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MULTIVARIATE SUPOU PROCESSES

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Univariate superpositions of Ornstein–Uhlenbeck-type processes (OU), called supOU processes, provide a class of continuous time processes capable of exhibiting long memory behavior. This paper introduces multivariate supOU processes and gives conditions for their existence and finiteness of moments. Moreover, the second-order moment structure is explicitly calculated, and examples exhibit the possibility of long-range dependence.

Our supOU processes are defined via homogeneous and factorizable Lévy bases. We show that the behavior of supOU processes is particularly nice when the mean reversion parameter is restricted to normal matrices and especially to strictly negative definite ones.

For finite variation Lévy bases we are able to give conditions for supOU processes to have locally bounded càdlàg paths of finite variation and to show an analogue of the stochastic differential equation of OU-type processes, which has been suggested in [2] in the univariate case. Finally, as an important special case, we introduce positive semi-definite supOU processes, and we discuss the relevance of multivariate supOU processes in applications.

1. Introduction. Lévy-driven Ornstein–Uhlenbeck-type processes (OU) are extensively used in applications as elements in continuous time models for observed time series. One area where they are often applied is mathematical finance (see, e.g., [15]), especially in the OU-type stochastic volatility model of [6]. An OU-type process is given as the solution of a stochastic differential equation of the form

$$dX_t = -aX_t\,dt + dL_t \tag{1.1}$$

with L being a Lévy process (see, e.g., [45] for a comprehensive introduction) and $a \in \mathbb{R}$. Typically, one is interested mainly in stationary solutions of (1.1). Provided $a > 0$ and $E(\ln(|L| \vee 1)) < \infty$, the SDE (1.1) has a unique stationary solution given by

$$X_t = \int_{-\infty}^{t} e^{-a(t-s)}\,dL_s.$$

However, in many applications the dependence structure exhibited by empirical data is found to be not in good accordance with that of OU-type processes which

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have autocorrelation functions of the form e^{-ah} for positive lags h. In many data sets a more complex and often a (quasi)long memory behavior of the autocorrelation function is encountered. OU-type processes could be replaced by fractional OU-type processes (see [30] or [31], for instance) to have long memory effects included in the model. However, in this case many desirable properties are lost, and, in particular, fractional OU-type processes no longer have jumps. An alternative to obtain long memory from OU-type processes and still to have jumps is to add up countably many independent OU-type processes, that is,

$$X_t = \sum_{k=1}^{\infty} w_i \int_{-\infty}^{t} e^{-a_i(t-s)}\, dL_{i,s}$$

with independent identically distributed Lévy processes $(L_i)_{i\in\mathbb{N}}$ and appropriate $a_i > 0$, $w_i > 0$ with $\sum_{i=1}^{\infty} w_i = 1$. Intuitively we can likewise "add" (i.e., integrate) up independent OU-type processes with all parameter values $a > 0$ possible. The resulting processes are called supOU processes and have been introduced in [2] where it has also been established that they may exhibit long-range dependence. For a comprehensive treatment regarding the theory and use of univariate supOU processes in finance we refer to [7].

So far supOU processes have only been considered in the univariate case. However, in many applications it is crucial to model several time series with a joint model, and so flexible multivariate models are important. Therefore, in this paper we introduce and study multivariate supOU processes. Due to the appearance of matrices and the related peculiarities our theory is not a straightforward extension of the univariate results. Multivariate (d-dimensional) OU-type processes (see, e.g., [26] or [46]) are defined as the solutions of SDEs of the form

$$dX_t = AX_t\, dt + dL_t \tag{1.2}$$

with L a d-dimensional Lévy process and A a $d \times d$ matrix. Provided $E(\ln(\|L\| \vee 1)) < \infty$ and all eigenvalues of A have strictly negative real part, we have again a unique stationary solution given by

$$X_t = \int_{-\infty}^{t} e^{A(t-s)}\, dL_s.$$

Intuitively our multivariate supOU processes are obtained by "adding up" independent OU-type processes with all possible parameters A; that is, we consider all $d \times d$ matrices A with eigenvalues of strictly negative real parts. It turns out later on that the behavior of supOU becomes more tractable when we restrict A to come only from a nice subset, like the negative definite matrices.

The remainder of this paper is structured as follows. The next section starts with a brief overview of important notation and conventions used in the paper and is followed in Section 2.2 by a comprehensive introduction into Lévy bases and the related integration theory, which will be needed to define supOU processes.

In Section 3 we first define multivariate supOU processes and provide existence conditions in Section 3.1. Thereafter, we discuss the existence of moments and derive the second-order structure. For the finite variation case we show important path properties in Section 3.3. Besides establishing that we have càdlàg paths of bounded variation, we give an analogue of the stochastic differential equation (1.2) for supOU processes and its proof. In particular, this proves a conjecture in [2], which has not yet been shown in any nondegenerate set-up. We conclude that section with several examples illustrating the behavior and properties of supOU processes and showing that they may exhibit long memory. In Section 4 we use our results to define positive semi-definite supOU processes and analyze their properties. These processes are important for applications like stochastic volatility modeling, since they may be used to describe the stochastic dynamics of a latent covariance matrix. Finally, this and other possible applications of supOU processes are discussed in Section 5.

2. Background and preliminaries.

2.1. *Notation.* We denote the set of real $m \times n$ matrices by $M_{m,n}(\mathbb{R})$. If $m = n$, we simply write $M_n(\mathbb{R})$ and denote the group of invertible $n \times n$ matrices by $GL_n(\mathbb{R})$, the linear subspace of symmetric matrices by $\mathbb{S}_n$, the (closed) positive semi-definite cone by $\mathbb{S}_n^+$ and the open positive definite cone by $\mathbb{S}_n^{++}$ (likewise $\mathbb{S}_n^{--}$ are the strictly negative definite matrices, etc.). I_n stands for the $n \times n$ identity matrix. The tensor (Kronecker) product of two matrices A, B is written as $A \otimes B$. vec denotes the well-known vectorisation operator that maps the $n \times n$ matrices to $\mathbb{R}^{n^2}$ by stacking the columns of the matrices below one another. For more information regarding the tensor product and vec operator we refer to [23], Chapter 4. The spectrum of a matrix is denoted by $\sigma(\cdot)$. Finally, A^* is the transpose (adjoint) of a matrix $A \in M_{m,n}(\mathbb{R})$, and A_{ij} stands for the entry of A in the ith row and jth column.

Norms of vectors or matrices are denoted by $\|\cdot\|$. If the norm is not further specified, then it is understood that we take the Euclidean norm or its induced operator norm, respectively. However, due to the equivalence of all norms none of our results really depends on the choice of norms.

For a complex number z we denote by $\Re(z)$ its real part. Moreover, the indicator function of a set A is written 1_A.

A mapping $f : V \to W$ is said to be $\mathscr{V}$–$\mathscr{W}$-measurable if it is measurable when the σ-algebra $\mathscr{V}$ is used on the domain V, and the σ-algebra $\mathscr{W}$ is used on the range W. The Borel σ-algebras are denoted by $\mathscr{B}(\cdot)$ and λ typically stands for the Lebesgue measure which in vector or matrix spaces is understood to be defined as the product of the coordinatewise Lebesgue measures.

Throughout we assume that all random variables and processes are defined on a given complete probability space $(\Omega, \mathscr{F}, P)$ equipped with an appropriate filtration when relevant.

Furthermore, we employ an intuitive notation with respect to the (stochastic) integration with matrix-valued integrators referring to any of the standard texts (e.g., [43]) for a comprehensive treatment of the theory of stochastic integration. Let $(A_t)_{t\in\mathbb{R}^+}$ in $M_{m,n}(\mathbb{R})$, $(B_t)_{t\in\mathbb{R}^+}$ in $M_{r,s}(\mathbb{R})$ be càdlàg and adapted processes and $(L_t)_{t\in\mathbb{R}^+}$ in $M_{n,r}(\mathbb{R})$ be a semi-martingale. Then we denote by $\int_0^t A_{s-}\,dL_s B_{s-}$ the matrix C_t in $M_{m,s}(\mathbb{R})$ which has ijth element $C_{ij,t} = \sum_{k=1}^n \sum_{l=1}^r \int_0^t A_{ik,s-}B_{lj,s-}\,dL_{kl,s}$. Equivalently such an integral can be understood in the sense of [34, 35] by identifying it with the integral $\int_0^t \mathbf{A}_{s-}\,dL_s$ with $\mathbf{A}_t$ being for each fixed t the linear operator $M_{n,r}(\mathbb{R}) \to M_{m,s}(\mathbb{R})$, $X \mapsto A_t X B_t$. Analogous notation is used in the context of integrals with respect to random measures.

Finally, integrals of the form $\int_A \int_B f(x,y)m(dx,dy)$ are understood to be over the set A in x and over B in y.

2.2. *Lévy bases.* To lay the foundations for the definition of vector-valued supOU processes, we give now a summary of Lévy bases and the related integration theory. In this context recall that a d-dimensional Lévy process can be understood as an $\mathbb{R}^d$-valued random measure on the real numbers. If $L = (L_t)_{t\in\mathbb{R}}$ is a d-dimensional Lévy process, this measure is simply determined by $L((a,b]) = L(b) - L(a)$ for all $a, b \in \mathbb{R}, a < b$.

Define now $M_d^- := \{X \in M_d(\mathbb{R}) : \sigma(X) \subset (-\infty, 0) + i\mathbb{R}\}$ and $\mathscr{B}_b(M_d^- \times \mathbb{R})$ to be the bounded Borel sets of $M_d^- \times \mathbb{R}$. Note that M_d^- is obviously a cone, but not a convex one (cf. [22], for instance). Moreover, we obviously have $\overline{M_d^-} = \{X \in M_d(\mathbb{R}) : \sigma(X) \subset (-\infty, 0] + i\mathbb{R}\}$.

DEFINITION 2.1. A family $\Lambda = \{\Lambda(B) : B \in \mathscr{B}_b(M_d^- \times \mathbb{R})\}$ of $\mathbb{R}^d$-valued random variables is called an *$\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$* if:

(a) the distribution of $\Lambda(B)$ is infinitely divisible for all $B \in \mathscr{B}_b(M_d^- \times \mathbb{R})$,
(b) for any $n \in \mathbb{N}$ and pairwise disjoint sets $B_1, \ldots, B_n \in \mathscr{B}_b(M_d^- \times \mathbb{R})$ the random variables $\Lambda(B_1), \ldots, \Lambda(B_n)$ are independent and
(c) for any pairwise disjoint sets $B_i \in \mathscr{B}_b(M_d^- \times \mathbb{R})$, $i \in \mathbb{N}$, satisfying $\bigcup_{n\in\mathbb{N}} B_n \in \mathscr{B}_b(M_d^- \times \mathbb{R})$ the series $\sum_{n=1}^\infty \Lambda(B_n)$ converges almost surely and it holds that $\Lambda(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n=1}^\infty \Lambda(B_n)$.

In the literature Lévy bases are also often called infinitely divisible independently scattered random measures (abbreviated i.d.i.s.r.m.) instead.

In the following we will only consider Lévy bases, which are homogeneous (in time) and factorizable (into the effects of one underlying infinitely divisible distribution and a probability distribution on M_d^-); that is, their characteristic function is of the form

$$E(\exp(iu^*\Lambda(B))) = \exp(\varphi(u)\Pi(B)) \tag{2.1}$$

for all $u \in \mathbb{R}^d$ and $B \in \mathscr{B}_b(M_d^-(\mathbb{R}) \times \mathbb{R})$. Here $\Pi = \pi \times \lambda$ is the product of a probability measure π on $M_d^-(\mathbb{R})$ and the Lebesgue measure λ on $\mathbb{R}$ and

$$\varphi(u) = iu^*\gamma - \frac{1}{2}u^*\Sigma u + \int_{\mathbb{R}^d} \big(e^{iu^*x} - 1 - iu^*x 1_{[0,1]}(\|x\|)\big)\nu(dx) \tag{2.2}$$

is the cumulant transform of an infinitely divisible distribution on $\mathbb{R}^d$ with Lévy–Khintchine triplet (γ, Σ, ν), that is, $\gamma \in \mathbb{R}^d$, $\Sigma \in \mathbb{S}_d^+$ and ν is a Lévy measure—a Borel measure on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(\|x\|^2 \wedge 1)\nu(dx) < \infty$. The quadruple $(\gamma, \Sigma, \nu, \pi)$ determines the distribution of the Lévy basis completely and is henceforth referred to as the "generating quadruple" (cf. [18]).

The Lévy process L defined by

$$L_t = \Lambda\big(M_d^- \times (0, t]\big) \quad \text{and} \quad L_{-t} = \Lambda\big(M_d^- \times (-t, 0)\big) \qquad \text{for } t \in \mathbb{R}^+,$$

has characteristic triplet (γ, Σ, ν) and is called "the underlying Lévy process."

For more information on $\mathbb{R}^d$-valued Lévy bases see [39] and [44].

A Lévy basis has a Lévy–Itô decomposition.

THEOREM 2.2 (Lévy–Itô decomposition). *Let Λ be a homogeneous and factorisable $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$. Then there exists a modification $\tilde{\Lambda}$ of Λ which is also a Lévy basis with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ such that there exists an $\mathbb{R}^d$-valued Lévy basis $\tilde{\Lambda}^G$ on $M_d^- \times \mathbb{R}$ with generating quadruple $(0, \Sigma, 0, \pi)$ and an independent Poisson random measure μ on $(\mathbb{R}^d \times M_d^- \times \mathbb{R}, \mathscr{B}(\mathbb{R}^d \times M_d^- \times \mathbb{R}))$ with intensity measure $\nu \times \pi \times \lambda$ which satisfy*

$$\begin{aligned}\tilde{\Lambda}(B) &= \gamma(\pi \times \lambda)(B) + \tilde{\Lambda}^G(B) \\ &\quad + \int_{\|x\| \le 1} \int_B x\big(\mu(dx, dA, ds) - ds\,\pi(dA)\nu(dx)\big) \\ &\quad + \int_{\|x\| > 1} \int_B x\mu(dx, dA, ds)\end{aligned} \tag{2.3}$$

for all $B \in \mathscr{B}_b(M_d^- \times \mathbb{R})$ and all $\omega \in \Omega$.

Provided $\int_{\|x\| \le 1} \|x\|\nu(dx) < \infty$, it holds that

$$\tilde{\Lambda}(B) = \gamma_0(\pi \times \lambda)(B) + \tilde{\Lambda}^G(B) + \int_{\mathbb{R}^d} \int_B x\mu(dx, dA, ds)$$

for all $B \in \mathscr{B}_b(M_d^- \times \mathbb{R})$ with

$$\gamma_0 := \gamma - \int_{\|x\| \le 1} x\nu(dx). \tag{2.4}$$

Furthermore, the integral with respect to μ exists as a Lebesgue integral for all $\omega \in \Omega$.

Here an $\mathbb{R}^d$-valued Lévy basis $\tilde{\Lambda}$ on $M_d^- \times \mathbb{R}$ is called a *modification* of a Lévy basis Λ if $\tilde{\Lambda}(B) = \Lambda(B)$ a.s. for all $B \in \mathscr{B}_b(M_d^- \times \mathbb{R})$. For the necessary background on the integration with respect to Poisson random measures we refer to [24], Section 2.1, and [27], Lemma 12.13.

