

Stationarity and Geometric Ergodicity of BEKK Multivariate GARCH Models

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Conditions for the existence of strictly stationary multivariate GARCH processes in the so-called BEKK parametrisation, which is the most general form of multivariate GARCH processes typically used in applications, and for their geometric ergodicity are obtained. The conditions are that the driving noise is absolutely continuous with respect to the Lebesgue measure and zero is in the interior of its support and that a certain matrix built from the GARCH coefficients has spectral radius smaller than one.

To establish the results semi-polynomial Markov chains are defined and analysed using algebraic geometry.

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1. Introduction

Generalised autoregressive conditionally heteroskedastic (GARCH) processes (originally introduced by [6, 16]) are heavily used in various areas of applications for the modelling of heteroskedastic time series data. Very often one has to model several interrelated time series with an appropriate multidimensional model. Since for multivariate GARCH processes the latent volatility process needs to take values in the positive semi-definite matrices, as it has to correspond to a covariance matrix at each point in time, the multivariate GARCH models are typically considerably more involved than the univariate one. For an overview over the various multivariate GARCH models existing and their applications we refer to [3, 31]. As always in time series modelling, existence and uniqueness of stationary solutions, as well as convergence to the stationary solution is of high importance. While

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this is not so hard a question for some multivariate GARCH specifications where – as in the Constant-Conditional-Correlation (CCC) Model or its extensions – the variances are modelled by univariate GARCH models, we address this question here for the very general Baba, Engle, Kraft and Kroner (BEKK) GARCH model introduced in [17] where all (co)variances influence each other in the time dynamics. The BEKK model is almost the most general multivariate GARCH model existing. Only the vec model also introduced in [17] is more general, but all vec models not representable in the BEKK parametrisation are somewhat degenerate (see [32]). Moreover, the restrictions on the parameters necessary to ensure a proper GARCH model are more or less not practicably formulatable in the vec model. Hence, the BEKK model is the most general one normally used.

To prove our results we employ Markov chain theory combined with algebraic geometry to obtain the proper state spaces and properties like irreducibility. In particular, we analyse stationarity and ergodicity for a general class of Markov chains which we call “semi-polynomial”, because they generalise the polynomial Markov chains of [27], and then apply the general results to the special case of multivariate GARCH processes. Our approach in the present paper is most similar to the PhD thesis of the first author [8], which, although it has never been published in an accessible way (except for a summary of the main results without proofs in [9]), has been relied upon in essential ways (see e.g. [10]). Unfortunately, the statements and proofs in that thesis contain some problematic issues and, hence, details of the proofs and statements given in the present paper deviate in essential ways. One difference, for instance, is the use of (weak) Feller chains, another regards the proper state spaces.

The remainder of this paper is structured as follows. Below we briefly summarise some general notation. In Section 2 we give a detailed definition of BEKK GARCH models, their vec and vech parametrisations, state our main result on the stationarity and ergodicity of multivariate GARCH processes and discuss its implications. Thereafter, we define and analyse semi-polynomial Markov chains in Section 3. Finally, we proof our main result for multivariate GARCH processes in Section 4. A brief summary of some notions of algebraic geometry necessary to understand the statements of our main results on GARCH processes is given in the appendix. There we have also collected some results from the theory of Markov chains which we are going to use in the proof of Theorem 3.12.

Notation

For the natural numbers excluding zero we write \mathbb{N}^* .

We denote the set of real $n \times d$ matrices by $M_{n \times d}(\mathbb{R})$, the vector space of real $d \times d$ matrices by $M_d(\mathbb{R})$, the linear subspace of symmetric matrices by \mathbb{S}_d , the positive semi-definite cone by \mathbb{S}_d^+ and the (strictly) positive definite matrices by \mathbb{S}_d^{++} . For a positive definite and positive semi-definite matrix $A \in \mathbb{S}_d$ we also write $A > 0$ and $A \geq 0$, respectively. The transpose of a matrix $A \in M_{n \times d}(\mathbb{R})$ will be denoted by A^t and the $d \times d$ identity matrix by I_d . If the dimension is obvious from the context, we sometimes neglect the subscript.

Every matrix $A \in M_{n \times d}(\mathbb{R})$ can be considered as a vector in \mathbb{R}^{nd} using the bijective vec transformation which stacks the columns of a matrix below one another beginning with the leftmost one. In the case of symmetric matrices, one often uses the vech transformation which maps \mathbb{S}_d bijectively to $\mathbb{R}^{d(d+1)/2}$ by stacking the lower triangular portion of a matrix. For instance, the matrix

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \in \mathbb{S}_3 \subseteq M_{3 \times 3}(\mathbb{R})$$

is mapped to the vector $(a, b, c, b, d, e, c, e, f)^t \in \mathbb{R}^9$ by the vec operator and to $(a, b, c, d, e, f)^t \in \mathbb{R}^6$

by the vech operator. Finally, we denote for two matrices $A \in M_{n \times d}(\mathbb{R})$ and $B \in M_{r \times m}(\mathbb{R})$ the tensor (Kronecker) product by $A \otimes B$.

For the relevant background on Markov chains and mixing we refer to any of the standard references, for instance, [13, 26].

2. Stationarity and Geometric Ergodicity of BEKK Multivariate GARCH Models

When one moves from a single-dimensional to a d -dimensional GARCH process, the univariate variance process becomes a $d \times d$ covariance matrix process Σ . In the so-called vec parametrisation (see [17]) the general multivariate GARCH(p, q) model with $p, q \in \mathbb{N}$ is given by

$$X_n = \Sigma_n^{1/2} \varepsilon_n, \quad (2.1)$$

$$\text{vec}(\Sigma_n) = \text{vec}(C) + \sum_{i=1}^q \tilde{A}_i \text{vec}(X_{n-i} X_{n-i}^t) + \sum_{j=1}^p \tilde{B}_j \text{vec}(\Sigma_{n-j}) \quad (2.2)$$

for $n \in \mathbb{N}^*$ where $(\varepsilon_n)_{n \in \mathbb{N}^*}$ is an \mathbb{R}^d -valued i.i.d. sequence and $\Sigma_n^{1/2}$ denotes the unique positive semi-definite square root of Σ_n (i.e. the unique element $\Sigma_n^{1/2}$ of \mathbb{S}_d^+ such that $\Sigma_n = \Sigma_n^{1/2} \Sigma_n^{1/2}$). To ensure the positive semi-definiteness of the covariance matrix process Σ the initial values $\Sigma_0, \dots, \Sigma_{1-p}$ and C have to be positive semi-definite and $\tilde{A}_1, \dots, \tilde{A}_q$ as well as $\tilde{B}_1, \dots, \tilde{B}_p$ need to be $d^2 \times d^2$ matrices mapping the vectorised positive semi-definite matrices into themselves. The initial values $X_0, X_{-1}, \dots, X_{1-q}$ may be arbitrary elements of \mathbb{R}^d .

The restriction on the linear operators \tilde{A}_i and \tilde{B}_j necessary to ensure positive semi-definiteness gave rise to the so-called BEKK model (see again [17]) which automatically ensures positive semi-definiteness:

$$X_n = \Sigma_n^{1/2} \varepsilon_n, \quad (2.3)$$

$$\Sigma_n = C + \sum_{i=1}^q \sum_{k=1}^{l_i} \bar{A}_{i,k} X_{n-i} X_{n-i}^t \bar{A}_{i,k}^t + \sum_{j=1}^p \sum_{r=1}^{s_j} \bar{B}_{j,r} \Sigma_{n-j} \bar{B}_{j,r}^t, \quad (2.4)$$

where $\bar{A}_{i,k}$ and $\bar{B}_{j,r}$ are now arbitrary elements of $M_d(\mathbb{R})$.

The BEKK model is equivalent to the vec model with $\tilde{A}_i = \sum_{k=1}^{l_i} \bar{A}_{i,k} \otimes \bar{A}_{i,k}$, $i = 1, \dots, q$, and $\tilde{B}_j = \sum_{r=1}^{s_j} \bar{B}_{j,r} \otimes \bar{B}_{j,r}$, $j = 1, \dots, p$. More details of the relations between vec and BEKK GARCH models are given in [32].

If we take the symmetry of the matrices Σ_n into account, we can write the vec (and thus the BEKK) model also in the vech representation:

$$X_n = \Sigma_n^{1/2} \varepsilon_n, \quad (2.5)$$

$$\text{vech}(\Sigma_n) = \text{vech}(C) + \sum_{i=1}^q A_i \text{vech}(X_{n-i} X_{n-i}^t) + \sum_{j=1}^p B_j \text{vech}(\Sigma_{n-j}), \quad (2.6)$$

where $A_i = H_d \tilde{A}_i K_d^t$, $B_j = H_d \tilde{B}_j K_d^t$, $i = 1, \dots, q$, $j = 1, \dots, p$ and the matrices H_d and K_d are the

unique $H_d, K_d \in M_{\frac{d(d+1)}{2} \times d^2}(\mathbb{R})$ such that

$$\text{vech}(D) = H_d \text{vec}(D), \text{vec}(D) = K_d^t \text{vech}(D) \quad \text{and} \quad H_d K_d^t = I_{d(d+1)/2}$$

for every $D \in \mathbb{S}_d$, whose existence and uniqueness follows immediately, since both vec and vech are linear operators.

Example 2.1. Let us give a simple example for the GARCH equations when $d = 2$. Consider the following BEKK model with $p = q = l_1 = s_1 = 1$:

$$\Sigma_n = C + \begin{pmatrix} a & c \\ b & d \end{pmatrix} X_{n-1} X_{n-1}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & g \\ f & h \end{pmatrix} \Sigma_{n-1} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad (2.7)$$

where a, b, c, d, e, f, g, h are arbitrary real numbers.

