_t)\right)_{t\in\mathbb{R}^+}.$$

Hence, we have characterised how linear transformations act on SVOU models in all cases.

4.2 SupOU stochastic volatility models

Now we turn to the effects of linear transformations on supOU stochastic volatility models. From the proof given above it is immediate that Proposition 4.3 remains valid for SVsupOU models. The same is true for Proposition 4.5 and Remark 4.6 (replacing Theorem 4.1 with the following Theorem 4.7).

Hence, it suffices to consider invertible linear transformations.

Theorem 4.7. *The classes of multivariate supOU type SV models with and without leverage are each preserved under invertible linear transformations, i.e. if X is $\mathrm{SVsupOU}(a,r,\psi,\gamma_0,\nu,\pi)$ and $B \in GL_d(\mathbb{R})$, then $Z = BX$ is $\mathrm{SVsupOU}(\tilde{a},\tilde{r},\tilde{\psi},\tilde{\gamma}_0,\tilde{\nu},\tilde{\pi})$ with:*

$$\tilde{a}_t = Ba_t\, \forall t \in \mathbb{R}^+, \quad \tilde{r} : \mathbb{S}_d^+ \to M_d(\mathbb{R}), X \mapsto Br(B^{-1}XB^{-*}), \quad \tilde{\psi} \in L(\mathbb{S}_d, \mathbb{R}^d) : X \mapsto B\psi(B^{-1}XB^{-*})$$
$$\tilde{\gamma}_0 = B\gamma_0 B^*, \quad \tilde{\nu}(dx) = \nu(B^{-1}dxB^{-*}), \quad \tilde{\pi}(dA) = \pi(B^{-1}dAB).$$

Proof. In view of the proof of Theorem 4.1 it suffices to consider $B\Sigma_t B^*$ and $BL_t B^*$. We have

$$\begin{aligned} B\Sigma_t B^* &= \int_{M_d^-}\int_{-\infty}^t Be^{A(t-s)}\Lambda(dA,ds)e^{A^*(t-s)}B^* = \int_{M_d^-}\int_{-\infty}^t e^{BAB^{-1}(t-s)}B\Lambda(dA,ds)B^*e^{B^{-*}A^*B^*(t-s)} \\ &= \int_{M_d^-}\int_{-\infty}^t e^{A(t-s)}\tilde{\Lambda}(dA,ds)e^{A^*(t-s)} \end{aligned}$$

where $\tilde{\Lambda}(dA,ds) := B\Lambda(B^{-1}dAB,ds)B^*$ is again a Lévy basis satisfying the conditions of Theorem 2.7. Setting $f : M_d^- \times \mathbb{R} \to M_d^- \times \mathbb{R}$, $(A,s) \mapsto (BAB^{-1},s)$ we see that for any $C \in \mathscr{B}_b(M_d^- \times \mathbb{R})$

$$\begin{aligned} E\left((\exp\left(i\mathrm{tr}(u\tilde{\Lambda}(C))\right)\right) =& E\left((\exp\left(i\mathrm{tr}(uB\Lambda(f^{-1}(C))B^*)\right)\right) = E\left((\exp\left(\varphi(B^*uB)(\pi\times\lambda)(f^{-1}(C))\right)\right) \\ =& E\left((\exp\left(\tilde{\varphi}(u)(\tilde{\pi}\times\lambda)(C)\right)\right) \end{aligned}$$

with $\tilde{\pi}(dA) = \pi(B^{-1}dAB)$ and $\tilde{\varphi}(u) = i\mathrm{tr}(u\tilde{\gamma}_0) + \int_{\mathbb{S}_d}\left(e^{i\mathrm{tr}(ux)} - 1\right)\tilde{\nu}(dx)$ where $\tilde{\gamma}_0 = B\gamma_0 B^*$ and $\tilde{\nu}(dx) = \nu(B^{-1}dxB^{-*})$.

Observing that $BL_tB^* = B\Lambda(M_d^- \times (0,t])B^* = \tilde{\Lambda}(M_d^- \times (0,t]) =: \tilde{L}_t$ now concludes the proof. □

Observe that the analogue of Remark 4.2 is also still valid.

5 Positive semidefinite OU type processes and factor modelling

In high-dimensions one typically reduces the complexity of models by introducing a small number of factors supposed to explain the dependencies between the individual assets. Typically, these factors are of a macroeconomic type or are representing the state of a branch of industry etc. Such factor modelling approaches can be included into positive semidefinite OU type models in several ways. Below we briefly outline two of them, noting that they have straightforward generalisations to the supOU case.