PROOF OF THEOREM 2.2. This follows immediately from [39], Theorem 4.5, because the control measure m is given by $m(B) = (\|\gamma\| + \mathrm{tr}(\Sigma) + \int_{\mathbb{R}^d}(1 \wedge \|x\|^2)\nu(dx))(\pi \times \lambda)(B)$ which is trivially continuous due to the presence of the Lebesgue measure. The second part is an immediate consequence, as no compensation for the small jumps is needed if $\int_{\|x\|\le 1} \|x\|\nu(dx) < \infty$. $\square$

From now on we assume without loss of generality that all Lévy bases are such that they have the Lévy–Itô decomposition (2.3).

In the following we need to define integrals of deterministic functions with respect to a Lévy basis Λ. Following [44], for simple functions $f : M_d^- \times \mathbb{R} \to M_d(\mathbb{R})$,

$$f(x) = \sum_{i=1}^{m} a_i 1_{B_i}(x)$$

with $m \in \mathbb{N}$, $a_i \in M_d(\mathbb{R})$ and $B_i \in \mathscr{B}_b(M_d^- \times \mathbb{R})$, and for every $B \in \mathscr{B}(M_d^- \times \mathbb{R})$ we define the integral

$$\int_B f(x)\Lambda(dx) = \sum_{i=1}^{m} a_i \Lambda(B \cap B_i).$$

A $\mathscr{B}(M_d^- \times \mathbb{R})$-$\mathscr{B}(M_d(\mathbb{R}))$-measurable function $f : M_d^- \times \mathbb{R} \to M_d(\mathbb{R})$ is said to be Λ-*integrable* if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ Lebesgue almost everywhere, and for all $B \in \mathscr{B}(M_d^- \times \mathbb{R})$ the sequence $\int_B f_n(x)\Lambda(dx)$ converges in probability. For Λ-integrable f we set $\int_B f(x)\Lambda(dx) = \mathrm{plim}_{n\to\infty} \int_B f_n(x)\Lambda(dx)$. As in [44] well definedness of the integral is ensured by [48].

The following result is a straightforward generalization of [44], Propositions 2.6 and 2.7, to $\mathbb{R}^d$-valued Lévy bases and follows along the same lines.

PROPOSITION 2.3. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic function of the form* (2.1) *and $f : M_d^- \times \mathbb{R} \to M_d(\mathbb{R})$ a $\mathscr{B}(M_d^- \times \mathbb{R})$-$\mathscr{B}(M_d(\mathbb{R}))$-measurable function. Then f is Λ-integrable if and only if*

$$\begin{aligned}\int_{M_d^-}\int_{\mathbb{R}}\bigg\| f(A,s)\gamma \\ + \int_{\mathbb{R}^d} f(A,s)x\big(1_{[0,1]}(\|f(A,s)x\|) \\ - 1_{[0,1]}(\|x\|)\big)\nu(dx)\bigg\| \, ds\,\pi(dA) < \infty,\end{aligned}\tag{2.5}$$

$$\int_{M_d^-}\int_{\mathbb{R}} \|f(A,s)\Sigma f(A,s)^*\| \, ds\,\pi(dA) < \infty,\tag{2.6}$$

$$\int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} (1 \wedge \|f(A,s)x\|^2)\nu(dx)\, ds\,\pi(dA) < \infty.\tag{2.7}$$

If f is Λ-integrable, the distribution of $\int_{M_d^-}\int_{\mathbb{R}^+} f(A,s)\Lambda(dA,ds)$ is infinitely divisible with characteristic function

$$\begin{aligned}E\bigg(\exp\bigg(iu^*\int_{M_d^-}\int_{\mathbb{R}^+} f(A,s)\Lambda(dA,ds)\bigg)\bigg) \\ = \exp\bigg(\int_{M_d^-}\int_{\mathbb{R}^+} \varphi(f(A,s)^*u)\, ds\,\pi(dA)\bigg)\end{aligned}\tag{2.8}$$

and characteristic triplet $(\gamma_{\text{int}}, \Sigma_{\text{int}}, \nu_{\text{int}})$ *given by*

$$\begin{aligned}\gamma_{\text{int}} = \int_{M_d^-}\int_{\mathbb{R}}\bigg(f(A,s)\gamma \\ + \int_{\mathbb{R}^d} f(A,s)x\big(1_{[0,1]}(\|f(A,s)x\|) \\ - 1_{[0,1]}(\|x\|)\big)\nu(dx)\bigg)\, ds\,\pi(dA),\end{aligned}\tag{2.9}$$

$$\Sigma_{\text{int}} = \int_{M_d^-}\int_{\mathbb{R}} f(A,s)\Sigma f(A,s)^*\, ds\,\pi(dA),\tag{2.10}$$

$$\nu_{\text{int}}(B) = \int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} 1_B(f(A,s)x)\nu(dx)\, ds\,\pi(dA) \qquad \forall B \in \mathscr{B}(\mathbb{R}^d).\tag{2.11}$$

When the underlying Lévy process has finite variation we can do ω-wise Lebesgue integration; that is, the integral can be obtained as a Lebesgue integral for each $\omega \in \Omega$.

PROPOSITION 2.4. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, 0, \nu, \pi)$ satisfying $\int_{\|x\|\le 1}\|x\|\nu(dx) < \infty$, and define γ_0 as in* (2.4), *that is, $\varphi(u) = iu^*\gamma_0 + \int_{\mathbb{R}^d}(e^{iu^*x} - 1)\nu(dx)$. Furthermore, let $f : M_d^- \times \mathbb{R} \to$*

$M_d(\mathbb{R})$ be a $\mathscr{B}(M_d^- \times \mathbb{R})$-$\mathscr{B}(M_d(\mathbb{R}))$-measurable function satisfying

$$\int_{M_d^-}\int_{\mathbb{R}} \|f(A,s)\|\,ds\,\pi(dA) < \infty, \tag{2.12}$$

$$\int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \bigl(1 \wedge \|f(A,s)x\|\bigr)\nu(dx)\,ds\,\pi(dA) < \infty. \tag{2.13}$$

Then

$$\begin{aligned}\int_{M_d^-}\int_{\mathbb{R}} f(A,s)\Lambda(dA,ds) &= \int_{M_d^-}\int_{\mathbb{R}} f(A,s)\gamma_0\,ds\,\pi(dA)\\ &\quad + \int_{\mathbb{R}^d}\int_{M_d^-}\int_{\mathbb{R}} f(A,s)x\mu(dx,dA,ds),\end{aligned} \tag{2.14}$$

and the right-hand side is a Lebesgue integral for every $\omega \in \Omega$ [conditions (2.12) and (2.13) are also necessary for this].

Moreover, the distribution of $\int_{M_d^-}\int_{\mathbb{R}} f(A,s)\Lambda(dA,ds)$ is infinitely divisible with characteristic function

$$\begin{aligned}&E\left(\exp\left(iu^*\int_{M_d^-}\int_{\mathbb{R}} f(A,s)\Lambda(dA,ds)\right)\right)\\ &\qquad = e^{iu^*\gamma_{\mathrm{int},0}+\int_{\mathbb{R}^d}(e^{iu^*x}-1)\nu_{\mathrm{int}}(dx)}, \qquad u \in \mathbb{R}^d,\end{aligned}$$

where

$$\gamma_{\mathrm{int},0} = \int_{M_d^-}\int_{\mathbb{R}} f(A,s)\gamma_0\,ds\,\pi(dA), \tag{2.15}$$

$$\nu_{\mathrm{int}}(B) = \int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} 1_B(f(A,s)x)\nu(dx)\,ds\,\pi(dA) \qquad \forall B \in \mathscr{B}(\mathbb{R}^d). \tag{2.16}$$

PROOF. Follows from the Lévy–Itô decomposition and the usual integration theory with respect to Poisson random measures (see [27], Lemma 12.13). □

REMARK 2.5. All results of this section remain valid when replacing M_d^- with $M_k(\mathbb{R})$, $k \in \mathbb{N}$, or any measurable subset of a finite-dimensional real vector space and when considering integration of functions $f : M_k(\mathbb{R}) \times \mathbb{R} \to M_{m,d}(\mathbb{R})$. We decided to state all our results with M_d^- as this set will be used mainly in the following and it reduces the notational burden.

3. Multidimensional supOU processes. In this section we introduce supOU processes taking values in $\mathbb{R}^d$ with $d \in \mathbb{N}$ and analyze their properties. This extends to a multivariate setting the theory of univariate supOU processes as introduced in [2] and studied further, for example, in [19].

Intuitively supOU processes are obtained by "adding up" independent OU-type processes with different mean reversion coefficient.

3.1. *Definition and existence.* We define a d-dimensional supOU process as a process of the form (3.4) below.

THEOREM 3.1. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ satisfying*

$$\int_{\|x\|>1} \ln(\|x\|)\nu(dx) < \infty, \tag{3.1}$$

and assume there exist measurable functions $\rho : M_d^- \to \mathbb{R}^+ \backslash \{0\}$ and $\kappa : M_d^- \to [1, \infty)$ such that

$$\|e^{As}\| \le \kappa(A) e^{-\rho(A)s} \qquad \forall s \in \mathbb{R}^+, \pi\text{-almost surely}, \tag{3.2}$$

and

$$\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty. \tag{3.3}$$

Then the process $(X_t)_{t\in\mathbb{R}}$ given by

$$X_t = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)} \Lambda(dA, ds) \tag{3.4}$$

is well defined for all $t \in \mathbb{R}$ and stationary. The distribution of X_t is infinitely divisible with characteristic triplet $(\gamma_X, \Sigma_X, \nu_X)$ given by

$$\begin{aligned}\gamma_X = \int_{M_d^-} \int_{\mathbb{R}^+} \biggl(e^{As}\gamma + \int_{\mathbb{R}^d} e^{As} x \bigl(1_{[0,1]}(\|e^{As}x\|) \\ - 1_{[0,1]}(\|x\|)\bigr)\nu(dx)\biggr) ds\, \pi(dA),\end{aligned} \tag{3.5}$$

$$\Sigma_X = \int_{M_d^-} \int_{\mathbb{R}^+} e^{As} \Sigma e^{A^* s}\, ds\, \pi(dA), \tag{3.6}$$

$$\begin{aligned}\nu_X(B) = \int_{M_d^-} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} 1_B(e^{As}x)\nu(dx)\, ds\, \pi(dA) \\ \text{for all Borel sets } B \subseteq \mathbb{R}^d.\end{aligned} \tag{3.7}$$

PROOF. The stationarity is obvious once the well definedness is shown. Using Proposition 2.3 it follows that necessary and sufficient conditions for the integral to exist are given by

$$\begin{aligned}\int_{M_d^-} \int_{\mathbb{R}^+} \biggl\| e^{As}\gamma + \int_{\mathbb{R}^d} e^{As} x \bigl(1_{[0,1]}(\|e^{As}x\|) \\ - 1_{[0,1]}(\|x\|)\bigr)\nu(dx)\biggr\| ds\, \pi(dA) < \infty,\end{aligned} \tag{3.8}$$

$$\int_{M_d^-}\int_{\mathbb{R}^+} \|e^{As}\Sigma e^{A^*s}\|\,ds\,\pi(dA) < \infty, \tag{3.9}$$

$$\int_{M_d^-}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} (1 \wedge \|e^{As}x\|^2)\nu(dx)\,ds\,\pi(dA) < \infty. \tag{3.10}$$

First we show (3.10)

$$\begin{aligned}
&\int_{M_d^-}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} (1 \wedge \|e^{As}x\|^2)\nu(dx)\,ds\,\pi(dA)\\
&\quad \le \int_{M_d^-}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \bigl(1 \wedge \kappa(A)^2 e^{-2\rho(A)s}\|x\|^2\bigr)\nu(dx)\,ds\,\pi(dA)\\
&\quad = \int_{M_d^-}\int_{\|x\|>1/\kappa(A)} \frac{\ln(\kappa(A)\|x\|)+1/2}{\rho(A)}\nu(dx)\pi(dA)\\
&\qquad + \int_{M_d^-}\int_{\|x\|\le 1/\kappa(A)} \frac{\kappa(A)^2\|x\|^2}{2\rho(A)}\nu(dx)\pi(dA).
\end{aligned}$$

The finiteness of the first integral follows from (3.1), (3.3), $\kappa(A) \ge 1$ and ν being a Lévy measure, which imply

$$\begin{aligned}
&\int_{M_d^-}\int_{\|x\|>1/\kappa(A)} \frac{\ln(\kappa(A)\|x\|)+1/2}{\rho(A)}\nu(dx)\pi(dA)\\
&\quad \le \int_{M_d^-}\int_{\|x\|>1} \frac{\ln(\kappa(A))+\ln(\|x\|)+1/2}{\rho(A)}\nu(dx)\pi(dA)\\
&\qquad + \int_{M_d^-}\int_{\|x\|\le 1} \frac{3\kappa(A)^2\|x\|^2}{2\rho(A)}\nu(dx)\pi(dA)\\
&\quad = \int_{M_d^-} \frac{\ln(\kappa(A))}{\rho(A)}\pi(dA)\int_{\|x\|>1}\nu(dx)\\
&\qquad + \int_{M_d^-} \frac{3\kappa(A)^2}{2\rho(A)}\pi(dA)\int_{\|x\|\le 1}\|x\|^2\nu(dx)\\
&\qquad + \int_{M_d^-} \frac{1}{\rho(A)}\pi(dA)\int_{\|x\|>1}\bigl(\ln(\|x\|)+1/2\bigr)\nu(dx) < \infty.
\end{aligned}$$

Likewise the finiteness of the second integral is implied by (3.3), $\kappa(A) \ge 1$ and $\int_{\|x\|\le 1}\|x\|^2\nu(dx) < \infty$, as ν is a Lévy measure.