From (2.7) one immediately derives the corresponding vec model:

$$\text{vec}(\Sigma_n) = \text{vec}(C) + \begin{pmatrix} a^2 & ac & ca & c^2 \\ ab & ad & cb & cd \\ ba & bc & da & dc \\ b^2 & bd & db & d^2 \end{pmatrix} \text{vec}(X_{n-1} X_{n-1}^t) + \begin{pmatrix} e^2 & eg & ge & g^2 \\ ef & eh & gf & gh \\ fe & fg & he & hg \\ f^2 & fh & hf & h^2 \end{pmatrix} \text{vec}(\Sigma_{n-1}). \quad (2.8)$$

Since the matrices H_2 and K_2 are given by

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the associated vech parametrisation of (2.7) and (2.8) is

$$\text{vech}(\Sigma_n) = \text{vech}(C) + \begin{pmatrix} a^2 & 2ac & c^2 \\ ab & ad + bc & cd \\ b^2 & 2bd & d^2 \end{pmatrix} \text{vech}(X_{n-1} X_{n-1}^t) + \begin{pmatrix} e^2 & 2eg & g^2 \\ ef & eh + fg & gh \\ f^2 & 2fh & h^2 \end{pmatrix} \text{vech}(\Sigma_{n-1}).$$

Definition 2.2. Let $(X_n)_{n \in \mathbb{N}^*}$ be a GARCH(p, q) process in the BEKK representation satisfying

- (i) C and the initial values $\Sigma_0, \dots, \Sigma_{1-p}$ are positive definite,
- (ii) $(\varepsilon_n)_{n \in \mathbb{N}^*}$ is an \mathbb{R}^d -valued i.i.d. sequence with distribution Γ , $\varepsilon_1 \in L^2$, $\mathbb{E}[\varepsilon_1] = 0$ and $\mathbb{E}[\varepsilon_1 \varepsilon_1^t] = I_d$,
- (iii) $(\varepsilon_n)_{n \in \mathbb{N}^*}$ is independent of $\mathcal{F}_0 = \sigma(X_{1-q}, X_{2-q}, \dots, X_0, \Sigma_{1-p}, \Sigma_{2-p}, \dots, \Sigma_0)$.

Then $(X_n)_{n \in \mathbb{N}^*}$ is called standard GARCH(p, q) process.

$\mathbb{E}[\varepsilon_1 \varepsilon_1^t] = I_d$ is a simple normalisation making the volatility process identifiable and, hence, not really a restriction.

Remark 2.3. In $X_n = \Sigma_n^{1/2} \varepsilon_n$ one could also choose another transformation $G(\Sigma_n)$ of the conditional covariance matrix Σ_n such that $G(\Sigma_n)^t G(\Sigma_n) = \Sigma_n$ instead of taking the square root. Boussama [8] takes $G(\Sigma_n)$ as a lower triangular matrix resulting from the Cholesky decomposition. However, the exact form of this transformation does not have any impact on our results concerning stationarity and ergodicity provided that G is an appropriate ‘‘smooth’’ transformation such that the GARCH process fits into the setting of semi-polynomial Markov chains studied in Section 3.

Using the vech representation we embed a standard GARCH(p, q) process into a Markov chain. Setting $\mathcal{C} := (\text{vech}(C)^t, 0, 0, \dots, 0)^t \in (\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q$ and defining the process $(Y_n)_{n \in \mathbb{N}^*}$ in $(\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q$ by $Y_n := (\text{vech}(\Sigma_n)^t, \text{vech}(\Sigma_{n-1})^t, \dots, \text{vech}(\Sigma_{n-p+1})^t, X_n^t, X_{n-1}^t, \dots, X_{n-q+1}^t)^t$, one easily sees that

$$Y_n = \mathcal{C} + \begin{pmatrix} \sum_{i=1}^q A_i \text{vech}(X_{n-i} X_{n-i}^t) + \sum_{j=1}^p B_j \text{vech}(\Sigma_{n-j}) \\ \text{vech}(\Sigma_{n-1}) \\ \vdots \\ \text{vech}(\Sigma_{n-p+1}) \\ X_n \\ X_{n-1} \\ \vdots \\ X_{n-q+1} \end{pmatrix}.$$

From this one obtains immediately a regular (in the sense of Definition A.7) map

$$\varphi : \left((\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q \right) \times \mathbb{R}^d \rightarrow \left((\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q \right) \text{ such that } Y_n = \varphi(Y_{n-1}, X_n).$$

Next we need to define the set W which the stationary standard GARCH(p, q) process takes its values in, as is to be seen. Set

$$\tilde{B} := \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M_{p \frac{d(d+1)}{2} + qd}(\mathbb{R}) \text{ with } B = \begin{pmatrix} B_1 & B_2 & \dots & B_{p-1} & B_p \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & I & 0 \end{pmatrix} \in M_{p \frac{d(d+1)}{2}}(\mathbb{R})$$

and consider the point T satisfying

$$T = \mathcal{C} + \tilde{B}T. \quad (2.9)$$

Existence and uniqueness of this point under the conditions of Theorem 2.4 is shown in Section 4.3. Furthermore, we define φ^n for $n \in \mathbb{N}^*$ recursively by $\varphi^1 := \varphi$ and

$$\begin{aligned} \varphi^{n+1} : \left((\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q \right) \times (\mathbb{R}^d)^{n+1} &\rightarrow \left((\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q \right), \\ \varphi^{n+1}(y, x_1, \dots, x_n, x_{n+1}) &= \varphi(\varphi^n(y, x_1, \dots, x_n), x_{n+1}) \end{aligned}$$

and set

$$W = \overline{\bigcup_{n \in \mathbb{N}^*} \varphi^n(T, (\mathbb{R}^d)^n)}$$

with \overline{S} denoting the closure of a set S in the Zariski topology (see appendix). Since Σ_n is always

positive definite, we define

$$U := \underbrace{\text{vech}(\mathbb{S}_d^{++}) \times \dots \times \text{vech}(\mathbb{S}_d^{++})}_p \times \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_q$$

and consider $W \cap U$ as the natural state space for $(Y_n)_{n \in \mathbb{N}^*}$. Finally, $\mathcal{B}(W \cap U)$ denotes the Borel σ -algebra over $W \cap U$ inherited from the standard Borel σ -algebra on $(\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q$, i.e. $\mathcal{B}(W \cap U)$ is related to the usual Euclidean topology.

Theorem 2.4. *Let $(X_n)_{n \in \mathbb{N}^*}$ be a standard GARCH(p, q) process.*

(i) *If*

(H1) *the distribution Γ of ε_1 is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d ,*

(H2) *the point zero is in the interior of $E := \text{supp}(\Gamma)$ and*

(H3) *the spectral radius of $\sum_{i=1}^q A_i + \sum_{j=1}^p B_j$ is less than 1,*

then the Markov chain $(Y_n)_{n \in \mathbb{N}^}$ is positive Harris recurrent and geometrically ergodic on the state space $(W \cap U, \mathcal{B}(W \cap U))$.*

The strictly stationary solution $(X_n)_{n \in \mathbb{Z}}$ of the standard GARCH(p, q) model associated with $(Y_n)_{n \in \mathbb{Z}}$ is unique and geometrically β -mixing. Furthermore, $X_n \in L^2$ and $\Sigma_n \in L^1$ for all $n \in \mathbb{Z}$. The strictly stationary solution is, hence, also weakly stationary.

(ii) *If there exists a weakly stationary solution for the standard GARCH(p, q) model, then the spectral radius of the matrix $\sum_{i=1}^q A_i + \sum_{j=1}^p B_j$ is less than 1.*

Remark 2.5. (i) If the initial values $\Sigma_0, \dots, \Sigma_{1-p}$ are in W and positive semi-definite but not positive definite, the geometric ergodicity still holds, since one easily sees that one then has $Y_p \in W \cap U$.

(ii) Obviously, our conditions for strict stationarity imply also weak (or second order) stationarity. In the univariate case it is well-known that for a driving noise ε with finite variance one may well choose the GARCH parameters in such a way that a unique stationary solution with infinite second moments exists (see [2, 7, 24], for example). Extending the above result to cover such cases seems not possible at the moment, although in principle the main problem is “only” to find an appropriate function for the Foster-Lyapunov drift criterion.

(iii) Theorem 2.4 should cover most of the BEKK multivariate GARCH models used in applications, since one usually wants a finite second moment and uses absolutely continuous noises ε (e.g. multivariate standard normal or standard t_ν -distributed noises with $\nu > 2$). Hence, (H3) will typically be the only condition that needs checking.

Note, however, that it is possible to weaken the assumptions (H1) and (H2) to the existence of a non-trivial absolutely continuous part of the innovation distribution with zero in the interior of its support (cf. [13, Section 2.4]). More precisely, if

(K1) the distribution Γ of ε_1 can be decomposed as $\Gamma = \Gamma_0 + \Gamma_1$ with Γ_0 being non-trivial and absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d (cf., for instance, [30, p. 174] where the Lebesgue decomposition of any σ -finite measure on $\mathcal{B}(\mathbb{R}^d)$ is explained),

(K2) the point zero is in the interior of $E = \text{supp}(\Gamma_0)$

and (H3) hold, then Theorem 2.4 remains valid.

We give a brief summary of the changes that have to be done in the following sections if one starts with (K1) and (K2) instead of (H1) and (H2): in the statements of Theorem 3.1, Proposition 3.2 and Theorem 3.3 one simply has to replace Γ by Γ_0 . Further the assumptions (A1) and (A2) in Section 3.2.1 have to be adapted in the same way as (H1) and (H2) have been weakened. A crucial point is to replace the right hand side in Proposition 3.9 by the subprobability measure $F^k(T, \otimes_{i=1}^k \Gamma_0)$. In the same manner we take ν in the proof of Proposition 3.10 equal to the subprobability measure $F^l(T, \otimes_{i=1}^l \Gamma_0)$. The rest of the paper is not affected.

- (iv) If we completely omitted the assumptions (K1) and (K2) on the innovations, the mixing result might be no longer true (cf. [1]). However, one could try to extend the idea of [15] where the existence of a τ -weakly dependent strictly stationary solution for a chain with infinite memory has been shown under a Lipschitz-type condition, but the expression of this condition in terms of the matrices appearing in the BEKK representation seems to be very delicate. In some cases, like for instance the upcoming Example 2.7, the contraction condition is easily verified (in this case with the Orlicz function $\Phi(x) = x^2$). Hence, in such cases [15, Theorem 3.1] yields the existence of a τ -weakly dependent strictly stationary solution of the standard GARCH model even if the innovation sequence does not possess an absolutely continuous component.