5.1 Factor modelling in the matrix subordinator

One possible approach is to specify the matrix subordinator L (or likewise the positive semidefinite Lévy basis in the supOU case) by using a factor approach. For instance, assume that $L^{(1)}$ is a $d\times d$ diagonal matrix subordinator with independent components and $L^{(2)}$ is a $k\times k$ matrix subordinator independent of $L^{(1)}$. Specifying the driving matrix subordinator L of an SVOU model as

$$L = L^{(1)} + FL^{(2)}F^*$$

with $F \in M_{d,k}(\mathbb{R})$ clearly introduces a factor structure into the shocks of the volatility process. In this model $L^{(1)}$ can be understood to resemble the shocks which are due to news affecting only one company, noting that by independence its components never jump together. Likewise $L^{(2)}$ can be interpreted as the shocks in macroeconomic and/or industry related variables, i.e. it resembles news affecting many/all companies at the same time. The matrix F can be understood as the factor loadings matrix of the individual companies with respect to the common factors in $L^{(2)}$. It should be noted that we have made basically no restrictions on $L^{(2)}$ so one can have a very sophisticated dependence structure for the common factors as well as orthogonal factors.

Provided the parameter A is chosen to be diagonal, the stochastic correlations

$$\rho_{ij,t} = \frac{\Sigma_{ij,t}}{\sqrt{\Sigma_{ii,t}\Sigma_{jj,t}}}$$

remain constant as long as there is no jump in L. However, jumps in both $L^{(1)}, L^{(2)}$ affect the correlations.

5.2 Factor modelling in the positive semidefinite OU type process

Another possible approach is the use of a positive semidefinite Ornstein-Uhlenbeck process as the stochastic factor volatility process. In this case one would choose Σ^F to be an $l \times l$ positive semidefinite Ornstein-Uhlenbeck process and the stochastic volatility process Σ is defined by

$$\Sigma = F\Sigma^F F^* \tag{5.1}$$

with $F \in M_{d,l}(\mathbb{R})$ being the factor loadings matrix.

An interesting special case of this specification arises by considering $l = d + k$ with $k \in \mathbb{N}$ and by specifying Σ in such a way that

$$\Sigma^F = \begin{pmatrix} \Sigma_{11} & 0 & \cdots & 0 & 0 \\ 0 & \Sigma_{22} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \Sigma_{dd} & 0 \\ 0 & \cdots & \cdots & 0 & \Sigma^{F,k} \end{pmatrix}$$

with the $\mathbb{S}_k$-valued OU type process $\Sigma^{F,k} \in$ and the univariate positive OU type processes Σ_{ii} for $i = 1, \dots, d$ all being independent.

Provided $F = (I_d, \tilde{F})$ the elements Σ_{ii} can be interpreted as the idiosyncratic factors of the stochastic volatility of the stocks, whereas $\Sigma^{F,k}$ resembles the common factors.

If $\Sigma^{F,k}$ is again diagonal and the components are independent, this model becomes the one of Hubalek and Nicolato (2009).

5.3 An Example

To illustrate the possibilities of factor modelling in connection with OU type processes further, we now consider a simple two dimensional example noting that extensions to higher dimensions are straightforward. Let $L^{(c)}, L^{(1)}, L^{(2)}$ be three independent univariate subordinators and $G \in \mathbb{S}_d^+$. Then we define the driving Lévy process L as

$$L_t = GL_t^{(c)} + \begin{pmatrix} L_t^{(1)} & 0 \\ 0 & L_t^{(2)} \end{pmatrix}.$$

Hence, we have a simple set-up of the form considered in Section 5.1. The common factor is given by $L^{(c)}$ and when it jumps there are jumps in the variances and the covariance, which are totally dependent, as the relation between the jump sizes is always given by G. $L^{(1)}, L^{(2)}$ are the idiosyncratic shocks to the individual variances. Whenever they have a jump, only the respective variance jumps.

For a general mean reversion matrix A, a jump in any of the Lévy processes $L^{(1)}, L^{(2)}$ may affect all components of the OU type process Σ in a continuous manner after the jump has occurred. So the easy factor interpretation works in general only for instantaneous changes, whereas the overall dependence structure may be considerably more complex, as it is also heavily influenced by A. Moreover, it should be noted that a jump in the idiosyncratic factor does not change the stochastic covariance Σ_{12}, but it reduces necessarily the absolute value of the stochastic correlation $\Sigma_{12}/\sqrt{\Sigma_{11}\Sigma_{22}}$.

Assume now that $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ is diagonal with $a_1, a_2 < 0$. Then a jump in $L^{(1)}$ or $L^{(2)}$ does not effect the other variance or the covariance component of Σ at all. Actually, we have in the stationary case

$$\Sigma_{ii,t} = G_{ii} \int_{-\infty}^{t} e^{2a_i(t-s)} dL_s^{(c)} + \int_{-\infty}^{t} e^{2a_i(t-s)} dL_s^{(i)},\ i = 1,2$$
$$\Sigma_{12,t} = G_{12} \int_{-\infty}^{t} e^{(a_1+a_2)(t-s)} dL_s^{(c)}.$$