Next (3.9) follows from (3.3) and

$$\begin{aligned}
\int_{M_d^-}\int_{\mathbb{R}^+} \|e^{As}\Sigma e^{A^*s}\|\,ds\,\pi(dA) &\le \|\Sigma\|\int_{M_d^-}\int_{\mathbb{R}^+} \kappa(A)^2 e^{-2\rho(A)s}\,ds\,\pi(dA)\\
&= \|\Sigma\|\int_{M_d^-} \frac{\kappa(A)^2}{2\rho(A)}\pi(dA).
\end{aligned}$$

Turning to (3.8) we have from (3.3) that

$$\int_{M_d^-}\int_{\mathbb{R}^+}\|e^{As}\gamma\|\,ds\,\pi(dA) \le \|\gamma\|\int_{M_d^-}\int_{\mathbb{R}^+}\kappa(A)e^{-\rho(A)s}\,ds\,\pi(dA) = \|\gamma\|\int_{M_d^-}\frac{\kappa(A)}{\rho(A)}\pi(dA) < \infty.$$

Moreover,

$$\begin{aligned}
&\int_{M_d^-}\int_{\mathbb{R}^+}\left\|\int_{\mathbb{R}^d} e^{As}x\big(1_{[-1,1]}(\|e^{As}x\|) - 1_{[-1,1]}(\|x\|)\big)\nu(dx)\right\| ds\,\pi(dA)\\
&\quad\le \int_{M_d^-}\int_{\mathbb{R}^+}\int_{\|x\|\le 1, \|e^{As}x\|\ge 1}\|e^{As}x\|\nu(dx)\,ds\,\pi(dA)\\
&\qquad + \int_{M_d^-}\int_{\mathbb{R}^+}\int_{\|x\|\ge 1, \|e^{As}x\|\le 1}\|e^{As}x\|\nu(dx)\,ds\,\pi(dA)\\
&\quad\le \int_{M_d^-}\int_{\mathbb{R}^+}\int_{\|x\|\le 1, \|e^{As}x\|\ge 1}\|e^{As}x\|^2\nu(dx)\,ds\,\pi(dA)\\
&\qquad + \int_{M_d^-}\int_{\mathbb{R}^+}\int_{\|x\|\in(1, e^{\rho(A)s/2})}\|x\|\kappa(A)e^{-\rho(A)s}\nu(dx)\,ds\,\pi(dA)\\
&\qquad + \int_{M_d^-}\int_{\mathbb{R}^+}\int_{\|x\|\ge e^{\rho(A)s/2}}\nu(dx)\,ds\,\pi(dA)\\
&\quad\le \int_{\|x\|\le 1}\|x\|^2\nu(dx)\int_{M_d^-}\frac{\kappa(A)^2}{2\rho(A)}\pi(dA) + \int_{M_d^-}\frac{2\kappa(A)}{\rho(A)}\pi(dA)\int_{\|x\|>1}\nu(dx)\\
&\qquad + \int_{M_d^-}\frac{2}{\rho(A)}\pi(dA)\int_{\|x\|>1}\ln(\|x\|)\nu(dx) < \infty
\end{aligned}$$

with the finiteness following from (3.1), (3.3) and ν being a Lévy measure.

That the distribution of X_t is infinitely divisible and has the stated characteristic triplet follows now immediately from Proposition 2.3. □

REMARK 3.2. (i) The necessary and sufficient conditions for the existence of the multivariate supOU process X are (3.8)–(3.10). However, as they are obviously very intricate to check in concrete situations, it seems to be appropriate to replace them by the sufficient conditions (3.1)–(3.3). One particular advantage of these conditions is that they involve only integrals with respect to either ν or π, but not with respect to both.

(ii) Note also that for $d = 1$ the conditions above become the necessary and sufficient conditions of [19], as we can then take $\kappa(A) = 1$ and $\rho(A) = -A$.

(iii) By looking at the Jordan decomposition one can see that pointwise there is for any $A \in M_d^-$ a constant $\kappa \in [1, \infty)$ and a $\rho \in (0, -\max(\Re(\sigma(A)))]$ such that $\|e^{As}\| \leq \kappa e^{-\rho s}$ for all $s \in \mathbb{R}^+$. If A is diagonalizable, it is possible to choose $\rho(A) = -\max(\Re(\sigma(A)))$ and $\kappa(A) = \|U\|\|U^{-1}\|$ with $U \in GL_d(\mathbb{C})$ being such that UAU^{-1} is diagonal. So (3.2) essentially demands that this choice has to be done measurably in A (see especially Example 3.5 for a concrete example).

It seems important, especially for applications, to understand how close our sufficient existence conditions are to necessity and to give also necessary conditions easier checkable than (3.8)–(3.10). To this end we need the concept of modulus of injectivity (see, e.g., [40], Section B.3). For a $Z \in M_d(\mathbb{R})$ the modulus of injectivity is defined as $j(Z) = \min_{\|x\|=1} \|Zx\|$. It is immediate to see that $0 \leq j(Z) \leq \|Z\|$ and $j(Z) = \|Z^{-1}\|^{-1}$ for $Z \in GL_d(\mathbb{R})$.

PROPOSITION 3.3. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ and assume there exist measurable functions $\tau : M_d^- \to \mathbb{R}^+\backslash\{0\}$ and $\vartheta : M_d^- \to (0, 1]$ such that*

$$j(e^{As}) \geq \vartheta(A)e^{-\tau(A)s} \qquad \forall s \in \mathbb{R}^+, \pi\text{-almost surely.} \tag{3.11}$$

Then necessary conditions for the integral (3.4) *to exist are*

$$\int_{\vartheta(A)\geq\varepsilon} \frac{1}{\tau(A)}\pi(dA) < \infty \tag{3.12}$$
$$\text{for any } \varepsilon \in (0, 1] \text{ with } \nu\big(\{\|x\| > 1/\varepsilon\}\big) > 0 \text{ and } \pi(\{\vartheta(A) \geq \varepsilon\}) > 0,$$

$$\int_{M_d^-} \frac{\vartheta(A)^2}{\tau(A)}\pi(dA) < \infty \qquad \text{provided } j(\Sigma) > 0 \text{ or } \nu(\{\|x\| \leq 1\}) > 0 \tag{3.13}$$

and

$$\int_{\|x\|>1} \ln(\|x\|)\nu(dx) < \infty. \tag{3.14}$$

This shows, in particular, that the logarithmic moment condition on ν is both necessary and sufficient and that (3.2), (3.3) together with (3.11)–(3.13) form a set of sufficient and necessary conditions which are as close as one can probably hope for conditions reasonably easy to work with.

PROOF OF PROPOSITION 3.3. In the case $j(\Sigma) > 0$ condition (3.13) follows immediately from (3.9).

Equation (3.10) implies that a necessary condition is

$$\begin{aligned}
\infty &> \int_{M_d^-} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \big(1 \wedge \vartheta(A)^2 e^{-2\tau(A)s}\|x\|^2\big)\nu(dx)\, ds\, \pi(dA) \\
&= \int_{M_d^-} \int_{\|x\|>1/\vartheta(A)} \frac{\ln(\vartheta(A)\|x\|) + 1/2}{\tau(A)}\nu(dx)\pi(dA) \\
&\quad + \int_{M_d^-} \int_{\|x\|\leq 1/\vartheta(A)} \frac{\vartheta(A)^2\|x\|^2}{2\tau(A)}\nu(dx)\pi(dA).
\end{aligned}$$

Since $\vartheta(A) \le 1$, the second summand implies (3.13) given $\nu(\{\|x\| \le 1\}) > 0$.

Choose now $\varepsilon > 0$ such that $\pi(\{\vartheta(A) \ge \varepsilon\}) > 0$. Then the first integral is bigger than

$$\int_{\vartheta(A)\ge\varepsilon}\int_{\|x\|>1/\varepsilon}\frac{\ln(\vartheta(A)\|x\|)+1/2}{\tau(A)}\nu(dx)\pi(dA).$$

This gives the necessity of (3.12). Moreover, the above integral is again greater than

$$\int_{\vartheta(A)\ge\varepsilon}\int_{\|x\|>1/\varepsilon}\frac{\ln(\varepsilon)+\ln(\|x\|)+1/2}{\tau(A)}\nu(dx)\pi(dA).$$

Since $\int_{\vartheta(A)\ge\varepsilon}\int_{\|x\|>1/\varepsilon}\frac{1}{\tau(A)}\nu(dx)\pi(dA)$ is necessarily finite, we have that

$$\int_{\vartheta(A)\ge\varepsilon}\int_{\|x\|>1/\varepsilon}\frac{\ln(\varepsilon)}{\tau(A)}\nu(dx)\pi(dA) > -\infty.$$

Hence, $\int_{\vartheta(A)\ge\varepsilon}\int_{\|x\|>1/\varepsilon}\frac{\ln(\|x\|)}{\tau(A)}\nu(dx)\pi(dA) < \infty$. Since $\int_{\vartheta(A)\ge\varepsilon}\frac{1}{\tau(A)}\pi(dA) > 0$ by construction, this implies the necessity of (3.14). $\square$

REMARK 3.4. (i) For $d = 1$ we have again recovered the necessary and sufficient conditions of [19], as we can then take $\vartheta(A) = 1$ and $\tau(A) = -A$.

(ii) Pointwise there is again for any $A \in M_d^-$ a constant $\vartheta \in (0, 1]$ and a $\tau \in [-\min(\Re(\sigma(A))), \infty)$ such that $j(e^{As}) \ge \vartheta e^{-\tau s}$ for all $s \in \mathbb{R}^+$. For A diagonalizable $\tau(A) = -\min(\Re(\sigma(A)))$ and $\vartheta(A) = j(U)j(U^{-1})$ can be chosen if UAU^{-1} is diagonal.

(iii) If ν has unbounded support, $\int_{\vartheta(A)\ge\varepsilon}\frac{1}{\tau(A)}\pi(dA) < \infty$ is necessary for all $\varepsilon > 0$. If $\pi(\{\vartheta(A) \ge \tilde{\varepsilon}\}) = 1$ for some $\tilde{\varepsilon} > 0$, then $\int_{M_d^-}\frac{1}{\tau(A)}\pi(dA) < \infty$ becomes a necessary condition, provided $\nu(\{\|x\| > 1/\tilde{\varepsilon}\}) > 0$.

In some applications like stochastic volatility modeling, for instance, one is particularly interested in the case where the underlying Lévy process is of finite variation and the supOU process is defined via ω-wise integration. The following result is proved using Proposition 2.4 together with variations of the arguments of the proofs of Theorem 3.1 and Proposition 3.3.

PROPOSITION 3.5. *(i) Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, 0, \nu, \pi)$ satisfying*

$$\int_{\|x\|>1}\ln(\|x\|)\nu(dx) < \infty \quad \textit{and} \quad \int_{\|x\|\le 1}\|x\|\nu(dx) < \infty \tag{3.15}$$

and assume there exist measurable functions $\rho : M_d^- \to \mathbb{R}^+\backslash\{0\}$ and $\kappa : M_d^- \to [1, \infty)$ such that

$$\|e^{As}\| \le \kappa(A)e^{-\rho(A)s} \qquad \forall s \in \mathbb{R}^+, \pi\textit{-almost surely}, \tag{3.16}$$

and

$$\int_{M_d^-} \frac{\kappa(A)}{\rho(A)} \pi(dA) < \infty. \tag{3.17}$$

Then the process $(X_t)_{t \in \mathbb{R}}$ *given by*

$$\begin{aligned} X_t &= \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \Lambda(dA, ds) \\ &= \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \gamma_0 \, ds \, \pi(dA) + \int_{\mathbb{R}^d} \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} x \mu(dx, dA, ds) \end{aligned} \tag{3.18}$$

is well defined as a Lebesgue integral for all $t \in \mathbb{R}$ *and* $\omega \in \Omega$ *and* X *is stationary.*

(ii) *If there exist measurable functions* $\tau : M_d^- \to \mathbb{R}^+ \backslash \{0\}$ *and* $\vartheta : M_d^- \to (0, 1]$ *such that* $j(e^{As}) \geq \vartheta(A) e^{-\tau(A)s}$ $\forall s \in \mathbb{R}^+$, π*-almost surely, then necessary conditions for the integral* (3.18) *to exist as a Lebesgue integral are*:

$$\int_{\vartheta(A) \geq \varepsilon} \frac{1}{\tau(A)} \pi(dA) < \infty \quad \text{for any } \varepsilon \in (0, 1] \text{ such that } \nu(\{\|x\| > 1/\varepsilon\}) > 0 \text{ and } \pi\big(\{\vartheta(A) \geq \varepsilon\}\big) > 0, \tag{3.19}$$

$$\int_{M_d^-} \frac{\vartheta(A)}{\tau(A)} \pi(dA) < \infty \qquad \text{provided } \gamma_0 \neq 0 \text{ or } \nu(\{\|x\| \leq 1\}) > 0, \tag{3.20}$$

$$\int_{\|x\|>1} \ln(\|x\|) \nu(dx) < \infty \quad \text{and} \quad \int_{\|x\| \leq 1} \|x\| \nu(dx) < \infty. \tag{3.21}$$

REMARK 3.6. If (3.3) is satisfied for a Lévy basis, then (3.17) is also satisfied.

We shall not develop the general case further, but consider two special cases which appear to be sufficient for most purposes. We define $M_d^{N-} := \{A \in M_d(\mathbb{R}) :$ A is normal and $\sigma(A) \subset (-\infty, 0) + i\mathbb{R}\}$.

PROPOSITION 3.7. (i) *Assume that* $\pi(M_d^{N-}) = 1$, *then* (3.2) *or* (3.16) *are satisfied with* $\kappa(A) = 1$ *and* $\rho(A) = -\max(\Re(\sigma(A)))$. *Moreover*, (3.3) *or* (3.17) *are implied by*

$$-\int_{M_d^{N-}} \frac{1}{\max(\Re(\sigma(A)))} \pi(dA) < \infty. \tag{3.22}$$

Likewise, (3.11) *is satisfied with* $\vartheta(A) = 1$ *and* $\tau(A) = -\min(\Re(\sigma(A)))$. *The necessary conditions* (3.12), (3.13), (3.19) *and* (3.20) *all become*

$$-\int_{M_d^{N-}} \frac{1}{\min(\Re(\sigma(A)))} \pi(dA) < \infty$$

[*assuming* $\nu(\|x\| > 1) > 0$ *for* (3.12), (3.19)].

(ii) *Assume that there are a* $K \in \mathbb{N}$ *and diagonalizable* $A_1, \dots, A_K \in M_d^-(\mathbb{R})$ *such that* $\pi(\{\lambda A_i : i = 1, \dots, K; \lambda \in \mathbb{R}^+ \backslash \{0\}\}) = 1$. *Then* (3.2) *or* (3.16) *are satisfied with* $\kappa(A) = C$ *for some* $C \in [1, \infty)$ *and* $\rho(A) = -\max(\Re(\sigma(A)))$. *Moreover,* (3.3) *or* (3.17) *are implied by*

$$-\int_{M_d^-} \frac{1}{\max(\Re(\sigma(A)))} \pi(dA) < \infty. \tag{3.23}$$

Likewise, (3.11) *is satisfied with* $\tau(A) = -\min(\Re(\sigma(A)))$ *and* $\vartheta(A) = c$ *for some* $c \in (0, 1]$ *and the necessary conditions* (3.12), (3.13), (3.19) *and* (3.20) *all become* $-\int_{M_d^-} \frac{1}{\min(\Re(\sigma(A)))} \pi(dA) < \infty$ [*assuming* $\nu(\|x\| > 1/c) > 0$ *for* (3.12), (3.19)].

In dimension one these are again the well-known necessary and sufficient conditions. Observe also that the eigenvalues are continuous (and hence measurable) in A because they are the zeros of the characteristic polynomial.

PROOF OF PROPOSITION 3.7. Part (i) follows immediately from the fact that all normal matrices are unitarily diagonalizable.

Likewise, (ii) is a consequence of the above mentioned pointwise bound and the fact that this can be turned into a global one because for fixed $i = 1, \dots, N$ the matrices $\{\lambda A_i\}_{\lambda \in \mathbb{R}^+ \backslash \{0\}}$ are all diagonalized by the same invertible matrices. □

In (i) the mean reversion parameter A of the superimposed OU-type processes is restricted to normal matrices and in (ii) to finitely many rays $\{\lambda A_i\}_{\lambda \in \mathbb{R}^+ \backslash \{0\}}$.