Finding an appropriate contraction condition in general appears highly non-trivial but may allow to show τ -dependence without assuming (K1) and (K2) and without using algebraic geometry. This would imply Central Limit Theorems (CLTs) and the validity of bootstrap procedures (see, for instance, [5]) for the strictly stationary solution. However, for simulation purposes as well as CLTs and Strong Laws of Large Numbers when not starting with the stationary distribution, geometric ergodicity and the “right” irreducible state space is very important (see again the upcoming Example 2.7). The latter seem not to be obtainable under τ -weak dependence conditions. Moreover, note also that τ -weak dependence is a weaker notion than strong mixing.

- (v) Strong mixing conditions are, as τ -dependence conditions, a way to derive limit theorems. Anyway, the CLT under strong mixing, even if the mixing coefficients decline exponentially fast (which is the case for all geometrically ergodic Markov chains), needs a stronger order moment condition than the second order ones obtained in Theorem 2.4 (see [19] and also [20, 22]). In [14] (see also [22, Corollary 3]) it has been shown that for a positive Harris recurrent and geometrically ergodic Markov chain $(X_t)_{t \in \mathbb{Z}}$ on a state space S with stationary distribution π , the CLT holds for any real-valued function f defined on S which satisfies $\int_S \pi(dx) f^2(x) \log^+ |f(x)| < \infty$. In our case these references together with the obtained results in particular imply the CLT for functions f such that $\int_{W \cap U} \pi(dx) f^2(x) \log^+ |f(x)| \leq C \cdot \int_{W \cap U} \pi(dx) V(x)$ (where V is the function specified in the upcoming proof of Theorem 4.9). Hence, for any $\varepsilon > 0$, the CLT can be applied to $(1/2 - \varepsilon)$ th and $(1 - \varepsilon)$ th absolute powers of Σ_n and X_n , respectively. Moreover, combining our conditions on geometric ergodicity with the fourth order moment conditions of [18] immediately gives sufficient conditions for the validity of more classical CLTs.

One might expect that the set $W \cap U$ spans the space $(\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q$ and that, hence, the state space of a stationary GARCH process is “non-degenerate”. However, this needs not to be true:

Example 2.6. Consider the following bivariate GARCH(1, 1) model:

$$\text{vech}(\Sigma_n) = \text{vech}(C) + A \text{vech}(X_{n-1} X_{n-1}^t) + B \text{vech}(\Sigma_{n-1}) \quad (2.10)$$

where A and B are two 3×3 matrices such that the spectral radius of $A + B$ is less than 1 and $BA = 0$. Such a GARCH model can be obtained from a BEKK model with $l_1 = s_1 = 1$ and $\bar{B}_{11}\bar{A}_{11} = 0$.

Starting from the initial point $T = (\text{vech}(\Sigma_0)^t, X_0^t)^t$ given by equation (2.9), we note that $\text{vech}(\Sigma_0) = \text{vech}(C) + B \text{vech}(\Sigma_0)$ and $X_0 = 0$ and obtain by iterating (2.10)

$$\text{vech}(\Sigma_n) = \text{vech}(\Sigma_0) + A \text{vech}(X_{n-1}X_{n-1}^t).$$

Let f be the regular map from \mathbb{R}^4 into \mathbb{R}^5 given by

$$(x_1, x_2, x_3, x_4) \mapsto f(x_1, x_2, x_3, x_4) := T + \begin{pmatrix} A(x_1^2, x_1x_2, x_2^2)^t \\ x_3 \\ x_4 \end{pmatrix}.$$

Then W is the Zariski closure of the semi-algebraic set $f(\mathbb{R}^4)$ (see [4, Theorem 2.3.4]) and W has to be strictly contained in \mathbb{R}^5 since $\dim f(\mathbb{R}^4) \leq 4 = \dim \mathbb{R}^4$.

Note that the problem of degeneracy in this example lies in the non-invertibility of at least one of the two matrices A, B . Indeed, it is easy to see that W is of full dimension and no such degeneracy as above can occur if A, B (or $\bar{A}_{11}, \bar{B}_{11}$ in the BEKK formulation) are both invertible and (H1), (H2) hold.

Moreover one has to be very careful when using GARCH models not to use them outside the state space $W \cap U$. Typically one would simulate a stationary GARCH process by starting with an arbitrary value and letting the process run. The values are only recorded after a burn-in period. The geometric ergodicity ensures that after an appropriately long burn-in period the obtained values can be basically regarded as coming from the stationary dynamics. To ensure that this approach works, our results show that one needs to start in $W \cap U$. One choice of the starting values always possible is T which is easily calculated from the parameters by solving a system of linear equations. Let us give an example where starting values outside $W \cap U$ indeed lead to a problem.

Example 2.7. Consider the set-up of Example 2.6 with $l_1 = s_1 = 1$ and

$$\bar{A}_{11} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \bar{B}_{11} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix},$$

with $|a| < 1, |b| < 1$ being non-zero real numbers. Obviously (H3) is satisfied.

It is then easy to see that the component corresponding to the second variance is constant in W , say it equals σ_{22} . If we start with an initial value Σ_0 with a second variance $(\Sigma_0)_{22}$, then one sees easily that $(\Sigma_n)_{22} = C_{22} + b^2(\Sigma_{n-1})_{22}$ for all $n \in \mathbb{N}^*$. Obviously, this equation has a unique fixed point which must be equal to σ_{22} and the right hand side corresponds to an injective map. By induction this implies $(\Sigma_n)_{22} \neq \sigma_{22}$ for all $n \in \mathbb{N}^*$ if the starting value satisfies $(\Sigma_0)_{22} \neq \sigma_{22}$.

Hence, for such a starting value $Y_n \notin W$ for all $n \in \mathbb{N}$ and thus the distribution of Y_n can never converge in the total variation sense to the stationary distribution π . This means that we can never have geometric ergodicity when allowing such starting values outside W but in U .

3. Stationarity and Geometric Ergodicity of Semi-polynomial Markov Chains

In this section we consider a general class of Markov chains and prove criteria for stationarity and geometric ergodicity. We will apply the results later on to the special case of multivariate GARCH processes, but the general results of this section seem also of interest of their own, since they should be applicable to different models as well.

We consider Markov chains in \mathbb{R}^n of the form $X_{t+1} = F(X_t, e_t)$ where $(e_t)_{t \in \mathbb{N}}$ is an m -dimensional i.i.d. sequence and F is an appropriate map as follows.

Let $V \subseteq \mathbb{R}^n$ be an algebraic variety (cf. Definition A.5) and U an open subset of \mathbb{R}^n and let $F : U \times \mathbb{R}^m \rightarrow U$ be a C^1 - map such that there exist a C^1 - map $f : U \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ and a map $\varphi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ satisfying:

$$(F1) \quad F(z, y) = \varphi(z, f(z, y)) \text{ for all } (z, y) \in U \times \mathbb{R}^m,$$

$$(F2) \quad \varphi(V \cap U \times \mathbb{R}^m) \subseteq V \cap U,$$

(F3) the map $(z, y) \mapsto \varphi(z, y)$ is regular in (z, y) (cf. Definition A.7) and

(F4) for all $z \in U$, the map $f_z(\cdot) = f(z, \cdot)$ is a C^1 - diffeomorphism from \mathbb{R}^m onto \mathbb{R}^m and the map $U \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, $(z, y) \mapsto f_z^{-1}(y)$ is continuous in (z, y) where $f_z^{-1}(\cdot)$ denotes the inverse map of $f_z(\cdot)$.

The case when F is a regular map, i.e. when f_z is the identity, has been considered in [27] under the name ‘‘polynomial Markov chains’’. Similarly to that paper we use extensively algebraic geometry (see the appendix for references and the most relevant definitions) and drift criteria to show the stationarity and ergodicity of these Markov chains, but the presence of the additional diffeomorphism f_z makes all proofs considerably more involved. Moreover, we always need to ensure that we stay in U .

3.1. Properties of the Image Measure

In a first step we consider how F acts for a fixed first argument on the noise distribution, which will lead to ψ -irreducibility conditions in the next section.

In general, the image of \mathbb{R}^m under $F_z(\cdot) := F(z, \cdot)$ is a semi-algebraic set in \mathbb{R}^n with dimension less than n (see [4, Theorem 2.3.4]). Thus the Lebesgue measure of this image is often zero.

Therefore we need to work with Hausdorff measures (see, for example, [12] for a detailed introduction). We suppose that the algebraic variety V is equipped with a regular measure μ_V defined on $(V, \mathcal{B}(V))$ where $\mathcal{B}(V)$ denotes again the Borel σ -algebra over V inherited from the usual Euclidean topology. Recall that μ_V is said to be regular if, for any $A \in \mathcal{B}(V)$ and any $\delta > 0$, there exist an open set $U \in \mathcal{B}(V)$ and a compact set $K \in \mathcal{B}(V)$ such that $K \subseteq A \subseteq U$ and $\mu_V(U \setminus K) < \delta$. In the following we assume that this measure μ_V is obtained by equipping the regular set $\mathcal{R}(V)$ of V (cf. Definition A.6) with an appropriate Hausdorff measure which is extended by zero to the singular set $\mathcal{S}(V) = V \setminus \mathcal{R}(V)$. Moreover we henceforth suppose that Γ is a measure absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^m with density γ . For $z \in V \cap U$ we denote by Γ_z the image measure of Γ under F_z in $V \cap U$. Furthermore we define $E := \text{supp}(\Gamma)$ which is essentially also the domain of positivity of the density γ . For the notion of smooth points we refer to [27, A 20] or [28, p. 42] and note that the definition makes sense for general C^1 - maps.