Within some restrictions on can now, like in Example 3.2, use Barndorff-Nielsen (2001, Theorem 3.1, Corollary 3.1), Fasen and Klüppelberg (2007, Remark 2.2) or Pigorsch and Stelzer (2009b, Theorem 4.9) to choose the driving Lévy processes $L^{(c)}, L^{(1)}, L^{(2)}$ such that the components of Σ have prescribed selfdecomposable distributions. To do so, one will typically have to restrict oneself to one family of selfdecomposable distributions which is highly tractable and where especially the effects of scalings and convolution are well understood. Examples of such families include (subclasses of) the Gamma and inverse Gaussian distribution. A particular example is the following. Let $a_1 = a_2$ and $G = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and assume we want to have that

$$\Sigma_{12,t} \sim \Gamma(\alpha_{12}, \beta), \tag{5.2}$$
$$\Sigma_{11,t} \sim \Gamma(\alpha_{12} + \alpha_{11}, \beta), \tag{5.3}$$
$$\Sigma_{22,t} \sim \Gamma(\alpha_{12} + \alpha_{22}, \beta), \tag{5.4}$$

with $\alpha_{11}, \alpha_{22}, \alpha_{12}, \beta > 0$. Then one simply uses Barndorff-Nielsen (2001, Theorem 3.1, Corollary 3.1) (or one of the variants cited above) to determine $L^{(c)}$ such that (5.2) holds and $L^{(1)}, L^{(2)}$ such that

$$\int_{-\infty}^{t} e^{2a_i(t-s)} dL_s^{(i)} \sim \Gamma(\alpha_{ii}, \beta)$$

holds. Note that this can also be understood as factor modelling on the level of stationary distributions.

Acknowledgements

This work was initiated during a visit of the authors to the Oxford-Man Institute at the University of Oxford in December 2007. The authors are very grateful for the hospitality and support. They thank Kevin Sheppard for a discussion leading them to consider (sup)OU models in the context of factor modelling.

References

Barndorff-Nielsen, O. E. (2001). Superposition of Ornstein–Uhlenbeck type processes, *Theory Probab. Appl.* **45**: 175–194.

Barndorff-Nielsen, O. E. and Pedersen, J. (2009). Representation and properties of CGPII processes, *ALEA Lat. Am. J. Probab. Math. Stat.* **6**: 179–197.

Barndorff-Nielsen, O. E. and Pérez-Abreu, V. (2008). Matrix subordinators and related Upsilon transformations, *Theory Probab. Appl.* **52**: 1–23.

Barndorff-Nielsen, O. E. and Shephard, N. (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics (with discussion), *J. R. Stat. Soc. B Statist. Methodol.* **63**: 167–241.

Barndorff-Nielsen, O. E. and Shephard, N. (2010). *Financial Volatility in Continuous Time*, Cambridge University Press, Cambridge. to appear.

Barndorff-Nielsen, O. E. and Stelzer, R. (2007). Positive-definite matrix processes of finite variation, *Probab. Math. Statist.* **27**: 3–43.

Barndorff-Nielsen, O. E. and Stelzer, R. (2009). Multivariate supOU processes, *Thiele Research Report 7*, Thiele Centre, Århus University.

Cont, R. and Tankov, P. (2004). *Financial Modelling with Jump Processes*, CRC Financial Mathematical Series, Chapman & Hall, London.

Fasen, V. and Klüppelberg, C. (2007). Extremes of supOU processes, *in* F. E. Benth, G. Di Nunno, T. Lindstrom, B. Øksendal and T. Zhang (eds), *Stochastic Analysis and Applications: The Abel Symposium 2005*, Vol. 2 of *Abel Symposia*, Springer, Berlin, pp. 340–359.

Guillaume, D. M., Dacorogna, M. M., Davé, R. D., Müller, U. A., Olsen, R. B. and Pictet, O. V. (1997). From the bird's eye to the microscope: a survey of new stylized facts of the intra-daily foreign exchange markets, *Finance Stoch.* **1**: 95–129.

Horn, R. A. and Johnson, C. R. (1991). *Topics in Matrix Analysis*, Cambridge University Press, Cambridge.

Hubalek, F. and Nicolato, E. (2009). On multivariate extensions of Lévy-driven Ornstein-Uhlenbeck type stochastic volatility models and multi-asset options. In preparation.

Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*, Vol. 288 of *Grundlehren der Mathematischen Wissenschaften*, 2nd edn, Springer, Berlin.

Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd edn, Springer, Berlin.

Métivier, M. and Pellaumail, J. (1980). *Stochastic Integration*, Academic Press, New York.

Øksendal, B. (1998). *Stochastic Differential Equations – An Introduction with Applications*, 5th edn, Springer.

Pedersen, J. (2003). The Lévy-Ito decomposition of an independently scattered random measure, *MaPhySto Research Report 2*, MaPhySto, Århus, Denmark. available from: `http://www.maphysto.dk`.

Pigorsch, C. and Stelzer, R. (2009a). A multivariate Ornstein-Uhlenbeck type stochastic volatility model, *submitted for publication*. available from: `http://www-m4.ma.tum.de`.

Pigorsch, C. and Stelzer, R. (2009b). On the definition, stationary distribution and second order structure of positive semi-definite Ornstein-Uhlenbeck type processes, *Bernoulli* **15**: 754–773.

Protter, P. (2004). *Stochastic Integration and Differential Equations*, Vol. 21 of *Stochastic Modelling and Applied Probability*, 2nd edn, Springer-Verlag, New York.

Rajput, B. S. and Rosinski, J. (1989). Spectral representations of infinitely divisible processes, *Probab. Theory Related Fields* **82**: 451–487.