REMARK 3.8. (i) Typically one will, in general, not consider normal matrices for A as in (i), but only negative definite ones, since this allows one to use well-known distributions on the positive definite matrices (see, e.g., [21]) for π. In the case (ii) possible π can be obtained by using arbitrary distributions on $\mathbb{R}^+$ along the rays and positive weights summing to one for the different rays.

(ii) Intuitively (3.22) and (3.23) and their necessary counterparts mean that π must not put too much mass on elements with very slow exponential decay rates.

3.2. *Finiteness of moments and second-order structure.* Before we look at the second-order structure, we give conditions ensuring the finiteness of moments.

THEOREM 3.9. *Let* X *be a stationary* d*-dimensional supOU process driven by a Lévy basis* Λ *satisfying the conditions of Theorem* 3.1.

(i) *If*

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty \tag{3.24}$$

for $r \in (0, 2]$, *then* X *has a finite* r*th moment*, *that is*, $E(\|X_t\|^r) < \infty$.

(ii) *If* $r \in (2, \infty)$ *and*

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty, \qquad \int_{M_d^-} \frac{\kappa(A)^r}{\rho(A)} \pi(dA) < \infty, \tag{3.25}$$

then X *has a finite* r*th moment, that is,* $E(\|X_t\|^r) < \infty$.

(iii) *Necessary conditions for* X *to have a finite* r*th moment are*

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty \tag{3.26}$$

in general and

$$\int_{\vartheta(A)\geq\varepsilon} \frac{\vartheta(A)^r}{\tau(A)} \pi(dA) < \infty \tag{3.27}$$

for any ε *such that* $\nu(\{\|x\| > 1/\varepsilon\}) > 0$ *and* $\pi(\{\vartheta(A) \geq \varepsilon\}) > 0$.

In connection with the above results observe that the underlying Lévy process L has an rth moment, that is, $E(\|L_1\|^r) < \infty$, for $r \in \mathbb{R}^+$ if and only if

$$\int_{\|x\|>1} \|x\|^r \nu_L(dx) \text{ is finite.}$$

PROOF OF THEOREM 3.9. Using [45], Corollary 25.8, we have to show $\int_{\|x\|>1} \|x\|^r \nu_X(dx) < \infty$ to establish (i) and (ii). Now,

$$\begin{aligned}
&\int_{\|x\|>1} \|x\|^r \nu_X(dx) \\
&\quad = \int_{M_d^-} \int_0^\infty \int_{\mathbb{R}^d} \|e^{As}x\|^r 1_{(1,\infty)}(\|e^{As}x\|) \nu(dx)\, ds\, \pi(dA) \\
&\quad \leq \int_{M_d^-} \int_0^\infty \int_{\mathbb{R}^d} \kappa(A)^r e^{-r\rho(A)s} \|x\|^r 1_{(1,\infty)}\bigl(\kappa(A) e^{-\rho(A)s}\|x\|\bigr) \nu(dx)\, ds\, \pi(dA) \\
&\quad = \int_{M_d^-} \int_{\|x\|>1/\kappa(A)} \int_0^{\ln(\kappa(A)\|x\|)/\rho(A)} \kappa(A)^r e^{-r\rho(A)s} \|x\|^r\, ds\, \nu(dx) \pi(dA) \\
&\quad = \int_{M_d^-} \int_{\|x\|>1/\kappa(A)} \frac{\kappa(A)^r \|x\|^r}{r\rho(A)} \left(1 - \frac{1}{\kappa(A)^r \|x\|^r}\right) \nu(dx) \pi(dA) \\
&\quad = \int_{M_d^-} \int_{\|x\|>1/\kappa(A)} \frac{\kappa(A)^r \|x\|^r - 1}{r\rho(A)} \nu(dx) \pi(dA).
\end{aligned}$$

That $\int_{M_d^-} \int_{\|x\|>1/\kappa(A)} \frac{1}{r\rho(A)} \nu(dx) \pi(dA) < \infty$ has already been shown in the proof of Theorem 3.1.

Moreover, we obtain

$$\begin{aligned}
&\int_{M_d^-}\int_{\|x\|>1/\kappa(A)} \frac{\kappa(A)^r\|x\|^r}{\rho(A)}\nu(dx)\pi(dA)\\
&\qquad \le \int_{M_d^-}\int_{\|x\|>1} \frac{\kappa(A)^r\|x\|^r}{\rho(A)}\nu(dx)\pi(dA)\\
&\qquad\quad + \int_{M_d^-}\int_{\|x\|\le 1} \frac{\kappa(A)^{r\vee 2}\|x\|^{r\vee 2}}{\rho(A)}\nu(dx)\pi(dA).
\end{aligned}$$

Hence, (i) and (ii) follow, since ν is a Lévy measure, using also (3.3) for (i).

Regarding the proof of (iii), analogous arguments give that

$$\int_{M_d^-}\int_{\|x\|>1/\vartheta(A)} \frac{\vartheta(A)^r\|x\|^r-1}{r\tau(A)}\nu(dx)\pi(dA)<\infty$$

is a necessary condition. An inspection of the proof of Proposition 3.3 shows that

$$\int_{M_d^-}\int_{\|x\|>1/\vartheta(A)} \big(1/\tau(A)\big)\nu(dx)\pi(dA)<\infty$$

is already necessary for X to exist. Hence,

$$\begin{aligned}
\infty &> \int_{M_d^-}\int_{\|x\|>1/\vartheta(A)} \frac{\vartheta(A)^r\|x\|^r}{r\tau(A)}\nu(dx)\pi(dA)\\
&\ge \int_{\vartheta(A)\ge\varepsilon}\int_{\|x\|>1/\varepsilon} \frac{\vartheta(A)^r\|x\|^r}{r\tau(A)}\nu(dx)\pi(dA)
\end{aligned}$$

is necessary for a finite rth moment of X. This implies (iii), since $\int_{\|x\|>1}\|x\|^r\nu(dx)$ is finite if and only if $\int_{\|x\|>c}\|x\|^r\nu(dx)<\infty$ for arbitrary $c>0$. $\square$

REMARK 3.10. In the set-up of Proposition 3.7(i) [and analogously in (ii)]

$$-\int_{M_d^-}\frac{1}{\max(\Re(\sigma(A)))}\pi(dA)<\infty, \qquad \int_{\|x\|>1}\|x\|^r\nu(dx)<\infty \tag{3.28}$$

imply (3.22) and (3.25) [resp. (3.24)].

Likewise, $-\int_{M_d^-}\frac{1}{\min(\Re(\sigma(A)))}\pi(dA)<\infty$, provided $\nu(\|x\|>1)>0$ or $\nu(\|x\|>1/c)>0$, respectively, and $\int_{\|x\|>1}\|x\|^r\nu(dx)<\infty$ become necessary conditions for X to exist and to have a finite rth moment.

In applications these results have important implications for modeling. If one wants to have certain moments finite and certain moments infinite, for example, because this is what observed data strongly suggests, one has to use a driving Lévy basis Λ having exactly the same moments finite.

Moreover, knowledge of the moments allows for the estimation of the model based on empirical observations by using the general method of moments (GMM) estimation procedure, for example. Often the inference on parameters is based on the second-order moment structure as, for instance, in the estimation of multivariate OU-type stochastic volatility models in [41] or the one of univariate supOU type models in [7]. To provide the foundations for such work and due to the importance in applications of understanding the temporal dependence structure, we now calculate the second-order moment structure.

THEOREM 3.11. *Let X be a stationary d-dimensional supOU process driven by a Lévy basis Λ satisfying the conditions of Theorem 3.1 and assume additionally that $\int_{\mathbb{R}^d} \|x\|^2 \nu(dx) < \infty$. Then $E(\|X_0\|^2) < \infty$ and we have*

$$E(X_0) = -\int_{M_d^-} A^{-1}\left(\gamma + \int_{|x|>1} x\nu(dx)\right)\pi(dA), \tag{3.29}$$

$$\operatorname{var}(X_0) = -\int_{M_d^-} (\mathscr{A}(A))^{-1}\left(\Sigma + \left(\int_{\mathbb{R}^d} xx^*\nu(dx)\right)\right)\pi(dA), \tag{3.30}$$

$$\operatorname{cov}(X_h, X_0) = -\int_{M_d^-} e^{Ah}(\mathscr{A}(A))^{-1}\left(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\right)\pi(dA) \qquad \text{for } h \in \mathbb{R}^+ \tag{3.31}$$

with $\mathscr{A}(A) : M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \mapsto AX + XA^*$.

Moreover, it holds that

$$\lim_{h\to\infty} \operatorname{cov}(X_h, X_0) = 0. \tag{3.32}$$

PROOF. The finiteness of the second moments follows from Theorem 3.9. Using the formulae of Theorem 3.1 and [45], Example 25.12, we obtain

$$E(X_0) = \gamma_X + \int_{\|x\|>1} x\nu_X(dx) = \int_{M_d^-}\int_{\mathbb{R}^+} e^{As}\left(\gamma + \int_{\|x\|>1} x\nu(dx)\right) ds\, \pi(dA).$$

Noting that $\frac{d}{ds}A^{-1}e^{As} = e^{As}$, integrating over s gives (3.29).

Likewise we get

$$\begin{aligned}\operatorname{var}(X_0) &= \Sigma_X + \int_{\mathbb{R}^d} xx^*\nu_X(dx)\\ &= \int_{M_d^-}\int_{\mathbb{R}^+} e^{As}\left(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\right)e^{A^*s}\, ds\, \pi(dA)\end{aligned}$$

which implies (3.30) by integrating over s.

Finally,

$$
\begin{aligned}
\operatorname{cov}(X_h, X_0) &= \operatorname{cov}\Big(\int_{M_d^-}\int_{-\infty}^{h} e^{A(h-s)}\Lambda(dA, ds), \int_{M_d^-}\int_{-\infty}^{0} e^{-As}\Lambda(dA, ds)\Big)\\
&= \operatorname{cov}\Big(\int_{M_d^-}\int_{-\infty}^{0} e^{A(h-s)}\Lambda(dA, ds), \int_{M_d^-}\int_{-\infty}^{0} e^{-As}\Lambda(dA, ds)\Big)\\
&= \int_{M_d^-} e^{Ah}\Big(\int_{-\infty}^{0} e^{-As}\Big(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big)e^{-A^*s}ds\Big)\pi(dA)\\
&= -\int_{M_d^-} e^{Ah}(\mathscr{A}(A))^{-1}\Big(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big)\pi(dA),
\end{aligned}
\tag{3.33}
$$

since Λ is a Lévy basis, and hence the random measures Λ on $M_d^- \times (0, h]$ and on $M_d^- \times (-\infty, 0]$ are independent.

From (3.33) one obtains

$$
\begin{aligned}
&\Big\|\int_{M_d^-} e^{Ah}\Big(\int_{-\infty}^{0} e^{-As}\Big(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big)e^{-A^*s}\,ds\Big)\pi(dA)\Big\|\\
&\qquad \le \int_{M_d^-}\int_{-\infty}^{0} \kappa(A)^2 e^{\rho(A)(2s-h)}\,ds\,\pi(dA)\Big\|\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big\|\\
&\qquad \le \int_{M_d^-} \frac{\kappa(A)^2}{2\rho(A)}\pi(dA)\Big\|\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big\| < \infty.
\end{aligned}
$$

Therefore $\lim_{h\to\infty} e^{Ah} = 0$ for all $A \in M_d^-$ and dominated convergence establish (3.32). □

3.3. *"SDE representation" and some important path properties.* In this section we show for a supOU process X a representation which generalises the SDE that governs OU-type processes, and we derive important path properties of X. The "SDE representation"—identity (3.38) below—has been conjectured in the univariate case in [2], where neither a proof nor conditions for its validity have been given. Below we are able to show these results for finite variation Lévy bases, which are naturally appearing in applications like stochastic volatility modeling. The properties which we establish are especially important in the context of integration, since they imply that, if X is the integrator, then pathwise Lebesgue integration can be carried out, and, when X is the integrand, the theory of stochastic integrals of càdlàg processes with respect to semimartingales (see [43], for instance) respectively the L^2-theory of, for example, [38] applies. Likewise, the integrated process is of importance in certain applications (see Section 5.2).

Below the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}}$ generated by Λ is defined by $\mathscr{F}_t$ being the σ-algebra generated by the set of random variables $\{\Lambda(B) : B \in \mathscr{B}(M_d^- \times (-\infty, t])\}$ for $t \in \mathbb{R}$.

THEOREM 3.12. *Let X be a supOU process as in Proposition 3.5. Then*:

(i) *$X_t(\omega)$ is $\mathscr{B}(\mathbb{R}) \times \mathscr{F}$ measurable as a function of $t \in \mathbb{R}$ and $\omega \in \Omega$ and adapted to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}}$ generated by Λ.*

(ii) *If*

$$\int_{M_d^-} \kappa(A)\pi(dA) < \infty, \tag{3.34}$$

the paths of X are locally uniformly bounded in t for every $\omega \in \Omega$.

Furthermore, $X_t^+ = \int_0^t X_s\,ds$ exists for all $t \in \mathbb{R}^+$ and

$$\begin{aligned} X_t^+ &= \int_{M_d^-} \int_{-\infty}^t A^{-1} e^{A(t-s)} \Lambda(dA, ds) - \int_{M_d^-} \int_{-\infty}^0 A^{-1} e^{-As} \Lambda(dA, ds) \\ &\quad - \int_{M_d^-} \int_0^t A^{-1} \Lambda(dA, ds). \end{aligned} \tag{3.35}$$

(iii) *Provided that*

$$\int_{M_d^-} \frac{(\|A\| \vee 1)\kappa(A)}{\rho(A)} \pi(dA) < \infty \tag{3.36}$$

and

$$\int_{M_d^-} \|A\|\kappa(A)\pi(dA) < \infty \tag{3.37}$$

it holds that

$$X_t = X_0 + \int_0^t Z_u\,du + L_t, \tag{3.38}$$

where L is the underlying Lévy process and

$$Z_u = \int_{M_d^-} \int_{-\infty}^u A e^{A(u-s)} \Lambda(dA, ds) \tag{3.39}$$

for all $u \in \mathbb{R}$ with the integral existing ω-wise.

Moreover, the paths of X are càdlàg and of finite variation on compacts.

PROOF. (i) is immediate from the definition of X_t as a Lebesgue integral and the measurability properties of the integrand $e^{A(t-s)} x 1_{\mathbb{R}^+}(t-s)$ which as a function of t, s, A, x is $\mathscr{B}(\mathbb{R} \times \mathbb{R} \times M_d^- \times \mathbb{R}^d)$-$\mathscr{B}(\mathbb{R}^d)$-measurable.