Theorem 3.1. *Suppose that $z_0 \in V \cap U$ and $F_{z_0}(\cdot)$ has a smooth point in \mathbb{R}^m . Then Γ_{z_0} is absolutely continuous with respect to the measure μ_V and has support $F_{z_0}(E)$.*

Proof. First we denote by Γ'_{z_0} the image measure $f_{z_0}(\Gamma)$. One obtains immediately by the Density Transformation Theorem that Γ'_{z_0} is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^m with density

$$\gamma'_{z_0}(y) := \gamma(f_{z_0}^{-1}(y)) \cdot \frac{1}{|\det Df_{z_0}(f_{z_0}^{-1}(y))|}, \quad y \in \mathbb{R}^m. \quad (3.1)$$

The support of Γ'_{z_0} is given by $f_{z_0}(E)$.

There exists $x_0 \in \mathbb{R}^m$ such that $F_{z_0}(x_0) = \varphi_{z_0}(y_0)$ is a regular point of V and $\text{rank}(DF_{z_0}(x_0)) = \dim V$ where $y_0 = f_{z_0}(x_0)$. Since $F_{z_0}(\cdot) = \varphi_{z_0}(f_{z_0}(\cdot))$, we have $DF_{z_0}(x_0) = D\varphi_{z_0}(y_0) \cdot Df_{z_0}(x_0)$. Since $f_{z_0}(\cdot)$ is a C^1 -diffeomorphism, the linear map $Df_{z_0}(x_0)$ is invertible. Thus $\text{rank}(D\varphi_{z_0}(y_0)) = \dim V$. Since $\varphi_{z_0}(y_0) \in \mathcal{R}(V)$ and $y_0 \in \mathcal{R}(\mathbb{R}^m)$ (\mathbb{R}^m is a smooth algebraic variety, i.e. $\mathcal{R}(\mathbb{R}^m) = \mathbb{R}^m$), the regular map $\varphi_{z_0}(\cdot)$ is smooth at y_0 and hence dominating (in the sense of [27, A23]). Applying [27, Theorem 3.1] gives the result. \square

Proposition 3.2. *Let $z_0 \in U$. Then, for every $\varepsilon > 0$, there exists $\alpha > 0$ such that*

$$|\Gamma'_z(B) - \Gamma'_{z_0}(B)| = |f_z(\Gamma)(B) - f_{z_0}(\Gamma)(B)| < \varepsilon$$

for all $B \in \mathcal{B}(\mathbb{R}^m)$ and every $z \in U$ with $\|z - z_0\| < \alpha$.

Proof. The image measure $\Gamma'_z = f_z(\Gamma)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^m with density γ'_z given by equation (3.1).

Let $\varepsilon > 0$ and $B \in \mathcal{B}(\mathbb{R}^m)$. The space of real-valued continuous functions on \mathbb{R}^m with compact support is dense in the L^1 sense in the space of all Lebesgue-integrable functions on \mathbb{R}^m . Thus, there exists a continuous function $\tilde{\gamma}: \mathbb{R}^m \rightarrow \mathbb{R}$ with compact support K such that

$$\int_{\mathbb{R}^m} |\gamma(x) - \tilde{\gamma}(x)| dx < \frac{\varepsilon}{3}. \quad (3.2)$$

Hence,

$$\begin{aligned} |\Gamma'_z(B) - \Gamma'_{z_0}(B)| &= \left| \int_B \gamma'_z(y) dy - \int_B \gamma'_{z_0}(y) dy \right| \\ &\leq \int_{\mathbb{R}^m} |\gamma(f_z^{-1}(y)) - \tilde{\gamma}(f_z^{-1}(y))| \cdot |\det Df_z^{-1}(y)| dy \\ &\quad + \int_{\mathbb{R}^m} |\tilde{\gamma}(f_z^{-1}(y))| |\det Df_z^{-1}(y)| - \tilde{\gamma}(f_{z_0}^{-1}(y)) |\det Df_{z_0}^{-1}(y)| \Big| dy \\ &\quad + \int_{\mathbb{R}^m} |\tilde{\gamma}(f_{z_0}^{-1}(y)) - \gamma(f_{z_0}^{-1}(y))| \cdot |\det Df_{z_0}^{-1}(y)| dy =: I_1 + I_2 + I_3. \end{aligned}$$

With (3.2) we obtain immediately by substitution $I_1 < \varepsilon/3$ and $I_3 < \varepsilon/3$.

$\tilde{\gamma}$ is bounded on \mathbb{R}^m by $\sup \tilde{\gamma}$ and for all $r > 0$ such that $\overline{B(z_0, r)} := \{z \in \mathbb{R}^n : \|z - z_0\| \leq r\} \subseteq U$, the set $C := \{(z, y) \in U \times \mathbb{R}^m : \|z - z_0\| \leq r \text{ and } f_z^{-1}(y) \in K\}$ is a compact set in $U \times \mathbb{R}^m$, since the map $\psi: U \times \mathbb{R}^m \rightarrow U \times \mathbb{R}^m$, $\psi(z, y) := (z, f_z^{-1}(y))$ is continuous and $C = \psi(\overline{B(z_0, r)} \times K)$. Thus, there is a real number $b > 0$ such that, for all $(z, y) \in C$, $|\det Df_z^{-1}(y)| < b$. The map $\tilde{\gamma}(f_z^{-1}(y)) \cdot |\det Df_z^{-1}(y)|$ is hence bounded on C by $b \cdot \sup \tilde{\gamma}$.

Let C_1 be the projection of C on \mathbb{R}^m and suppose without loss of generality $\|z - z_0\| \leq r$. Then, for all $y \notin C_1$, we have $\tilde{\gamma}(f_z^{-1}(y)) = \tilde{\gamma}(f_{z_0}^{-1}(y)) = 0$ which implies

$$I_2 = \int_{C_1} |\tilde{\gamma}(f_z^{-1}(y))| |\det Df_z^{-1}(y)| - \tilde{\gamma}(f_{z_0}^{-1}(y)) |\det Df_{z_0}^{-1}(y)| \Big| dy.$$

This integrand is dominated by $2b \sup \tilde{\gamma}$ and converges pointwise to zero if z converges to z_0 (cf. (F4)). Since b and $\sup \tilde{\gamma}$ are finite constants and C_1 is compact the dominant $2b \sup \tilde{\gamma}$ is integrable over C_1 . Hence, we can apply the Dominated Convergence Theorem and get $I_2 \rightarrow 0$ as $z \rightarrow z_0$, i.e. there exists $0 < \alpha < r$ such that $I_2 < \varepsilon/3$ for all $z \in U$ with $\|z - z_0\| < \alpha$. \square

Theorem 3.3. *Suppose that $z_0 \in V \cap U$ and $F_{z_0}(\cdot)$ has a smooth point in \mathbb{R}^m . Then*

$$\liminf_{\substack{z \rightarrow z_0 \\ z \in V \cap U}} \Gamma_z(A) \geq \Gamma_{z_0}(A) \quad (3.3)$$

for every $A \in \mathcal{B}(V \cap U)$ (the Borel σ -algebra inherited from the usual Euclidean topology).

Proof. Let $A \in \mathcal{B}(V \cap U)$ and $\varepsilon > 0$. Since $\varphi_{z_0}(\cdot)$ is dominating and $\Gamma'_{z_0} = f_{z_0}(\Gamma)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^m (cf. proof of Theorem 3.1), [27, Theorem 3.2] yields a neighbourhood V_0 of z_0 in $V \cap U$ such that $\varphi_z(\Gamma'_{z_0})(A) \geq \varphi_{z_0}(\Gamma'_{z_0})(A) - \frac{\varepsilon}{2}$ for all $z \in V_0$ which is equivalent to

$$f_z(\Gamma)(\varphi_z^{-1}(A)) \geq f_{z_0}(\Gamma)(\varphi_{z_0}^{-1}(A)) - \frac{\varepsilon}{2} \quad \forall z \in V_0. \quad (3.4)$$

Due to Proposition 3.2, there exists $\alpha > 0$ such that $f_z(\Gamma)(B) \geq f_{z_0}(\Gamma)(B) - \frac{\varepsilon}{2}$ for all $B \in \mathcal{B}(\mathbb{R}^m)$ and every $z \in U$ with $\|z - z_0\| < \alpha$. We choose $B = \varphi_z^{-1}(A)$ and deduce for every $z \in U$, $\|z - z_0\| < \alpha$,

$$f_z(\Gamma)(\varphi_z^{-1}(A)) \geq f_{z_0}(\Gamma)(\varphi_{z_0}^{-1}(A)) - \frac{\varepsilon}{2}. \quad (3.5)$$

With (3.4) and (3.5) we obtain for all $z \in V_0$ with $\|z - z_0\| < \alpha$ that $f_z(\Gamma)(\varphi_z^{-1}(A)) \geq f_{z_0}(\Gamma)(\varphi_{z_0}^{-1}(A)) - \varepsilon$. Since $\varphi_z(\Gamma'_z) = F_z(\Gamma) = \Gamma_z$, this is equivalent to

$$\Gamma_z(A) \geq \Gamma_{z_0}(A) - \varepsilon \quad \forall z \in V_0 \cap \{z \in \mathbb{R}^n : \|z - z_0\| < \alpha\}.$$

This shows (3.3) since $\varepsilon > 0$ can be chosen arbitrarily small. \square

3.2. Stationarity and Ergodicity

Culminating in Theorem 3.11 we now gradually show Harris recurrence, geometric ergodicity and β -mixing for semi-polynomial Markov chains.

3.2.1. Assumptions

Concerning the sequence $(e_t)_{t \in \mathbb{N}}$ we make the following additional assumptions for our semi-polynomial Markov chain:

- (A1) Every e_t has distribution Γ which is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^m with density γ . Let E denote the support of Γ .

We define for all $k \in \mathbb{N}^*$, $k > 1$ the functions $F^k(z, y_1, \dots, y_k) := F(F^{k-1}(z, y_1, \dots, y_{k-1}), y_k)$ where $z \in U$, $(y_1, \dots, y_k) \in (\mathbb{R}^m)^k$ and set $F^1 = F$.

With this notation we introduce for $z \in V \cap U$ the orbit

$$S_z := \bigcup_{k \in \mathbb{N}^*} \left\{ F^k(z, y_1, \dots, y_k) : y_1, \dots, y_k \in E \right\}.$$

To prove the desired properties for semi-polynomial Markov chains we assume:

(A2) There is a point $a \in \text{int}(E)$ and a point $T \in V \cap U$ such that, for all $z \in V \cap U$, the sequence $(X_t^z)_{t \in \mathbb{N}}$ defined by $X_0^z = z$ and $X_t^z = F(X_{t-1}^z, a)$ for $t \geq 1$ converges to the point T .

T is called *attracting point* of the chain $(X_t)_{t \in \mathbb{N}}$.

We set $W := \overline{S_T}^Z$ the Zariski closure of the orbit S_T . Note that obviously $T \in W$ and $W \subseteq V$. To show uniqueness of the strictly stationary solution we need the assumption:

(A3) Any strictly stationary solution of the Markov chain $X_{t+1} = F(X_t, e_t)$ takes its values in the algebraic variety $W \cap U$.

Remark 3.4. (i) If (A2) is satisfied, then T is a fixed point of $F(\cdot, a)$, since F is continuous.

(ii) It is obvious that W is an algebraic set since it is the Zariski closure of S_T . In fact, it is even irreducible (cf. the upcoming Section 3.2.2).

Strictly speaking $W \cap U$ is not necessarily an algebraic variety, but, as it is the intersection of an algebraic variety in \mathbb{R}^n and the set U where our Markovian dynamics are defined, we refer to it as an algebraic variety.

3.2.2. Algebraic Variety of States

In this subsection we suppose that the assumptions (A1) and (A2) hold. We will show that W , defined as above, is indeed an algebraic variety which we will call the Markov chain's *algebraic variety of states*.

Let $(D_k)_{k \in \mathbb{N}^*}$ be the sequence of subsets of U defined by $D_k := F^k(T, E^k)$. Since $F(T, a) = T$ (cf. Remark 3.4 (i)) we obtain $D_k = F^k(F(T, a), E^k) = F^{k+1}(T, \{a\} \times E^k) \subseteq D_{k+1}$, i.e. the sequence $(D_k)_{k \in \mathbb{N}^*}$ is an ascending sequence of subsets of \mathbb{R}^n .

We set $W_k := \overline{F^k(T, (\mathbb{R}^m)^k)}^Z$. Then we have $W_k = \overline{\varphi^k(T, (\mathbb{R}^m)^k)}^Z$ (defining φ^k analogously to F^k) since $f_T(\cdot)$ is a C^1 -diffeomorphism.

Lemma 3.5. For all $k \in \mathbb{N}^*$ we have $W_k = \overline{D_k}^Z$.

Proof. To this end consider the map $f_T^{(k)} : (\mathbb{R}^m)^k \rightarrow (\mathbb{R}^m)^k$, $(y_1, \dots, y_k) \mapsto (x_1, \dots, x_k)$ where $x_1 = f_T(y_1)$, $x_2 = f_{F(T, y_1)}(y_2)$, \dots , $x_k = f_{F^{k-1}(T, y_1, \dots, y_{k-1})}(y_k)$.

Due to the properties of f and F (in particular (F4)), it is clear that $f_T^{(k)}$ is bijective, continuous and its inverse is continuous as well, i.e. $f_T^{(k)}$ is a homeomorphism.

Assumption (A2) implies that E^k contains an open ball of $(\mathbb{R}^m)^k$. Thus, since $f_T^{(k)}$ is homeomorphic, $f_T^{(k)}(E^k)$ contains an open ball of $(\mathbb{R}^m)^k$.

From [4, Corollary 3.4.5] we obtain $\overline{f_T^{(k)}(E^k)}^Z = (\mathbb{R}^m)^k$. This shows

$$W_k = \overline{F^k(T, (\mathbb{R}^m)^k)}^Z = \overline{\varphi^k(T, (\mathbb{R}^m)^k)}^Z = \overline{\varphi^k\left(T, \overline{f_T^{(k)}(E^k)}^Z\right)}^Z.$$

Since $\varphi^k(T, \cdot)$ is regular (cf. (F3)), $\varphi^k(T, \cdot)$ is continuous with respect to the Zariski topology due to Proposition A.8. Hence, $W_k = \overline{\varphi^k\left(T, \overline{f_T^{(k)}(E^k)}^Z\right)}^Z = \overline{F^k(T, E^k)}^Z = \overline{D_k}^Z$ which proves W_k to be the Zariski closure of D_k . \square

Lemma 3.6. W_k is irreducible for all $k \in \mathbb{N}^*$.

Proof. If we suppose that there is $k \in \mathbb{N}^*$ such that $W_k = V_1 \cup V_2$ where V_1 and V_2 are algebraic sets with $V_1 \subsetneq W_k$ and $V_2 \subsetneq W_k$, then

$$(\mathbb{R}^m)^k = \left(\varphi_T^k\right)^{-1}(W_k) = \underbrace{\left(\varphi_T^k\right)^{-1}(V_1)}_{(*)} \cup \underbrace{\left(\varphi_T^k\right)^{-1}(V_2)}_{(**)}$$

where $\varphi_T^k(\cdot) = \varphi^k(T, \cdot)$. Now $(*)$ and $(**)$ are algebraic sets, because V_1 and V_2 are algebraic sets and $\varphi_T^k(\cdot)$ is continuous with respect to the Zariski topology (see proof of Lemma 3.5). Since $\left(\varphi_T^k\right)^{-1}(V_i) \subsetneq (\mathbb{R}^m)^k$ for $i = 1, 2$ (otherwise $W_k = \overline{\varphi^k(T, (\mathbb{R}^m)^k)} \subseteq \overline{V_i} = V_i$ which would be a contradiction to $V_i \subsetneq W_k$), this would prove $(\mathbb{R}^m)^k$ to be reducible which is a contradiction. \square

Proposition 3.7. *There exists $l \in \mathbb{N}^*$ such that $W_k = W_l$ for all $k \geq l$ and $W = W_l$. In particular, W is an algebraic variety.*

Proof. From [4, Corollary 3.4.5] it follows that if $V_1 \subseteq V_2 \subseteq V_3 \subseteq \dots$ is an ascending sequence of algebraic varieties in \mathbb{R}^n then there exists $l \in \mathbb{N}^*$ such that $V_k = V_l$ for all $k \geq l$.

Lemma 3.5 and Lemma 3.6 show that $(W_k)_{k \in \mathbb{N}^*}$ is an ascending sequence of algebraic varieties and so there exists $l \in \mathbb{N}^*$ such that $W_k = W_l$ for all $k \geq l$. We then observe that

$$S_T = \bigcup_{k \in \mathbb{N}^*} \underbrace{\left\{ F^k(T, y_1, \dots, y_k) : y_1, \dots, y_k \in E \right\}}_{=F^k(T, E^k)} = \bigcup_{k \in \mathbb{N}^*} D_k.$$

Since

$$\overline{\bigcup_{k \in \mathbb{N}^*} D_k} \subseteq \overline{\bigcup_{k \in \mathbb{N}^*} \overline{D_k}} = \overline{W_l} = W_l$$

$$= \bigcup_{k \in \mathbb{N}^*} W_k = W_l$$

and $W_l = \overline{D_l} \subseteq \overline{\bigcup_{k \in \mathbb{N}^*} D_k}$, we obtain $W = \overline{S_T} = W_l$. \square

Lemma 3.8. *For all $k \in \mathbb{N}^*$ we have $F^k(W \cap U, (\mathbb{R}^m)^k) \subseteq W \cap U$. Hence, the Markov chain can be restricted to the variety of states $W \cap U$.*

Proof. With the definition of the subsets D_k and W_k , respectively, one has for all $k \in \mathbb{N}^*$

$$\varphi(D_k, \mathbb{R}^m) = F(D_k, \mathbb{R}^m) = F\left(\underbrace{F^k(T, E^k)}_{\subseteq F^k(T, (\mathbb{R}^m)^k)}, \mathbb{R}^m\right) \subseteq F^{k+1}(T, (\mathbb{R}^m)^{k+1}) \subseteq W_{k+1} \cap U \subseteq W \cap U. \quad (3.6)$$

The continuity of regular maps with respect to the Zariski topology and the regularity of φ yield

$$F(W \cap U, \mathbb{R}^m) = \varphi(W \cap U, \mathbb{R}^m) \subseteq \varphi(W, \mathbb{R}^m) = \varphi(\overline{D_l}, \mathbb{R}^m) \subseteq \overline{\varphi(\overline{D_l}, \mathbb{R}^m)} = \overline{\varphi(D_l, \mathbb{R}^m)} \stackrel{(3.6)}{\subseteq} W.$$

Since we assume $F : U \times \mathbb{R}^m \rightarrow U$, we have $F(W \cap U, \mathbb{R}^m) \subseteq W \cap U$. By induction we have $F^k(W \cap U, (\mathbb{R}^m)^k) \subseteq W \cap U$ for all $k \in \mathbb{N}^*$. Hence we can restrict the Markov chain to the variety of states $W \cap U$. \square

Proposition 3.9. For all $A \in \mathcal{B}(W \cap U)$ and all $k \geq l$

$$\liminf_{\substack{z \rightarrow T \\ z \in W \cap U}} P^k(z, A) \geq P^k(T, A),$$

where P^k is the k -step transition probability kernel of the Markov chain $(X_t)_{t \in \mathbb{N}}$.

Proof. Since $\varphi_T^k(\cdot) = \varphi^k(T, \cdot)$ is regular and dominating for all $k \geq l$ (since $\overline{Z\varphi^k(T, (\mathbb{R}^m)^k)} = W_k = W$ for all $k \geq l$, cf. Proposition 3.7), [27, A 23] implies that $\varphi_T^k(\cdot)$ has a smooth point.