(ii) We first show local uniform boundedness of X. Choose arbitrary $T_1, T_2 \in \mathbb{R}$ with $T_1 < T_2$. Then

$$\begin{aligned} f_{T_1,T_2}(A, s, x) &:= \sup_{t\in[T_1,T_2]} \left\| e^{A(t-s)} x 1_{\mathbb{R}^+}(t-s) \right\| \\ &\le \left(\kappa(A) e^{-\rho(A)(T_1-s)} 1_{(-\infty,T_1]}(s) + \kappa(A) 1_{(T_1,T_2]}(s)\right) \|x\| \end{aligned}$$

for all $A \in M_d^-(\mathbb{R})$, $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$ and

$$\sup_{t\in[T_1,T_2]} \|X_t\| \le \int_{M_d^-} \int_{\mathbb{R}} f_{T_1,T_2}(A, s, \gamma_0)\, ds\, \pi(dA) \\ + \int_{\mathbb{R}^d} \int_{M_d^-} \int_{\mathbb{R}} f_{T_1,T_2}(A, s, x)\mu(dx, dA, ds).$$

Therefore we only have to show the ω-wise existence and finiteness of the integral on the right-hand side. This is, however, an immediate consequence of the above upper bound, (3.34), Proposition 2.4 and arguments as in the proof of Theorem 3.1 noting that

$$\int_{M_d^-} \int_{T_1}^{T_2} \int_{\mathbb{R}^d} \big(1 \wedge \kappa(A)\|x\|\big)\nu(dx)\, ds\, \pi(dA) \\ \le (T_2 - T_1)\bigg(\int_{M_d^-} \int_{\|x\|\le 1} \kappa(A)\|x\|\nu(dx)\pi(dA) \\ + \int_{M_d^-} \int_{\|x\|>1} 1\nu(dx)\pi(dA)\bigg).$$

Turning to X_t^+ the existence follows immediately from the local boundedness. Noting that we have actually proved the local boundedness of

$$\int_{M_d^-} \int_{-\infty}^{t} \|e^{A(t-s)}\gamma_0\|\, ds\, \pi(dA) + \int_{\mathbb{R}^d} \int_{M_d^-} \int_{-\infty}^{t} \|e^{A(t-s)}x\|\mu(dx, dA, ds)$$

above, we can use Fubini to obtain

$$X_t^+ = \int_{M_d^-} \int_{-\infty}^{t} \int_{0\vee u}^{t} e^{A(s-u)}\gamma_0\, ds\, du\, \pi(dA) \\ + \int_{\mathbb{R}^d} \int_{M_d^-} \int_{-\infty}^{t} \int_{0\vee u}^{t} e^{A(s-u)}x\, ds\, \mu(dx, dA, du) \\ = \int_{M_d^-} \int_{-\infty}^{t} A^{-1}e^{A(s-u)}\gamma_0\big|_{s=(0\vee u)}^{t}\, du\, \pi(dA) \\ + \int_{\mathbb{R}^d} \int_{M_d^-} \int_{-\infty}^{t} A^{-1}e^{A(s-u)}x\big|_{s=(0\vee u)}^{t}\, ds\, \mu(dx, dA, du),$$

which establishes (3.35) by straightforward calculations.

(iii) Using similar calculations as before, the existence of Z_u as an ω-wise integral follows from Proposition 2.4 and (3.36). Similarly to (ii) one sees that under

(3.37) Z is locally uniformly bounded in u. Hence, one can use Fubini to obtain

$$\begin{aligned}\int_0^t Z_u\,du &= \int_{\mathbb{R}^d}\int_{M_d^-}\int_{-\infty}^t\int_{0\vee s}^t Ae^{A(u-s)}x\,du\,\mu(dx,dA,ds)\\ &\quad+\int_{M_d^-}\int_{-\infty}^t\int_{0\vee s}^t Ae^{A(u-s)}\gamma_0\,du\,ds\,\pi(dA)\\ &=\int_{\mathbb{R}^d}\int_{M_d^-}\int_{-\infty}^t e^{A(u-s)}x\big|_{u=(0\vee s)}^t\mu(dx,dA,ds)\\ &\quad+\int_{M_d^-}\int_{-\infty}^t e^{A(u-s)}\gamma_0\big|_{u=(0\vee s)}^t\,ds\,\pi(dA)=X_t-X_0-L_t\end{aligned}$$

which establishes (3.38). That X has cádlág paths of finite variation is now an immediate consequence of this integral representation. □

REMARK 3.13. (i) Condition (3.34) is always true if π is concentrated on the normal matrices or on finitely many rays and hence especially in dimension $d=1$. Moreover, it could be replaced by the weaker but rather impracticable condition that $f_{[T_1,T_2]}(A,s,x)\wedge 1$ is integrable with respect to $\pi\times\lambda\times\nu$ and that, for any fixed x, $f_{[T_1,T_2]}(A,s,x)$ is integrable with respect to $\pi\times\lambda$ (cf. [29], Proposition 2.1, for a very related result whose proof is similar in spirit to ours, but uses a series representation instead of the Lévy–Itô decomposition).

(ii) Intuitively (3.37) means that π does not place too much mass on the elements of M_d^- with high norm and thus very fast exponential decay rates.

If π is concentrated on the normal matrices or finitely many diagonalizable rays, then (3.36) and (3.37) become

$$-\int_{M_d^-}\frac{(\|A\|\vee 1)}{\max\Re(\sigma(A))}\pi(dA)<\infty \quad\text{and}\quad \int_{M_d^-}\|A\|\pi(dA)<\infty. \tag{3.40}$$

In particular, the second condition simply means that π has a finite first moment.

If π is concentrated on $\mathbb{S}_d^{--}$, then we have $\|A\|=-\min(\sigma(A))$, and (3.36) becomes

$$\int_{\mathbb{S}_d^{--}}\frac{(\min(\sigma(A))\wedge -1)}{\max(\sigma(A))}\pi(dA)<\infty, \tag{3.41}$$

so it can be seen as a condition on the spread between the different exponential decay rates measured by the eigenvalues. It is easy to see that in dimension $d=1$, it is equivalent to $\int_{\mathbb{R}^-}(-1/A)\pi(dA)<\infty$, which is part of the necessary and sufficient conditions for the existence of the supOU process.

(iii) It is very easy to construct examples when our sufficient conditions for X to exist as an ω-wise integral are satisfied, but neither (3.36) nor (3.37) for Z above. Take, for example, π concentrated on $v_n=\begin{pmatrix}-n & 0\\ 0 & -1\end{pmatrix}$ with $\pi(v_n)=6/(\pi^2n^2)$. In such a case we unfortunately do not know whether Z exists because our previously employed techniques seem, at best, to give a necessary condition of the type (3.36) involving $j(A) \wedge 1$, and hence the necessary conditions for Z to exist would be implied by the sufficient ones for X. Therefore, we also refrained from giving necessary conditions in this section.

3.4. *Examples and long-range dependence.* Like in the univariate case, the expression (3.31) does not imply that we necessarily have an exponential decay of the autocovariance function and thus a short memory process. On the contrary we can easily obtain a long memory process, as the following examples exhibit. Note that this illustrates that (3.32) is not obvious and indeed requires a detailed proof as above.

Apart from showing that multivariate supOU processes may exhibit long-range dependence, the purpose of this section is to analyze some concrete examples and their properties.

Regarding long-range dependence, there is, unfortunately, basically no general theory developed in the multivariate case so far. Below we mean by long-range dependence (or long memory) simply that at least one element of the autocovariance function decays asymptotically like $h^{-\alpha}$ for the lag h going to infinity and for some $\alpha \in (0; 1)$. Intuitively this should clearly be a case when one may appropriately speak of long-range dependence. Establishing a general theory for multivariate long-range dependence seems to be very important, but is beyond the scope of this paper.

EXAMPLE 3.1. Let Λ be a d-dimensional Lévy basis with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ with ν satisfying $\int_{\mathbb{R}^d} \|x\|^2 \nu(dx) < \infty$ and π being given as the distribution of RB with a diagonalizable $B \in M_d^-$ and R being a real $\Gamma(\alpha, \beta)$-distributed random variable with $\alpha > 1, \beta \in \mathbb{R}^+ \backslash \{0\}$. Hence, R has probability density $f(r) = \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r} 1_{\mathbb{R}^+}(r)$, and from

$$
\begin{aligned}
&-\int_{M_d^-} \frac{1}{\max(\Re(\sigma(A)))} \pi(dA) \\
&\quad = \frac{-\beta^\alpha}{\max(\Re(\sigma(B)))\Gamma(\alpha)} \int_{\mathbb{R}^+} r^{\alpha-2} e^{-\beta r}\, dr \\
&\quad = \frac{-\beta^\alpha}{\max(\Re(\sigma(B)))\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha-1)}{\beta^{\alpha-1}} = \frac{-\beta}{\alpha \max(\Re(\sigma(B)))}
\end{aligned}
$$

we conclude that (3.28) holds. Hence, the process $X_t = \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \Lambda(dA, ds)$ exists, is stationary and has finite second moments. Similar calculations imply that $\alpha > 1$ is also necessary for X_t to exist.

For the autocovariance function at positive lags h we find

$$\begin{aligned}\operatorname{cov}(X_h, X_0) &= -\int_{M_d^-} e^{Ah}(\mathscr{A}(A))^{-1}\operatorname{vec}\Bigl(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Bigr)\pi(dA)\\ &= \int_{\mathbb{R}^+} e^{Bhr-\beta I_d r} r^{\alpha-2}\,dr\,\Bigl(-\frac{\beta^\alpha}{\Gamma(\alpha)}\mathscr{B}^{-1}\Bigl(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Bigr)\Bigr)\end{aligned}$$

with $\mathscr{B}: M_d(\mathbb{R}) \to M_d(\mathbb{R}), X \mapsto BX + XB^*$. Let now $U \in GL_d(\mathbb{C})$ and $\lambda_1, \lambda_2, \dots, \lambda_d \in (-\infty, 0) + i\mathbb{R}$ be such that

$$UBU^{-1} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \lambda_d\end{pmatrix}.$$

Then, from $\int_0^\infty t^{z-1}e^{-kt}\,dt = \Gamma(z)k^{-z}$ for all $z, k \in (0, \infty) + i\mathbb{R}$, where the power is defined via the principal branch of the complex logarithm (see [1], page 255), we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^+} e^{Bhr-\beta I_d r} r^{\alpha-2}\,dr\\ &\quad = U\int_{\mathbb{R}^+} \exp\left(-r\left(\beta I_d - \begin{pmatrix} \lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \lambda_d\end{pmatrix} h\right)\right) r^{\alpha-2}\,dr\,U^{-1}\\ &\quad = \Gamma(\alpha-1)U\left(\beta I_d - \begin{pmatrix} \lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \lambda_d\end{pmatrix} h\right)^{1-\alpha} U^{-1}\\ &\quad = \Gamma(\alpha-1)(\beta I_d - Bh)^{1-\alpha}.\end{aligned}$$

Above the $(1-\alpha)$th power of a matrix is understood to be defined via spectral calculus as usual.

Hence,

$$\operatorname{cov}(X_h, X_0) = -\frac{\beta^\alpha}{\alpha-1}(\beta I_d - Bh)^{1-\alpha}\mathscr{B}^{-1}\Bigl(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Bigr),$$

and thus we have a polynomially decaying autocovariance function. For $\alpha \in (1, 2)$ we obviously get long memory.

Another question is whether we have the nice path properties of Theorem 3.12. Hence, assume additionally that $\int_{\|x\|\le 1}\|x\|\nu(dx) < \infty$. In our example condition

(3.34) is trivially satisfied and so the paths of X are locally uniformly bounded in t. Regarding condition (3.40) the second part is equivalent to

$$\|B\| \int_0^\infty r^\alpha e^{-\beta r}\, dr < \infty,$$

which is always true, as any Gamma distribution has a finite mean. Denoting the density of the $\Gamma(\alpha, \beta)$-distribution by $f_{\alpha,\beta}(r)$, one obtains for the first part

$$\begin{aligned} -\int_0^\infty \frac{(r\|B\| \vee 1)}{r \max(\Re(\sigma(B)))} f_{\alpha,\beta}(r)\, dr &= -\int_0^{\|B\|^{-1}} \frac{1}{r \max(\Re(\sigma(B)))} f_{\alpha,\beta}(r)\, dr \\ &\quad - \int_{\|B\|^{-1}}^\infty \frac{\|B\|}{\max(\Re(\sigma(B)))} f_{\alpha,\beta}(r)\, dr, \end{aligned}$$

which is obviously finite. Hence, the conditions of Theorem 3.12(iii) are satisfied and thus the paths are càdlàg and of finite variation, and (3.38) is valid.

EXAMPLE 3.2. The previous example has an immediate extension to the case when π is concentrated on several rays instead of a single one as above. Assume we have $w_1, \dots, w_m \in [0, 1]$ with $\sum_{i=1}^m w_i = 1$ and diagonalizable $B_1, \dots, B_m \in M_d^-$, and define π_i to be the probability measure of the random variable $R_i B_i$ with R_i being $\Gamma(\alpha_i, \beta_i)$ distributed with $\alpha_i > 1$, $\beta_i \in \mathbb{R}^+ \backslash \{0\}$. If ν is as above and $\pi = \sum_{i=1}^m w_i \pi_i$, we get for the multivariate supOU process X

$$\operatorname{cov}(X_h, X_0) = -\sum_{i=1}^m \left(\frac{w_i \beta_i^{\alpha_i}}{\alpha_i - 1} (\beta_i I_d - B_i h)^{1-\alpha_i} \mathscr{B}_i^{-1} \right) \left(\Sigma + \int_{\mathbb{R}^d} x x^* \nu(dx) \right)$$

with $\mathscr{B}_i : M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \mapsto B_i X + X B_i^*$.

Assuming now $\int_{\|x\| \le 1} \|x\| \nu(dx) < \infty$, it is likewise straightforward to see that conditions (3.34) and (3.40) are satisfied. Hence, the paths of X are locally uniformly bounded in t, càdlàg and of finite variation, and (3.38) is valid.

EXAMPLE 3.3. A similar result can be obtained if we restrict the mean reversion parameter A to the strictly negative definite matrices $\mathbb{S}_d^{--}$ and define π as a probability distribution on the proper convex cone $\mathbb{S}_d^{--}$ as follows. Let $\mathbf{S}_d^{--}$ denote the intersection of the unit sphere in $\mathbb{S}_d$ with $\mathbb{S}_d^{--}$, let $\alpha : \mathbf{S}_d^{--} \to (1, \infty)$, $\beta : \mathbf{S}_d^{--} \to (0, \infty)$ be measurable mappings and w a probability distribution on $\mathbf{S}_d^{--}$ such that

$$-\int_{\mathbf{S}_d^{--}} \frac{\beta(v)}{\alpha(v) \max(\sigma(v))} w(dv) < \infty. \tag{3.42}$$

Now define π via

$$\pi(B) = \int_{\mathbf{S}_d^{--}} \int_0^\infty 1_B(rv) \frac{\beta(v)^{\alpha(v)}}{\Gamma(\alpha(v))} r^{\alpha(v)-1} e^{-\beta(v) r}\, dr\, w(dv)$$

for any Borel set $B \in M_d^-(\mathbb{R})$. Then π is a probability distribution concentrated on $\mathbb{S}_d^{--}$.