Similar to the proof of Theorem 3.1 we can now show that the map $F^k(T, \cdot)$ has a smooth point in $(\mathbb{R}^m)^k$. Let $x_0 \in (\mathbb{R}^m)^k$ be the smooth point of $\varphi_T^k(\cdot)$, i.e. $\varphi^k(T, x_0) \in \mathcal{R}(W)$ and $\text{rank}(D\varphi_T^k(x_0)) = \dim W$. Then $F^k\left(T, \left(f_T^{(k)}\right)^{-1}(x_0)\right) = \varphi^k(T, x_0)$ and

$$DF^k(T, \cdot)\left(\left(f_T^{(k)}\right)^{-1}(x_0)\right) = D\varphi_T^k(x_0) \cdot Df_T^{(k)}\left(\left(f_T^{(k)}\right)^{-1}(x_0)\right)$$

where the linear map $Df_T^{(k)}(x)$ is invertible for all $x \in (\mathbb{R}^m)^k$ (to this end note that $f_T^{(k)}$ is not only continuous but also differentiable and that $Df_T^{(k)}$ is a block matrix with lower triangle structure where the blocks on the diagonal are invertible). Hence the matrix on the left hand side has also rank $\dim W$ and $\left(f_T^{(k)}\right)^{-1}(x_0)$ is a smooth point of $F^k(T, \cdot)$.

Finally, note that $P^k(z, A) = F^k(z, \otimes_{i=1}^k \Gamma)(A)$ where Γ is the distribution of every e_t (cf. (A1)) and we conclude with Theorem 3.3. \square

3.2.3. Harris Recurrence, Ergodicity and β - Mixing

In this subsection we will prove the promised properties of semi-polynomial Markov chains under a Foster-Lyapunov-condition. First we show irreducibility on the algebraic variety of states and aperiodicity. As usual, ψ denotes a maximal irreducibility measure.

Proposition 3.10. Suppose that (A1) and (A2) hold. Then the semi-polynomial Markov chain $(X_t)_{t \in \mathbb{N}}$ is ψ - irreducible and aperiodic on the state space $(W \cap U, \mathcal{B}(W \cap U))$.

Moreover, the support of ψ has non-empty interior.

Proof. (1) Due to Proposition 3.9 we have for all $A \in \mathcal{B}(W \cap U)$

$$\liminf_{\substack{z \rightarrow T \\ z \in W \cap U}} P^l(z, A) \geq P^l(T, A). \quad (3.7)$$

We define a probability measure ν on the state space $(W \cap U, \mathcal{B}(W \cap U))$ by $\nu(A) := P^l(T, A)$, $A \in \mathcal{B}(W \cap U)$. Then, for every $A \in \mathcal{B}(W \cap U)$ with $\nu(A) \neq 0$, there exists due to (3.7) a neighbourhood W_1 of T in $W \cap U$ such that

$$P^l(z, A) \geq \frac{\nu(A)}{2} \forall z \in W_1. \quad (3.8)$$

(2) Let $K = \{z_1, \dots, z_r\} \subseteq W \cap U$ for some $r \in \mathbb{N}^*$. We are going to show that there is a $q \in \mathbb{N}^*$ such that $P^q(z_i, W_1) > 0 \forall i \in \{1, \dots, r\}$. To this end, consider for $i = 1, \dots, r$ the sequences $(X_t^{z_i})_{t \in \mathbb{N}}$ defined by

$$X_0^{z_i} = z_i \quad \text{and} \quad X_t^{z_i} = F(X_{t-1}^{z_i}, a), \quad t \geq 1$$

where $a \in \text{int}(E)$ as in (A2).

Due to assumption (A2) there is $q \in \mathbb{N}^*$ such that $X_q^{z_i} = F^q(z_i, a, \dots, a) \in W_1 \forall i \in \{1, \dots, r\}$. Since $F^q : W \cap U \times (\mathbb{R}^m)^q \rightarrow W \cap U$ is continuous, there exists for every $i \in \{1, \dots, r\}$ a neighbourhood U_i of (z_i, a, \dots, a) in $W \cap U \times (\mathbb{R}^m)^q$ such that

$$F^q(y, y_1, \dots, y_q) \in W_1 \forall (y, y_1, \dots, y_q) \in U_i.$$

Then, for all $i \in \{1, \dots, r\}$, U_i contains $U_i' \times U_{(a, \dots, a)}^i$ where U_i' and $U_{(a, \dots, a)}^i$ are suitable neighbourhoods of z_i in $W \cap U$ and (a, \dots, a) in $(\mathbb{R}^m)^q$, respectively.

We define $U_{(a, \dots, a)} := \bigcap_{i=1}^r U_{(a, \dots, a)}^i$ which is clearly also a neighbourhood of (a, \dots, a) in $(\mathbb{R}^m)^q$. Then we have $F^q(z_i, y_1, \dots, y_q) \in W_1$ for all $i \in \{1, \dots, r\}$ and $(y_1, \dots, y_q) \in U_{(a, \dots, a)}$. Since $U_{(a, \dots, a)}$ contains itself $U_a \times \dots \times U_a$ where U_a is an appropriate neighbourhood of a in \mathbb{R}^m , we deduce for all $i \in \{1, \dots, r\}$:

$$P^q(z_i, W_1) \geq \mathbb{P}((e_1, \dots, e_q) \in U_{(a, \dots, a)}) \geq \mathbb{P}(e_1 \in U_a)^q = \Gamma(U_a)^q. \quad (3.9)$$

(3) Let $A \in \mathcal{B}(W \cap U)$ with $\nu(A) \neq 0$. As in (1), W_1 denotes the neighbourhood of T in $W \cap U$ such that $P^l(z, A) \geq \nu(A)/2$ for all $z \in W_1$. Using the Chapman-Kolmogorov equation (cf. [26] Theorem 3.4.2), we obtain for every $i \in \{1, \dots, r\}$

$$\begin{aligned} P^{q+l}(z_i, A) &= \int_{W \cap U} P^q(z_i, dy) P^l(y, A) \geq \int_{W_1} P^q(z_i, dy) P^l(y, A) \stackrel{(3.8)}{\geq} \frac{\nu(A)}{2} \cdot \int_{W_1} P^q(z_i, dy) \\ &= \frac{\nu(A)}{2} \cdot P^q(z_i, W_1) \stackrel{(3.9)}{\geq} \frac{\Gamma(U_a)^q}{2} \cdot \nu(A). \end{aligned}$$

Due to assumption (A2) U_a contains an open set of E . Thus $\Gamma(U_a) > 0$. This implies that the chain $(X_t)_{t \in \mathbb{N}}$ is ν -irreducible (and thus also ψ -irreducible due to [26, Proposition 4.2.2]).

(4) To show aperiodicity, we suppose the chain to be periodic with period d . Due to [26, Theorem 5.4.4] there exist disjoint sets $D_1, \dots, D_d \in \mathcal{B}(W \cap U)$ such that

$$(i) P(z, D_{(i \bmod d)+1}) = 1 \forall i = 1, \dots, d \text{ and } z \in D_i \quad \text{and} \quad (ii) \psi((\cup_{i=1}^d D_i)^c) = 0.$$

Since $\psi \succ \nu$ (cf. [26, Proposition 4.2.2]), $(\cup_{i=1}^d D_i)^c$ is also a ν -null set. Obviously there must be a set D_i with positive ν -measure, let this set be D_1 without loss of generality.

Let $x \in D_1$ and $y \in D_d$. For $K := \{x, y\}$ we have just shown in step (3) that

$$P^{q+l}(x, D_1) > 0 \quad \text{and} \quad P^{q+l}(y, D_1) > 0$$

for some $q \in \mathbb{N}^*$. Hence, the integers $q+l$ and $q+l-1$ are divisible by d . Consequently $d = 1$.

(5) We have shown in step (3) that $(X_t)_{t \in \mathbb{N}}$ is $P^l(T, \cdot)$ -irreducible. Since $F^l(T, \cdot)$ has a smooth point in $(\mathbb{R}^m)^l$ (cf. proof of Proposition 3.9) Theorem 3.1 implies that $P^l(T, \cdot) = F^l(T, \otimes_{i=1}^l \Gamma)$ is absolutely continuous with respect to the measure μ_W . Hence, $\text{int}(\text{supp } P^l(T, \cdot)) \neq \emptyset$ and we also obtain that $\text{int}(\text{supp } \psi) \neq \emptyset$. \square

We can now state our main result for semi-polynomial Markov chains. Therefore we use the standard notation $PV(x) := \mathbb{E}[V(X_1)|X_0 = x] = \mathbb{E}_x[V(X_1)]$.

Theorem 3.11. *Suppose (A1) and (A2) are valid. If in addition the Foster-Lyapunov-condition holds, i.e. there exist a small set $C \in \mathcal{B}(W \cap U)$, positive constants $\alpha < 1$, $b < \infty$ and a function $V \geq 1$ such*

that

$$PV(x) \leq \alpha \cdot V(x) + b \cdot 1_C(x) \forall x \in W \cap U, \quad (\text{FL})$$

then the semi-polynomial Markov chain $(X_t)_{t \in \mathbb{N}}$ is positive Harris recurrent and geometrically ergodic on the algebraic variety of states $W \cap U$. Furthermore, the strictly stationary process $(X_t)_{t \in \mathbb{Z}}$ is geometrically β -mixing and $\pi(V) := \mathbb{E}[V(X_t)] < \infty$.

Proof. Due to Proposition 3.10 and the assumptions (A1) and (A2), $(X_t)_{t \in \mathbb{N}}$ is ψ -irreducible and aperiodic on $(W \cap U, \mathcal{B}(W \cap U))$. We conclude by using the following Theorem 3.12. \square

Theorem 3.12. *Let $(X_t)_{t \in \mathbb{N}}$ be a ψ -irreducible Markov chain on a state space $(S, \mathcal{B}(S))$ with transition probability kernel P . If the chain is aperiodic and the Foster-Lyapunov-condition holds, i.e. there exist a small set $C \in \mathcal{B}(S)$, positive constants $\alpha < 1$, $b < \infty$ and a function $V \geq 1$ such that*

$$PV(x) \leq \alpha \cdot V(x) + b \cdot 1_C(x) \forall x \in S,$$

then $(X_t)_{t \in \mathbb{N}}$ is positive Harris recurrent, geometrically ergodic and the strictly stationary process $(X_t)_{t \in \mathbb{Z}}$ is geometrically β -mixing. Furthermore, $\pi(V) < \infty$.