Using this π in the above set-up means that the mean reversion parameter is no longer necessarily restricted to finitely many rays. Moreover, similar calculations to the ones in Example 3.1 give

$$-\int_{M_d^-} \frac{1}{\max(\Re(\sigma(A)))} \pi(dA) = \int_{\mathbf{S}_d^{--}} \frac{-\beta(v)}{\alpha(v)\max(\sigma(v))} w(dv) < \infty.$$

Hence, (3.28) holds and the process $X_t = \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \Lambda(dA, ds)$ exists, is stationary and has finite second moments. Likewise we get for the autocovariance function

$$\begin{aligned}\operatorname{cov}(X_h, X_0) = -\bigg(\int_{\mathbf{S}_d^{--}} \frac{\beta(v)^{\alpha(v)}}{\alpha(v)-1} \Big(\beta(v) I_d - vh\Big)^{1-\alpha(v)} (\mathscr{V}(v))^{-1} w(dv)\bigg) \\ \times \left(\Sigma + \int_{\mathbb{R}^d} xx^* \nu(dx)\right)\end{aligned}$$

with $\mathscr{V}(v) : M_d(\mathbb{R}) \to M_d(\mathbb{R}),\, X \mapsto vX + Xv^*$.

Turning to the path properties, assume now that $\int_{\|x\| \le 1} \|x\| \nu(dx) < \infty$. Again condition (3.34) is trivially satisfied and so the paths of X are locally uniformly bounded in t. Regarding condition (3.40) the second part becomes

$$\int_{M_d^-} \|A\| \pi(dA) = \int_{\mathbf{S}_d^{--}} \int_0^\infty r f_{\alpha(v),\beta(v)}(r)\, dr\, w(dv) = \int_{\mathbf{S}_d^{--}} \frac{\alpha(v)}{\beta(v)} w(dv),$$

and for the first part one obtains

$$\begin{aligned}&-\int_{M_d^-} \frac{(\|A\| \vee 1)}{\max \Re(\sigma(A))} \pi(dA) \\ &\quad = -\int_{\mathbf{S}_d^{--}} \int_0^1 \frac{1}{r \max(\sigma(v))} f_{\alpha(v),\beta(v)}(r)\, dr\, w(dv) \\ &\qquad - \int_{\mathbf{S}_d^{--}} \int_1^\infty \frac{1}{\max(\sigma(v))} f_{\alpha(v),\beta(v)}(r)\, dr\, w(dv).\end{aligned} \tag{3.43}$$

The first summand is finite due to (3.42) and the second one is finite if the integral $-\int_{\mathbf{S}_d^{--}} (1/\max(\sigma(v))) w(dv)$ is finite. Hence, provided

$$-\int_{\mathbf{S}_d^{--}} \frac{1}{\max(\sigma(v))} w(dv) < \infty \quad \text{and} \quad \int_{\mathbf{S}_d^{--}} \frac{\alpha(v)}{\beta(v)} w(dv) < \infty,$$

the conditions of Theorem 3.12(iii) are satisfied, and thus the paths are càdlàg and of finite variation and (3.38) is valid.

Based on this we can easily give an example where we know that the supOU process exists due to Proposition 3.5, but the conditions of Theorem 3.12(iii) are not satisfied. Assume w is a discrete distribution concentrated on the points

$$v_n = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1+(3n)^{-1} & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, \qquad n \in \mathbb{N},$$

and that $w(v_n) = \frac{6}{\pi^2 n^2}$, $\alpha(v_n) = 2$ and $\beta(v_n) = n^{-1}$. Then we have that

$$-\int_{\mathbf{S}_d^{--}} \frac{\beta(v)}{\alpha(v) \max(\sigma(v))} w(dv) = \frac{6}{\pi^2} \sum_{n=1}^{\infty} n^{-3} < \infty,$$

but

$$\int_{M_d^-} \|A\| \pi(dA) = \frac{12}{\pi^2} \sum_{n=1}^{\infty} n^{-1} = \infty,$$

and hence condition (3.37) is not satisfied. Observe that this means that the probability measure π we have constructed does not have a first moment, although it is defined via a polar representation where the radial parts are all univariate Gamma distributions. Moreover, it is easy to see that (3.43) is finite and so Z exists and thus it is only the local uniform boundedness our sufficient conditions fail to provide when trying to show (3.38). Showing that (3.38) is indeed not valid seems to be a very delicate issue, as already remarked.

EXAMPLE 3.4. Let Λ be now a two-dimensional Lévy basis with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ with ν satisfying $\int_{\mathbb{R}^2} \|x\|^2 \nu(dx) < \infty$. We restrict the mean reversion parameter A to $\mathbb{D}_2^{--}$, the 2×2 diagonal matrices with strictly negative entries on the diagonal. Hence, π is a measure on $\mathbb{D}_2^{--}$, which can be identified with $(\mathbb{R}^{--})^2$, and we assume that π has Lebesgue density

$$\begin{aligned} &\pi(da_1, da_2) \\ &\quad = \frac{\beta_1^{\alpha_1} \beta_2^{\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (-a_1)^{\alpha_1 - 1} (-a_2)^{\alpha_2 - 1} e^{\beta_1 a_1 + \beta_2 a_2} 1_{(\mathbb{R}^{--})^2}(a_1, a_2)\, da_1\, da_2 \end{aligned}$$

with $\alpha_1, \alpha_2 > 1$ and $\beta_1, \beta_2 > 0$. So the diagonal elements are independent, and their absolute values follow Gamma distributions. We obtain

$$\begin{aligned} &-\int_{\mathbb{D}_2^{--}} \frac{1}{\max(\Re(\sigma(A)))} \pi(dA) \\ &\quad = \int_0^\infty \int_0^\infty \frac{1}{\min(a_1, a_2)} \frac{\beta_1^{\alpha_1} \beta_2^{\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (a_1)^{\alpha_1 - 1} (a_2)^{\alpha_2 - 1} e^{-\beta_1 a_1 - \beta_2 a_2}\, da_1\, da_2 \\ &\quad \le \int_0^\infty \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} (a_1)^{\alpha_1 - 2} e^{-\beta_1 a_1}\, da_1 \int_0^\infty \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} (a_2)^{\alpha_2 - 1} e^{-\beta_2 a_2}\, da_2 \\ &\qquad + \int_0^\infty \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} (a_1)^{\alpha_1 - 1} e^{-\beta_1 a_1}\, da_1 \int_0^\infty \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} (a_2)^{\alpha_2 - 2} e^{-\beta_2 a_2}\, da_2 < \infty. \end{aligned}$$

Hence, (3.28) holds, and the process $X_t = \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \Lambda(dA, ds)$ exists, is stationary and has finite second moments.

Let us now consider the individual components $X_{1,t}$, $X_{2,t}$ of X_t. Denote by $P_1 : \mathbb{R}^2 \to \mathbb{R}$, $(x_1, x_2)^* \mapsto x_1$ the projection onto the first coordinate and define an $\mathbb{R}$-valued Lévy basis Λ_1 on $\mathbb{R}^{--} \times \mathbb{R}$ via $\Lambda_1(da_1, ds) = P(\Lambda(P_1^{-1}(da_1), ds)$ and a Lévy measure ν_1 on $\mathbb{R}$ via $\nu_1(dx_1) = \nu(P_1^{-1}(dx_1))$. Then Λ_1 has characteristic quadruple $(\gamma_1, \Sigma_{11}, \nu_1, \pi_1)$ with π_1 having Lebesgue density

$$\pi_1(da_1) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1))} (-a_1)^{\alpha_1 - 1} e^{\beta_1 a_1} 1_{(\mathbb{R}^{--})}(a_1)\, da_1$$

and

$$X_{1,t} = \int_{\mathbb{R}^{--}} \int_{-\infty}^t e^{a_1(t-s)} \Lambda_1(da_1, ds).$$

For the autocovariance function of the first component we get

$$\operatorname{cov}(X_{1,h}, X_{1,0}) = \frac{\beta_1^{\alpha_1}}{2(\alpha_1 - 1)} (\beta_1 + h)^{1-\alpha_1} \biggl(\Sigma_{11} + \int_{\mathbb{R}} x_1^2 \nu_1(dx_1) \biggr), \qquad h \in \mathbb{R}^+.$$

An analogous result holds for the second component $X_{2,t}$ and we have long memory in both components provided $\alpha_1, \alpha_2 \in (1, 2)$.

The importance of this example is, however, that we can model the stationary distributions of X_1 and X_2, that is, the margins of the stationary distribution of X_t, very explicitly by specifying the margins of ν, that is, ν_1 and ν_2. From [2], Theorem 3.1, Corollary 3.1, and [19], Remark 2.2, we know that it is exactly all nondegenerate self-decomposable distributions on $\mathbb{R}$ which arise as the stationary distributions of the components. Moreover, these authors provide formulae to calculate ν_1 (or ν_2) if one wants to obtain a given stationary distribution for the component (alternatively [3], Lemma 5.1, or the refinement [42], Theorem 4.9, can be used). Hence, one can specify a two-dimensional supOU process with prescribed stationary distributions of the components by calculating the required ν_1 and ν_2 and choosing ν accordingly. The easiest way to get a possible ν is by specifying $\nu(dx_1, dx_2) = \nu_1(dx_1) \times \delta_0(x_2) + \delta_0(x_1) \times \nu_2(dx_2)$ with δ_0 denoting the Dirac distribution with unit mass at zero. In this case the components of X are independent. An easy way to get an appropriate ν and allowing for dependence is to combine ν_1 and ν_2 using a Lévy copula (see [4, 28]).

Likewise it is again interesting to look at the path properties of Theorem 3.12. Assuming again $\int_{\|x\| \le 1} \|x\| \nu(dx) < \infty$, condition (3.34) is trivially satisfied, and so the paths of X are locally uniformly bounded in t. Regarding the second part of condition (3.40) we have that

$$\int_{M_d^-} \|A\| \pi(dA) < \infty$$

is equivalent to

$$\int_{(\mathbb{R}^+)^2} \max(a_1, a_2) f_{\alpha_1,\beta_1}(a_1) f_{\alpha_2,\beta_2}(a_2)\, da_1\, da_2$$
$$\leq \int_{\mathbb{R}^+} a_1 f_{\alpha_1,\beta_1}(a_1)\, da_1 + \int_{\mathbb{R}^+} a_2 f_{\alpha_2,\beta_2}(a_2)\, da_2 < \infty,$$

which is always true. Turning to (3.41), it is implied by

$$\int_{(\mathbb{R}^+)^2} \frac{\max(a_1, a_2)}{\min(a_1, a_2)} f_{\alpha_1,\beta_1}(a_1) f_{\alpha_2,\beta_2}(a_2)\, da_1\, da_2$$
$$\leq \int_{(\mathbb{R}^+)^2} \frac{a_1 + a_2}{a_1} f_{\alpha_1,\beta_1}(a_1) f_{\alpha_2,\beta_2}(a_2)\, da_1\, da_2$$
$$+ \int_{(\mathbb{R}^+)^2} \frac{a_1 + a_2}{a_2} f_{\alpha_1,\beta_1}(a_1) f_{\alpha_2,\beta_2}(a_2)\, da_1\, da_2 < \infty,$$

which is easily seen to be always true. Hence, the conditions of Theorem 3.12(iii) are satisfied, and thus the paths are càdlàg and of finite variation and (3.38) is valid.

Obviously this example has a straightforward extension to general dimension d.

EXAMPLE 3.5. So far we have only studied cases where we could use Proposition 3.7 and did especially never have to bother with $\kappa(A)$ in the conditions of Theorem 3.1.

In this example we will present a case where the behavior of $\kappa(A)$ is crucial and where we show how κ and ρ can be specified in a measurable way. We define the following sets:

$$\mathscr{D}_d^- = \{X \in M_d(\mathbb{R}) : X \text{ is diagonal; all diagonal elements are strictly negative,}$$
$$\text{pairwise distinct and ordered such that } x_{ii} < x_{jj}\ \forall 1 \leq i \leq j \leq d\};$$
$$\mathscr{S}_d = \{X \in GL_d(\mathbb{R}) : \text{ the first nonzero element in each column is } 1\};$$
$$\mathscr{M}_d^- = \{SDS^{-1} : S \in \mathscr{S}_d, D \in \mathscr{D}_d^-\}.$$

If $A = SDS^{-1}$ is in $\mathscr{M}_d^-$, the matrix D consists of the eigenvalues of A, and the columns of S are the eigenvectors of A. In principle there are many possible S and D if we only demand $A = SDS^{-1}$. However, if we restrict ourselves to $S \in \mathscr{S}_d$, $D \in \mathscr{D}_d^-$, then S, D are unique, as elementary linear algebra shows. This means that the map

$$\mathfrak{M} : \mathscr{S}_d \times \mathscr{D}_d^- \to \mathscr{M}_d^-, (S, D) \mapsto SDS^{-1}$$

is bijective (and obviously continuous). We denote by $\mathfrak{M}^{-1} = (\mathfrak{S}, \mathfrak{D})$ the inverse mapping. Since computing eigenvectors and eigenvalues are measurable procedures as are the orderings and normalizations involved in obtaining the diagonal matrix in $\mathscr{D}_d^-$ and the eigenvector matrix in $\mathscr{S}_d$, all these mappings are measurable. Note also that $\mathscr{D}_d^-$, $\mathscr{S}_d$, $\mathscr{M}_d^-$ are Borel sets.

Defining $\kappa : \mathscr{M}_d^- \to [1,\infty), A \mapsto \|\mathfrak{S}(A)\|\|(\mathfrak{S}(A))^{-1}\|$, $\rho(A) = -\max(\Re(\sigma(A)))$ gives, therefore, measurable mappings on $\mathscr{M}_d^-$ satisfying $\|e^{As}\| \le \kappa(A)e^{-\rho(A)s}$. Using these definitions for κ and ρ one could now specify probability distributions π on $\mathscr{M}_d^-$ and check whether condition (3.3) is satisfied, and the associated supOU process therefore exists.