Proof. Since the Foster-Lyapunov-condition holds, the non-negative functions $V' := V - 1$, $f := 1 - \alpha$ and $s := b1_C$ satisfy the assumption of Theorem B.1. Hence we obtain for the first return time to C , denoted by τ_C ,

$$\mathbb{E}_x[\tau_C] = \frac{1}{1 - \alpha} \mathbb{E}_x \left[\sum_{k=0}^{\tau_C - 1} f(X_k) \right] \leq \frac{1}{1 - \alpha} \left(V(x) - 1 + \underbrace{\mathbb{E}_x \left[\sum_{k=0}^{\tau_C - 1} s(X_k) \right]}_{=b1_C(x)} \right) < \infty \forall x \in S$$

and thus obviously $L(x, C) = \mathbb{P}_x(\tau_C < \infty) = 1$ for all $x \in S$. Since every small set is also petite (cf. [25] or [26]), Proposition B.2 yields Harris recurrence of $(X_t)_{t \in \mathbb{N}}$. Again the Foster-Lyapunov-condition shows that $V' := (1 - \alpha)^{-1}V$, $f := V$ and $b' := (1 - \alpha)^{-1}b$ satisfy (ii) of Theorem B.3 which implies that $(X_t)_{t \in \mathbb{N}}$ is positive and $\pi(V) < \infty$. It is once more the same condition that yields directly geometric ergodicity by virtue of Theorem B.4. Finally, combining [11, Proposition 1 (1)] with (B.1) and $\pi(V) < \infty$, we deduce that the strictly stationary process $(X_t)_{t \in \mathbb{Z}}$ is geometrically β -mixing. \square

Theorem 3.13. *Suppose the setting of Theorem 3.11 and assume in addition that (A3) holds. Then the strictly stationary process is unique.*

Proof. If there is another P -invariant probability measure π' , then $\text{supp}(\pi') \subseteq W \cap U$ due to (A3). Since the chain $(X_t)_{t \in \mathbb{N}}$ is recurrent on $W \cap U$ (Theorem 3.11), it has at most one P -invariant probability measure on $(W \cap U, \mathcal{B}(W \cap U))$ (cf. [26, Theorem 10.4.4]). Therefore the strictly stationary solution is unique. \square

Remark 3.14. Note that the whole theory from algebraic geometry has only been used to prove irreducibility and aperiodicity on the algebraic variety of states $W \cap U$. The results thereafter (Theorems 3.12 and 3.13) are consequences of the theory of Markov chains.

4. Proof of the Main Theorem 2.4

In this section we gradually prove our main result for multivariate GARCH processes, Theorem 2.4.

4.1. GARCH Processes as Semi-Polynomial Markov Processes

First we show that the autoregressive representation of standard GARCH processes involving the function φ leads to a semi-polynomial Markov chain.

Lemma 4.1. *The mapping $G : \mathbb{S}_d^+ \rightarrow \mathbb{S}_d^+$ which maps Σ to $\Sigma^{1/2}$ is a C^1 - diffeomorphism on \mathbb{S}_d^{++} .*

Proof. We define $f_d : \mathbb{S}_d^{++} \rightarrow \mathbb{S}_d^{++}$, $X \mapsto X \cdot X$. Since every $X \in \mathbb{S}_d^{++}$ has full rank, f_d is well-defined. We will show that f_d is bijective and that, for every $X \in \mathbb{S}_d^{++}$, the differential $df_d(X)$ is a linear homeomorphism.

By [21, Theorem 7.2.6], a positive definite matrix has a unique positive definite square root. Hence f_d is bijective. Let $X \in \mathbb{S}_d^{++}$. The differential of f_d at the point X is given by

$$\forall H \in \mathbb{S}_d \quad df_d(X)H = \left. \frac{\partial}{\partial t} f_d(X + tH) \right|_{t=0} = HX + XH.$$

Our aim is to show that $H = 0$ whenever $df_d(X)H = 0$. In fact, this is a simple consequence of [29, Theorem 1] where the solutions X of the general matrix quadratic equation $0 = A + BX + XB^t - CXX$ for fixed $A, B, C \in M_d(\mathbb{R})$ have been analysed. \square

Since $X_n = \Sigma_n^{1/2} \varepsilon_n$, we thus obtain that

$$f : U \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad (Y_{n-1}, \varepsilon_n) \mapsto G(\Sigma_n) \varepsilon_n = X_n$$

is a C^1 - map from $U \times \mathbb{R}^d$ into \mathbb{R}^d where U is the open set in $(\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q$ defined by

$$U := \underbrace{\text{vech}(\mathbb{S}_d^{++}) \times \dots \times \text{vech}(\mathbb{S}_d^{++})}_p \times \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_q.$$

Due to the assumption that C and the initial values $\Sigma_0, \dots, \Sigma_{1-p}$ are positive definite, every Σ_n and $\Sigma_n^{1/2}$ is also positive definite and thus $G(\Sigma_n) = \Sigma_n^{1/2}$ is always an invertible matrix. Hence, for every $Y \in U$, the map $f_Y(\cdot) = f(Y, \cdot)$ is linear bijective from \mathbb{R}^d onto \mathbb{R}^d , i.e. $f_Y(\cdot)$ is a C^1 - diffeomorphism. Moreover the map $U \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $(Y, \varepsilon) \mapsto f_Y^{-1}(\varepsilon)$ is continuous in (Y, ε) where $f_Y^{-1}(\cdot)$ denotes the inverse of $f_Y(\cdot)$.

Altogether we are thus in our setting of semi-polynomial Markov chains. For the Markovian representation Y_n of a standard GARCH(p, q) process X_n we have

$$Y_n = F(Y_{n-1}, \varepsilon_n) := \varphi(Y_{n-1}, \underbrace{f_{Y_{n-1}}(\varepsilon_n)}_{=X_n}) \quad (4.1)$$

where F is a C^1 - map from $U \times \mathbb{R}^d$ into U . Moreover, it is obvious that (2.5), (2.6) have a stationary solution if and only if (4.1) has one.

4.2. Some Results from Linear Algebra

In this section we will show some results from linear algebra which will be necessary to establish the Foster-Lyapunov-condition for multivariate GARCH models.

Let $n \in \mathbb{N}^*$ and $(F_i)_{1 \leq i \leq n}$ be elements of $M_d(\mathbb{R})$. We set

$$\xi : M_d(\mathbb{R}) \rightarrow M_d(\mathbb{R}), \quad \xi(M) := \sum_{i=1}^n F_i M F_i^t.$$

This map is obviously linear. We can consider ξ a linear map from \mathbb{R}^{d^2} into \mathbb{R}^{d^2} using the vec operator as follows :

$$\text{vec}(\xi(M)) = \left(\sum_{i=1}^n F_i \otimes F_i \right) \text{vec}(M) = F \text{vec}(M) \text{ where } F := \sum_{i=1}^n F_i \otimes F_i.$$

Note that we have $\xi(\mathbb{S}_d) \subseteq \mathbb{S}_d$, i.e. the symmetric $d \times d$ matrices are mapped into themselves by ξ . We denote by $\tilde{\xi}$ the restriction of ξ to the linear subspace \mathbb{S}_d . Using the vech operator, we obtain, for all $M \in \mathbb{S}_d$,

$$\text{vech}(\tilde{\xi}(M)) = \text{vech}(\xi(M)) = H_d \text{vec}(\xi(M)) = H_d F \text{vec}(M) = H_d F K_d^t \text{vech}(M).$$

Since we can identify \mathbb{S}_d via the vech operator with $\mathbb{R}^{d(d+1)/2}$, the transformation matrix of $\tilde{\xi}$ is given by

$$\tilde{F} := H_d F K_d^t.$$

We obtain the following lemma:

Lemma 4.2. *Let $C \in \mathbb{S}_d^{++}$. The following statements are equivalent:*

- (i) *The spectral radius of ξ is less than 1.*
- (ii) *The spectral radius of $\tilde{\xi}$ is less than 1.*
- (iii) *There is $\Sigma \in \mathbb{S}_d^{++}$ such that $\Sigma = C + \xi(\Sigma)$.*

Proof. (i) \Rightarrow (ii): Obvious since $\tilde{\xi}$ is a restriction of ξ .

(ii) \Rightarrow (iii): If the spectral radius of $\tilde{\xi}$ is less than 1, then the Neumann series $\sum_{n=0}^{\infty} \tilde{\xi}^n$ is convergent with respect to a suitable operator norm. We define

$$\Sigma := \sum_{n=0}^{\infty} \tilde{\xi}^n(C). \quad (4.2)$$

Clearly, Σ is symmetric. Moreover, for all $M \in \mathbb{S}_d^+$, we have $\tilde{\xi}(M) \in \mathbb{S}_d^+$ by the definition of $\tilde{\xi}$ and ξ , respectively. By iteration we obtain that $\tilde{\xi}^n(M) \in \mathbb{S}_d^+$ for all $n \in \mathbb{N}^*$. Thus the matrix $\Sigma - \tilde{\xi}^0(C) = \Sigma - C$ is symmetric and positive semi-definite. This implies that Σ is positive definite.

Since ξ and $\tilde{\xi}$ coincide on \mathbb{S}_d , we deduce $\Sigma = \sum_{n=0}^{\infty} \xi^n(C) = C + \xi\left(\sum_{n=1}^{\infty} \xi^{n-1}(C)\right) = C + \xi(\Sigma)$.