However, in concrete situations it seems easier to specify a Borel probability measure $\pi_{\mathscr{S}_d \times \mathscr{D}_d^-}$ on $\mathscr{S}_d \times \mathscr{D}_d^-$ and define π as its image under $\mathfrak{M}$, i.e. $\pi(B) = \pi_{\mathscr{S}_d \times \mathscr{D}_d^-}(\mathfrak{M}^{-1}(B))$ for all Borel sets B. Assume $\pi_{\mathscr{S}_d \times \mathscr{D}_d^-} = \pi_{\mathscr{S}_d} \times \pi_{\mathscr{D}_d^-}$ is the product of two probability measures $\pi_{\mathscr{S}_d}$ on $\mathscr{S}_d$ and $\pi_{\mathscr{D}_d^-}$ on $\mathscr{D}_d^-$. Then we have

$$\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty \quad \Longleftrightarrow \quad \int_{\mathscr{S}_d} \|S\|^2 \|S^{-1}\|^2 \pi_{\mathscr{S}_d}(dS) < \infty \quad \text{and}$$
$$-\int_{\mathscr{D}_d^-} \frac{1}{\max(\Re(\sigma(D)))} \pi_{\mathscr{D}_d^-}(dD) < \infty.$$

That $\int_{\mathscr{S}_d} \|S\|^2 \|S^{-1}\|^2 \pi_{\mathscr{S}_d}(dS)$ can be finite or infinite depending on the choice of $\pi_{\mathscr{S}_d}$ is exhibited by the following example. Let $\pi_{\mathscr{S}_2}$ be a discrete measure concentrated on the points

$$S_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \quad \text{and} \quad p_n := \pi_{\mathscr{S}_2}(S_n) = C_\alpha n^{-\alpha} \qquad \forall n \in \mathbb{N}$$

with $\alpha > 1$ and $C_\alpha = 1/\sum_{n=1}^\infty n^{-\alpha}$. Then

$$S_n^{-1} = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}.$$

Using the equivalence of all norms we get that

$$\int_{\mathscr{S}_2} \|S\|^2 \|S^{-1}\|^2 \pi_{\mathscr{S}_2}(dS) < \infty \quad \Longleftrightarrow \quad C_\alpha \sum_{n=1}^\infty n^4 p_n < \infty \quad \Longleftrightarrow \quad \alpha > 5.$$

Returning to the general example with π given via $\pi_{\mathscr{S}_d} \times \pi_{\mathscr{D}_d^-}$ and turning to path properties, we assume again $\int_{\|x\|\le 1} \|x\|\nu(dx) < \infty$. In this finite variation case the existence conditions (3.17) become

$$\int_{\mathscr{S}_d} \|S\|\|S^{-1}\| \pi_{\mathscr{S}_d}(dS) < \infty \quad \text{and} \quad -\int_{\mathscr{D}_d^-} \frac{1}{\max(\Re(\sigma(D)))} \pi_{\mathscr{D}_d^-}(dD) < \infty.$$

Furthermore, condition (3.34) is always satisfied when the existence conditions are satisfied and so the paths of X are locally uniformly bounded in t. Straightforward arguments show that the conditions of Theorem 3.12(iii) are satisfied, and thus the paths are càdlàg and of finite variation and (3.38) is valid if

$$\int_{\mathscr{S}_d} \|S\|^2 \|S^{-1}\|^2 \pi_{\mathscr{S}_d}(dS) < \infty, \qquad -\int_{\mathscr{D}_d^-} \frac{\|D\|}{\max(\Re(\sigma(D)))} \pi_{\mathscr{D}_d^-}(dD) < \infty$$

and

$$\int_{\mathscr{D}_d^-} \|D\| \pi_{\mathscr{D}_d^-}(dD) < \infty.$$

By polarly decomposing $\pi_{\mathscr{D}_d^-}$ into a measure on the unit sphere in the diagonal matrices and a radial part, the long memory specifications of the foregoing examples have straightforward extensions to this set-up.

4. Positive semi-definite supOU processes. Based on the previous section we now consider supOU processes which are positive semi-definite at all times. The importance of such processes is that they can be used to describe the random evolution of a latent covariance matrix over time and, hence, they can be used in multivariate models for heteroskedastic data, for example, the stochastic volatility model of [9].

Let us briefly recall that a $d \times d$ positive semi-definite OU-type process (see [8]) is defined as the unique càdlàg solution of the SDE

$$d\Sigma_t = (A\Sigma_t + \Sigma_t A^*)\,dt + dL_t, \qquad \Sigma_0 \in \mathbb{S}_d^+,$$

with $A \in M_d(\mathbb{R})$ and L being a $d \times d$ matrix subordinator (see [5]), that is, a Lévy process in $\mathbb{S}_d$ with $L_t - L_s \in \mathbb{S}_d^+$ $\forall s, t \in \mathbb{R}^+, s < t$. If $E(\ln(\max(\|L_1\|, 1))) < \infty$ and $\max(\Re(\sigma(A))) < 0$, the above SDE has the unique stationary solution

$$\Sigma_t = \int_{-\infty}^t e^{A(t-s)}\,dL_s e^{A^*(t-s)}.$$

That the linear operators $\mathbb{S}_d \to \mathbb{S}_d$ of the form $Z \mapsto AZ + ZA^*$ with some $A \in M_d(\mathbb{R})$ are the ones to be used for positive semi-definite OU-type processes has been established in [42].

As just recalled, one has to restrict the driving Lévy process to matrix subordinators in order to obtain OU-type processes taking values in the positive semi-definite matrices. Below we need to impose a comparable condition on the Lévy basis to get positive semi-definite supOU processes. Note that for a $d \times d$ matrix-valued Lévy basis Λ we denote by $\mathrm{vec}(\Lambda)$ the $\mathbb{R}^{d^2}$-valued Lévy basis given by $\mathrm{vec}(\Lambda)(B) = \mathrm{vec}(\Lambda(B))$ for all Borel sets B. Moreover, observe that $\mathrm{tr}(XY^*)$ [with $X, Y \in M_d(\mathbb{R})$ and tr denoting the usual trace functional] defines a scalar product on $M_d(\mathbb{R})$ and that the vec operator is a Hilbert space isometry between $M_d(\mathbb{R})$ equipped with this scalar product and $\mathbb{R}^{d^2}$ with the usual Euclidean scalar product.

Positive semi-definite supOU processes are defined as processes of the form (4.4) below, which is the analogue of (3.4).

THEOREM 4.1. *Let Λ be an $\mathbb{S}_d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, 0, \nu, \pi)$ with $\gamma_0 := \gamma - \int_{\|x\|\le 1} x\nu(dx) \in \mathbb{S}_d^+$ and ν being a Lévy measure on $\mathbb{S}_d$ satisfying $\nu(\mathbb{S}_d \backslash \mathbb{S}_d^+) = 0$,*

$$\int_{\|x\|>1} \ln(\|x\|)\nu(dx) < \infty \quad \text{and} \quad \int_{\|x\|\le 1} \|x\|\nu(dx) < \infty. \tag{4.1}$$

Moreover, assume there exist measurable functions $\rho: M_d^- \to \mathbb{R}^+\backslash\{0\}$ and $\kappa: M_d^- \to [1,\infty)$ such that

$$\|e^{As}\| \le \kappa(A)e^{-\rho(A)s} \qquad \forall s \in \mathbb{R}^+, \pi\text{-almost surely}, \tag{4.2}$$

and

$$\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty. \tag{4.3}$$

Then the process $(\Sigma_t)_{t\in\mathbb{R}}$ given by

$$\begin{aligned} \Sigma_t &= \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \Lambda(dA, ds) e^{A^*(t-s)} \\ &= \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \gamma_0 e^{A^*(t-s)}\, ds\, \pi(dA) \\ &\quad + \int_{\mathbb{S}_d} \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} x e^{A^*(t-s)} \mu(dx, dA, ds) \end{aligned} \tag{4.4}$$

is well defined as a Lebesgue integral for all $t \in \mathbb{R}$ and $\omega \in \Omega$ and Σ is stationary.

Moreover,

$$\operatorname{vec}(\Sigma_t) = \int_{M_d^-} \int_{-\infty}^t e^{(A\otimes I_d + I_d \otimes A)(t-s)} \operatorname{vec}(\Lambda)(dA, ds), \tag{4.5}$$

$\Sigma_t \in \mathbb{S}_d^+$ for all $t \in \mathbb{R}$ and the distribution of Σ_t is infinitely divisible with characteristic function

$$E(\exp(i \operatorname{tr}(u\Sigma_t))) = \exp\biggl(i \operatorname{tr}(u\gamma_{\Sigma,0}) + \int_{\mathbb{S}_d} (e^{i \operatorname{tr}(ux)} - 1)\nu_\Sigma(dx)\biggr), \qquad u \in \mathbb{S}_d,$$

where

$$\gamma_{\Sigma,0} = \int_{M_d^-} \int_0^\infty e^{As} \gamma_0 e^{A^* s}\, ds\, \pi(dA), \tag{4.6}$$

$$\nu_\Sigma(B) = \int_{M_d^-} \int_0^\infty \int_{\mathbb{S}_d^+} 1_B(e^{As} x e^{A^* s}) \nu(dx)\, ds\, \pi(dA) \tag{4.7}$$

for all Borel sets $B \subseteq \mathbb{S}_d$.

PROOF. The equivalence of (4.5) and (4.4) follows from standard results on the vectorization operator and the tensor product (see [23]).

Next we note that $e^{(A\otimes I_d+I_d\otimes A)(t-s)} = e^A \otimes e^A$ and that $\|e^A \otimes e^A\| = \|e^A\|^2$ (using the operator norm associated with the Euclidean norm). Hence, all assertions except $\Sigma_t \in \mathbb{S}_d^+$ for all $t \in \mathbb{R}^+$ follow immediately from Propositions 2.4 and 3.5.

However, $\Sigma_t \in \mathbb{S}_d^+$ for all $t \in \mathbb{R}^+$ is now immediate, since the integral exists ω-wise, $e^{As} X e^{A^*s} \in \mathbb{S}_d^+$ $\forall A \in M_d(\mathbb{R})$, $X \in \mathbb{S}_d^+$, $s \in \mathbb{R}$ and $\mathbb{S}_d^+$ is a closed convex cone. $\square$

REMARK 4.2. (i) As in Proposition 3.7, $\kappa(A)$ can be replaced by 1 and $\rho(A)$ by $-\max(\Re(\sigma(A)))$ in (4.3) [and also in (4.14) and (4.15) below] provided π is concentrated on the normal matrices or finitely many diagonalizable rays.

(ii) Throughout this section we refrain from stating necessary conditions, as they can be immediately inferred from the foregoing sections and the arguments presented for the sufficient conditions.

Most importantly in the context of stochastic volatility models, which involve stochastic integrals with Σ as integrand, Theorem 3.12 also has an analogue for positive semi-definite supOU processes.

THEOREM 4.3. *Let Σ be the positive semi-definite supOU process of Theorem 4.1. Then*:

(i) $\Sigma_t(\omega)$ *is* $\mathscr{B}(\mathbb{R}) \times \mathscr{F}$ *measurable as a function of* $t \in \mathbb{R}$ *and* $\omega \in \Omega$ *and adapted to the filtration* $(\mathscr{F}_t)_{t\in\mathbb{R}}$ *generated by* Λ.

(ii) *If*

$$\int_{M_d^-} \kappa(A)^2 \pi(dA) < \infty, \tag{4.8}$$

the paths of Σ *are locally uniformly bounded in* t *for every* $\omega \in \Omega$.

Furthermore, $\Sigma_t^+ = \int_0^t \Sigma_s\, ds$ *exists for all* $t \in \mathbb{R}^+$ *and*

$$\begin{aligned}
\Sigma_t^+ &= \int_{M_d^-} \int_{-\infty}^t (\mathbf{A}(A))^{-1} \big(e^{A(t-s)} \Lambda(dA, ds) e^{A^*(t-s)}\big) \\
&\quad - \int_{M_d^-} \int_{-\infty}^0 (\mathbf{A}(A))^{-1} (e^{-As} \Lambda(dA, ds) e^{-A^*s}) \\
&\quad - \int_{M_d^-} \int_0^t (\mathbf{A}(A))^{-1} \Lambda(dA, ds)
\end{aligned} \tag{4.9}$$

with $\mathbf{A}(A) : \mathbb{S}_d \to \mathbb{S}_d$, $X \mapsto AX + XA^*$.

(iii) *Provided that*

$$-\int_{M_d^-} \frac{(\|A\| \vee 1)\kappa(A)^2}{\rho(A)} \pi(dA) < \infty \tag{4.10}$$

and

$$\int_{M_d^-} \|A\|\kappa(A)^2 \pi(dA) < \infty \tag{4.11}$$

it holds that

$$\Sigma_t = \Sigma_0 + \int_0^t Z_u du + L_t, \tag{4.12}$$

where L is the underlying matrix subordinator and

$$\begin{aligned} Z_u = \int_{M_d^-} \int_{-\infty}^{u} \bigl(& A e^{A(u-s)} \Lambda(dA, ds) e^{A^*(u-s)} \\ & + e^{A(u-s)} \Lambda(dA, ds) e^{A^*(u-s)} A^*\bigr) \end{aligned} \tag{4.13}$$

for all $u \in \mathbb{R}$ with the integral existing ω-wise.

Moreover, the paths of Σ are càdlàg and of finite variation on compacts.

Formula (4.9) is of particular interest in connection with stochastic volatility modeling, as in this case the integrated volatility Σ_t^+ is a quantity of fundamental importance (see Section 5.2).

Finally, we consider the existence of moments and the second-order structure which follow immediately from Theorems 3.9 and 3.11.

PROPOSITION 4.4. *Let Σ be a stationary $\mathbb{S}_d^+$-valued supOU process driven by a Lévy basis Λ satisfying the conditions of Theorem 4.1.*

(i) *If*

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty \tag{4.14}$$

for $r \in (0, 1]$, then Σ has a finite rth moment, that is, $E(\|\Sigma_t\|^r) < \infty$.

(ii) *If $r \in (1, \infty)$ and*

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty, \qquad \int_{M_d^-} \frac{\kappa(A)^{2r}}{\rho(A)} \pi(dA) < \infty, \tag{4.15}$$

then Σ has a finite rth moment, that is, $E(\|\Sigma_t\|^r) < \infty$.

(iii) *If the conditions given in* (ii) *are satisfied for* $r = 2$, *then the second-order structure of* Σ *is given by*

$$E(\Sigma_0) = -\int_{M_d^-} \mathbf{A}(A)^{-1}\left(\gamma_0 + \int_{\mathbb{S}_d} x\nu(dx)\right)\pi(dA),$$

$$\operatorname{var}(\operatorname{vec}(\Sigma_0)) = -\int_{M_d^-} (\mathscr{A}(A))^{-1}\left(\int_{\mathbb{S}_d} \operatorname{vec}(x)\operatorname{vec}(x)^*\nu(dx)\right)\pi(dA),$$

$$\begin{aligned}\operatorname{cov}(\operatorname{vec}(\Sigma_h), \operatorname{vec}(\Sigma_0)) = -\int_{M_d^-} & e^{(A\otimes I_d + I_d\otimes A)h}(\mathscr{A}(A))^{-1} \\ & \times\left(\int_{\mathbb{S}_d} \operatorname{vec}(x)\operatorname{vec}(x)^*\nu(dx)\right)\pi(dA) \\ & \qquad \forall h\in\mathbb{R}^+,\end{aligned}$$

with $\mathbf{A}(A): M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \mapsto AX + XA^*$ *and* $\mathscr{A}(A): M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R})$, $X \mapsto (A\otimes I_d + I_d\otimes A)X + X(A^*\otimes I_d + I_d\otimes A^*)$.

Examples 3.1–3.5 can all be immediately adapted to the positive semi-definite set-up. More examples in connection with stochastic volatility modeling can be found in [9].

5. Areas of applications. In this section we discuss possible applications for our model and the relevance of our results for them. Some of these applications are already developed further in other work.

5.1. *Time series modeling.* In many areas of applications (e.g., telecommunication, hydrology, economics, finance) one is confronted with time series exhibiting a long memory behavior (see [16], for instance) or at least a decay of the autocovariance appearing to be a polynomial decay rather than the exponential as typically encountered in models of Markovian nature. Moreover, often the relevant data series are multidimensional, and multivariate models are needed in order to understand and adequately model the dependence effects of the observed data. For irregularly-spaced or high-frequency data as well as when intending to look at the data at more than one frequency, it is often advisable not to use discrete-time models (like AR(FI)MA, see, e.g., [12]), but continuous-time models. Such models can exhibit both continuous and discontinuous sample paths. Among the ones with continuous sample paths are continuous-time counterparts of ARFIMA, like FICARMA (see [13, 31, 47]), which also exhibit long memory. Often it is, however, appropriate to use models with discontinuous sample paths. In such situations it appears adequate to use multivariate supOU processes. It should be noted that the individual autocovariances of multivariate supOU processes do not necessarily have to decay monotonically like in the univariate case, but may exhibit damped sinusoidal-like and comparable behavior due to the involved matrix exponentials. Hence, one can reproduce second-order moment behavior which in the univariate case calls for the use of CARMA(p, q)-based models.

As for all such models it is a delicate issue to estimate multivariate supOU processes from observed data, and this poses many challenging questions. Our calculation of the second-order moment structure clearly opens the door for (general) method-of-moments-based techniques. Let us illustrate this in a simple example.

Consider the set-up of Example 3.1 with B restricted to have only real eigenvalues. Note that in this set-up the parameters cannot be identified, because replacing β by βc and B by cB leaves the law of the Lévy basis invariant for any $c > 0$. So we assume $\beta = 1$ without loss of generality. Recall that

$$\operatorname{acov}(h) := \operatorname{cov}(X_h, X_0) = -\frac{(I_d - Bh)^{1-\alpha}}{\alpha - 1}\mathscr{B}^{-1}\left(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\right)$$

with $\mathscr{B}: M_d(\mathbb{R}) \to M_d(\mathbb{R}), X \mapsto BX + XB^*$. Assuming that $\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)$ is invertible we can define

$$\Gamma_h = \operatorname{acov}(h)\operatorname{acov}(0)^{-1} = (I_d - Bh)^{1-\alpha}$$

which has eigenvalues of the form $f(h) = (1 - \lambda h)^{1-\alpha}$ with $\lambda \in (-\infty, 0)$. So we can obtain an estimator $\hat{\alpha}$ by calculating the empirical autocovariance function from data and fitting f to the maximum (or any other) eigenvalue of $\widehat{\operatorname{acov}}(h)\widehat{\operatorname{acov}}(0)^{-1}$ for $h = n\Delta$ with $n \in \mathbb{N}$ and $\Delta > 0$ being the distance between subsequent observations (assumed to be constant) by nonlinear least squares, for instance. Thereafter, we can get an estimator for B by $\hat{B} = I_d - (\widehat{\operatorname{acov}}(h)\widehat{\operatorname{acov}}(0)^{-1})^{\hat{\alpha}-1}$. Now it is straightforward to also get estimators for $\gamma + \int_{\|x\|>1} x\nu(dx)$ and $\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)$ from the empirical mean and variance. One thus obtains the first two moments of the underlying Lévy process. If we restrict the allowed Lévy bases such that the first two moments of the underlying Lévy process identify the parameters γ, Σ, ν, we have thus obtained a procedure to estimate all parameters of our model.

To prove consistency and asymptotic normality (or other asymptotic distributions in the case of true long memory) of the estimators one obviously needs to understand the (highly non-Markovian) dependence structure of our processes and establish mixing conditions or appropriate substitutes. We hope to address this issue in future work.

In a set-up like in Example 3.4 the estimation becomes much easier because all parameters except the ones describing the dependence between the components can be inferred from the univariate marginal distributions and the univariate moment structure. In particular, the parameters β_i and α_i can be estimated from the autocovariances of the individual one-dimensional series; hence, one does not have to compute eigenvalues or matrix powers as above.

Finally, it should be noted that in Theorem 3.1 or formula (2.8) we have given formulae for the characteristic function of multivariate supOU processes. Hence, one can also use estimation techniques based on the empirical characteristic function. This is obviously best done in cases where the driving Lévy basis is chosen such that the integrals in Theorem 3.1 or formula (2.8) can be calculated analytically. Moreover, these formulae can be used to calculate higher order cumulants for the methods of moments based estimation.

5.2. *Finance and econometrics.* Lévy-based stochastic volatility models are very successfully applied in both financial mathematics and financial econometrics, since they capture many of the stylized facts (nonconstant, stochastic volatility exhibiting jumps, heavy tails, volatility clustering, leverage effect, ...; see, e.g., [15, 20]) of financial returns very well. One model often employed is the Ornstein–Uhlenbeck-type stochastic volatility model introduced in [6]. The use of supOU processes as the volatility process, that is, the process modelling the instantaneous (co)variance, allows one to introduce also long memory, which is another important stylized fact, but not covered by most models, into the model. Let us illustrate this in a simple set-up. Let Σ be a positive semi-definite supOU process as introduced in Section 4 and satisfying the conditions of Theorem 4.3(ii) and (iii). Then the log-returns of d financial assets (stocks or currencies, for instance) are given by

$$Y_t = Y_0 + \int_0^t (\mu + \Sigma_s \beta)\, ds + \int_0^t \Sigma_s^{1/2}\, dW_s + \rho\, dL_t \tag{5.1}$$

with $\mu, \beta \in \mathbb{R}^d$, initial log prices Y_0 independent of Λ, $\rho : \mathbb{S}_d \to \mathbb{R}^d$ a linear operator and L being the underlying Lévy process. It should be noted that in this model jumps in the price and the volatility always occur together which is reasonable for financial data (see [25]).

An extension of the above model has been investigated in-depth in [9] where it is, in particular, shown that long memory in Σ causes long-range dependence in Y and explicit formulae for the moment structure of the (squared) returns are obtained. This allows an in-depth econometric analysis and estimation of the supOU stochastic volatility model comparable to what has been done in [41] for the multivariate OU-type stochastic volatility model, where it was shown that that model can be estimated and fits well to observed data, both from the stock and foreign exchange markets.

In financial mathematics one is often interested in calculating prices of derivatives from a given model and in determining the parameters from option prices observed on the markets, referred to as calibration. In most reasonably realistic models, unlike the Black–Scholes model, one cannot obtain closed-form formulae for the prices of derivatives. However, calculating prices via Monte Carlo simulations is typically too time consuming. Whenever possible, better techniques to calculate derivative prices, that is, conditional expectations of future payoffs, are called for. One technique which proved to be very adequate in many situations is the calculation of the prices by inverting the Laplace transform (see [14, 17, 37]). For the multivariate OU model this technique is successfully applied in [36].

Also, in the above given model (5.1) one can calculate the conditional Fourier transform of the future prices. Since some complex values will arise, one has to be careful with the scalar products. We use the scalar product $\langle x_1, x_2\rangle = x_2^* x_1$ on $\mathbb{R}^d$ and $\langle X_1, X_2\rangle = \operatorname{tr}(X_2^* X_1)$ on $\mathbb{S}_d$ denoting by x^* the Hermitian and by tr the trace of a matrix. For a linear operator * also denotes the adjoint operator in the following. Moreover, we assume that $E(\exp(i \operatorname{tr}(\Lambda(B)^* u)) = \exp(\varphi_\Lambda(u)\Pi(B))$ for all $u \in \mathbb{S}_d$ with φ being the cumulant transform of the underlying Lévy process and $\Pi = \pi \times \Lambda$. Actually, $E(\exp(i \operatorname{tr}(\Lambda(B)^* u))$ exists for all $u \in M_d(\mathbb{R}) + i\mathbb{S}_d^+$ and φ can be extended to this domain as well. Then it follows by similar arguments as in [36, 41] that

$$
\begin{aligned}
E(e^{iY_t^* u} | Y_0) = \exp\Biggl\{ & i(Y_0 + \mu t)^* u \\
& + \int_{M_d^-} \int_{-\infty}^{t} \varphi_\Lambda \biggl[e^{A^*(t-s)} \Bigl(\mathbf{A}(A)^{-*} \Bigl(u\beta^* + \frac{i}{2} u u^* \Bigr) \Bigr) e^{A(t-s)} \\
& - 1_{(-\infty,0]}(s) e^{-A^* s} \Bigl(\mathbf{A}(A)^{-*} \Bigl(u\beta^* + \frac{i}{2} u u^* \Bigr) \Bigr) e^{-As} \\
& - 1_{(0,t]}(s) \Bigl(\mathbf{A}(A)^{-*} \Bigl(u\beta^* + \frac{i}{2} u u^* \Bigr) - \rho^* u \Bigr) \biggr] \, ds \, \pi(dA) \Biggr\}
\end{aligned}
\tag{5.2}
$$

for all $u \in \mathbb{R}^d$ and $t \in (0, \infty)$. Here $\mathbf{A}(A)^{-*} := (\mathbf{A}(A)^{-1})^*$ is the linear operator on $M_d(\mathbb{R})$ given by $X \mapsto A^* X + XA$. If one restates the above formula by representing φ_Λ in terms of the Lévy–Khintchine triplet, one can calculate some of the integrals with respect to ds in the drift and Brownian covariance matrix part explicitely, since $\int_{-\infty}^t e^{A^*(t-s)} (\mathbf{A}(A)^{-*} X) e^{A(t-s)} \, ds = X$, for example. However, since this results in rather lengthy formulae, especially in the part coming from the Lévy measure, we refrain from giving further details. Note that we only condition on Y_0, since Σ_0 is highly non-Markovian and, hence, not informative regarding future values of Σ, and that equation (4.9) is essential to obtain (5.2).

However, it seems to be an important question what to condition upon in such a non-Markovian setting and to the best of our knowledge this issue has not been addressed so far. In (5.2) we basically assume that we only know the current price. One could also assume that one knows all historic prices and, hence, the historic values of Σ, since they are given by the continuous quadratic variation of the prices. Unfortunately, it seems extremely hard to understand what happens if one conditions on all these historic prices. Another point of view would be to say that W and Λ resemble the information arriving at the markets and that market participants observe all this information precisely. In that case it is appropriate to condition upon the σ-algebra $\mathscr{G}_0$ generated by Λ up to time zero and Y_0 (which now is only assumed to be independent of future values of Λ and W) and one obtains

$$
\begin{aligned}
&E(e^{iY_t^*u}|\mathscr{G}_0)\\
&\quad = \exp\bigg\{i\bigg[(Y_0+\mu t)^*u\\
&\qquad\qquad + \operatorname{tr}\bigg(\int_{M_d^-}\int_{-\infty}^0 \mathbf{A}(A)^{-1}(e^{A(t-s)}\Lambda(dA,ds)e^{A^*(t-s)}\\
&\qquad\qquad\qquad - e^{-As}\Lambda(dA,ds)e^{-A^*s})\Big(u\beta^*+\frac{i}{2}uu^*\Big)\bigg)\bigg]\\
&\qquad\qquad + \int_{M_d^-}\int_0^t \varphi_\Lambda\bigg[e^{A^*(t-s)}\Big(\mathbf{A}(A)^{-*}\Big(u\beta^*+\frac{i}{2}uu^*\Big)\Big)e^{A(t-s)}\\
&\qquad\qquad\qquad - \Big(\mathbf{A}(A)^{-*}\Big(u\beta^*+\frac{i}{2}uu^*\Big)-\rho^*u\Big)\bigg]\,ds\,\pi(dA)\bigg\}
\end{aligned} \tag{5.3}
$$

for all $u\in\mathbb{R}^d$ and $t\in(0,\infty)$.

It is clear that under appropriate technical conditions the conditional Laplace transform given Y_0 or $\mathscr{G}_0$, respectively, exists in a neighborhood of zero and (5.2) or (5.3) can be extended to hold on this neighborhood. Then one can use Laplace transform techniques to calculate prices of financial derivatives. Like for the standard OU-type model in [36] specifications are called for under which some of the integrals can be calculated explicitly, since otherwise the numerical integration takes too long to make pricing and especially calibration feasible in reasonable time. Observe that in (5.3) one would set

$$Z_t := \int_{M_d^-}\int_{-\infty}^0 \mathbf{A}(A)^{-1}\big(e^{A(t-s)}\Lambda(dA,ds)e^{A^*(t-s)} - e^{-As}\Lambda(dA,ds)e^{-A^*s}\big)$$

and determine Z_t also by calibration to option prices. Obviously, one can do this only for derivatives with a fixed maturity $t\in\mathbb{R}^+$, unless one increases the number of parameters one calibrates.

5.3. *Multivariate* supCAR(*MA*). Ornstein–Uhlenbeck-type processes are a special case of the so-called (multivariate) continuous time autoregressive moving-average (CARMA) processes (see [10, 11, 33]). As the continuous time analogue of ARMA processes, CARMA processes, are a fundamental class of processes for time series modeling in continuous time. A d-dimensional supCAR(p) process Y can be defined by

$$Y_t = (I_d,0,\dots,0)\int_{M_{dp}^-}\int_{-\infty}^t e^{A(t-s)}(0,\dots,0,I_d)^{\mathrm{T}}\Lambda(dA,ds),$$

where Λ is an $\mathbb{R}^d$-valued Lévy basis on $M_{dp}^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ with π concentrated on the matrices in M_{dp}^- of the form

$$\begin{pmatrix} 0 & I_d & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & I_d \\ -A_p & -A_{p-1} & \cdots & -A_2 & -A_1 \end{pmatrix} \tag{5.4}$$

with appropriate $d \times d$ matrices A_i. Clearly, "$\int_{-\infty}^t e^{A(t-s)}(0, \dots, 0, I_d)^{\mathrm{T}} \Lambda(dA, ds)$" is for A fixed a CAR(p) process (note that this is in contrast to [32] who define supCARMA processes differently), so it is appropriate to call this process supCAR(p). Obviously many properties for Y follow from our results immediately, since it is basically given by the first d-coordinates of a high-dimensional supOU process, and this definition gives a possibility to extend CAR processes allowing for long memory and jumps.

Using our techniques one can define supCARMA(p, q) process with $q < p$ using an $\mathbb{R}^d$-valued Lévy basis on $M_{dp}^- \times M_{dp,d} \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$. To obtain proper supCARMA processes one demands $\pi(\mathscr{A} \times M_{dp,d}) = 1$, denoting the set of matrices of the form (5.4) by $\mathscr{A}$, and one sets

$$Y_t = (I_d, 0, \dots, 0) \int_{M_{dp}^-} \int_{-\infty}^t e^{A(t-s)} B \Lambda(dA, dB, ds).$$

Adapting and extending our arguments one can easily obtain results for this class of processes. For the interpretation as CARMA processes note that the moving average coefficients have to be calculated from the $d \times d$ blocks of B by inverting the formulae given in [33], Theorem 3.12.

6. Conclusion.

In this paper we introduced multivariate supOU processes and obtained various important properties of them. Furthermore, some areas of application have been outlined and we are currently considering their use in stochastic volatility modelling beginning in [9]. However, there are still many important issues concerning the supOU processes themselves which we hope to address in future work. Of particular interest is, for example, the development of good estimators for supOU models and to show properties like consistency and asymptotic normality for them. This is related to understanding better the dependence structure of supOU processes, which are clearly not Markovian.

Likewise, we have shown that supOU processes allow to model long memory effects (in a specific sense). However, a detailed theory of multivariate long-range dependence needs to be developed.

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