(iii) \Rightarrow (i): Suppose that there exists $\Sigma \in \mathbb{S}_d^{++}$ such that $\Sigma = C + \xi(\Sigma)$. We denote the complex $d \times d$ matrices by $M_d(\mathbb{C})$ and the conjugate transpose of a vector $x \in \mathbb{C}^d$ by x^* . For every $P \in M_d(\mathbb{C})$ we define

$$\|P\|_{\Sigma} := \sup_{x \in \mathbb{C}^d, x^* \Sigma x = 1} |x^* P x|$$

which is a norm on $M_d(\mathbb{C})$ since $\Sigma \in \mathbb{S}_d^{++}$.

Then, for all $x \in \mathbb{C}^d$, $|x^*Px| \leq \|P\|_\Sigma (x^*\Sigma x)$. Since the unit sphere $\{x \in \mathbb{C}^d : x^*\Sigma x = 1\}$ is compact, there exists, for every $P \in M_d(\mathbb{C})$, a vector $x_p \in \mathbb{C}^d$ such that $\|P\|_\Sigma = |x_p^*Px_p|$ with $x_p^*\Sigma x_p = 1$. Let now λ be an eigenvalue of ξ . Then there is an $M \in M_d(\mathbb{C})$, $M \neq 0$ such that $\lambda M = \xi(M) = \sum_{i=1}^n F_i M F_i^t$.

For every $x \in \mathbb{C}^d$, we deduce

$$|\lambda| \cdot |x^*Mx| = \left| \sum_{i=1}^n x^* F_i M F_i^t x \right| \leq \sum_{i=1}^n |(F_i^t x)^* M (F_i^t x)| \leq \|M\|_\Sigma \sum_{i=1}^n x^* F_i \Sigma F_i^t x = \|M\|_\Sigma x^* \underbrace{\left(\sum_{i=1}^n F_i \Sigma F_i^t \right)}_{=\xi(\Sigma)=\Sigma-C} x.$$

If we choose x_M such that $\|M\|_\Sigma = |x_M^* M x_M|$ and $x_M^* \Sigma x_M = 1$, we obtain $|\lambda| \leq 1 - x_M^* C x_M < 1$ (note that $\|M\|_\Sigma \neq 0$). Hence the spectral radius of ξ is less than 1. \square

We consider now the families of matrices $(\bar{A}_{i,k}, \bar{B}_{j,r})$ and (A_i, B_j) which occur in the BEKK and vech representation of a standard GARCH(p, q) model, respectively.

Proposition 4.3. *The spectral radius of $\sum_{i=1}^q A_i + \sum_{j=1}^p B_j$ is less than 1 if and only if there exists $\Sigma \in \mathbb{S}_d^{++}$ such that*

$$\Sigma = C + \sum_{i=1}^q \sum_{k=1}^{l_i} \bar{A}_{i,k} \Sigma \bar{A}_{i,k}^t + \sum_{j=1}^p \sum_{r=1}^{s_j} \bar{B}_{j,r} \Sigma \bar{B}_{j,r}^t. \quad (4.3)$$

Proof. Define $\xi : M_d(\mathbb{R}) \rightarrow M_d(\mathbb{R})$ by $\xi(M) = \sum_{i=1}^q \sum_{k=1}^{l_i} \bar{A}_{i,k} M \bar{A}_{i,k}^t + \sum_{j=1}^p \sum_{r=1}^{s_j} \bar{B}_{j,r} M \bar{B}_{j,r}^t$. Then the transformation matrix of ξ is

$$F = \sum_{i=1}^q \sum_{k=1}^{l_i} \bar{A}_{i,k} \otimes \bar{A}_{i,k} + \sum_{j=1}^p \sum_{r=1}^{s_j} \bar{B}_{j,r} \otimes \bar{B}_{j,r} = \sum_{i=1}^q \tilde{A}_i + \sum_{j=1}^p \tilde{B}_j.$$

Note that the transformation matrix of $\tilde{\xi}$ (restriction of ξ to the linear subspace \mathbb{S}_d) is $H_d F K_d^t = \sum_{i=1}^q A_i + \sum_{j=1}^p B_j$. Due to Lemma 4.2 the spectral radius of $\sum_{i=1}^q A_i + \sum_{j=1}^p B_j$ is less than 1 if and only if there is $\Sigma \in \mathbb{S}_d^{++}$ such that (4.3) holds. \square

Remark 4.4. By a simple transposition argument one can equivalently state that the spectral radius of $\sum_{i=1}^q A_i + \sum_{j=1}^p B_j$ is less than 1 if and only if there exists $\Sigma \in \mathbb{S}_d^{++}$ such that

$$\Sigma = C + \sum_{i=1}^q \sum_{k=1}^{l_i} \bar{A}_{i,k}^t \Sigma \bar{A}_{i,k} + \sum_{j=1}^p \sum_{r=1}^{s_j} \bar{B}_{j,r}^t \Sigma \bar{B}_{j,r}$$

which we are going to use in the upcoming proof of Theorem 4.9.

In the following we consider the block matrix B defined as in Section 2.

Proposition 4.5. (i) *If the spectral radius of the matrix $\sum_{j=1}^p B_j$ is less than 1, then the one of B is also less than 1.*

(ii) *If the spectral radius of the matrix $\sum_{i=1}^q A_i + \sum_{j=1}^p B_j$ is less than 1, then the one of $\sum_{j=1}^p B_j$ is also less than 1.*

Proof. (i) Suppose that the spectral radius of $\sum_{j=1}^p B_j$ is less than 1. Then there exists due to Lemma 4.2 a symmetric positive definite matrix $\tilde{\Sigma} \in \mathbb{S}_d^{++}$ such that

$$\tilde{\Sigma} = C + \sum_{j=1}^p \sum_{r=1}^{s_j} \bar{B}_{j,r} \tilde{\Sigma} \bar{B}_{j,r}^t. \quad (4.4)$$

Let λ be an eigenvalue of B associated with the eigenvector $h = (h_1^t, \dots, h_p^t)^t \in (\mathbb{R}^{d(d+1)/2})^p$. Then $\lambda h_1 = \sum_{j=1}^p B_j h_j$ and $\lambda h_j = h_{j-1}$ for $2 \leq j \leq p$. Thus $h_p \neq 0$ (otherwise h would be zero) and $\lambda^p h_p = \lambda(\lambda^{p-1} h_p) = \lambda h_1 = \sum_{j=1}^p B_j h_j = \sum_{j=1}^p \lambda^{p-j} B_j h_p$. Let $M \in \mathbb{S}_d$ such that $\text{vech}(M) = h_p$. Then $\lambda^p M = \sum_{j=1}^p \sum_{r=1}^{s_j} \lambda^{p-j} \bar{B}_{j,r} M \bar{B}_{j,r}^t$. We define the norm $\|\cdot\|_{\tilde{\Sigma}}$ on $M_d(\mathbb{C})$ as in the proof of Lemma 4.2 by $\|P\|_{\tilde{\Sigma}} := \sup_{x \in \mathbb{C}^d, x^* \tilde{\Sigma} x = 1} |x^* P x|$, $P \in M_d(\mathbb{C})$. Then, for all $x \in \mathbb{C}^d$,

$$\begin{aligned} |\lambda|^p \cdot |x^* M x| &= \left| \sum_{j=1}^p \sum_{r=1}^{s_j} \lambda^{p-j} x^* \bar{B}_{j,r} M \bar{B}_{j,r}^t x \right| \leq \sum_{j=1}^p \sum_{r=1}^{s_j} |\lambda|^{p-j} |x^* \bar{B}_{j,r} M \bar{B}_{j,r}^t x| \\ &\leq \|M\|_{\tilde{\Sigma}} \sum_{j=1}^p \sum_{r=1}^{s_j} |\lambda|^{p-j} (x^* \bar{B}_{j,r} \tilde{\Sigma} \bar{B}_{j,r}^t x). \end{aligned}$$

If we assume that there is an eigenvalue λ of B with $|\lambda| \geq 1$, then we obtain, taking the vector x such that $x^* \tilde{\Sigma} x = 1$ and $|x^* M x| = \|M\|_{\tilde{\Sigma}}$ and using (4.4), that

$$\begin{aligned} |\lambda|^p &\leq \sum_{j=1}^p \sum_{r=1}^{s_j} |\lambda|^{p-j} (x^* \bar{B}_{j,r} \tilde{\Sigma} \bar{B}_{j,r}^t x) \leq |\lambda|^{p-1} \left[x^* \left(\sum_{j=1}^p \sum_{r=1}^{s_j} \bar{B}_{j,r} \tilde{\Sigma} \bar{B}_{j,r}^t \right) x \right] \\ &= |\lambda|^{p-1} [x^* (\tilde{\Sigma} - C) x] = |\lambda|^{p-1} (1 - x^* C x). \end{aligned}$$

Since C is symmetric positive definite, one has $x^* C x > 0$. Hence, $|\lambda|^p < |\lambda|^{p-1}$, i.e. $|\lambda| < 1$ which is a contradiction. Thus the spectral radius of B has to be less than 1.

(ii) Suppose that the spectral radius of the matrix $\sum_{i=1}^q A_i + \sum_{j=1}^p B_j$ is less than 1. Then, due to Proposition 4.3, there exists $\Sigma \in \mathbb{S}_d^{++}$ such that $\Sigma = C + \sum_{i=1}^q \sum_{k=1}^{l_i} \bar{A}_{i,k} \Sigma \bar{A}_{i,k}^t + \sum_{j=1}^p \sum_{r=1}^{s_j} \bar{B}_{j,r} \Sigma \bar{B}_{j,r}^t$. We set $\tilde{C} := C + \sum_{i=1}^q \sum_{k=1}^{l_i} \bar{A}_{i,k} \Sigma \bar{A}_{i,k}^t$. Now, \tilde{C} is symmetric positive definite and $\Sigma = \tilde{C} + \sum_{j=1}^p \sum_{r=1}^{s_j} \bar{B}_{j,r} \Sigma \bar{B}_{j,r}^t$. Using again Proposition 4.3 we deduce that the spectral radius of $\sum_{j=1}^p B_j$ is less than 1. \square

Remark 4.6. The matrix $\tilde{\Sigma}$ in (4.4) is the limit of a Neumann series (cf. proof of Lemma 4.2). Thus

$$\text{vech}(\tilde{\Sigma}) = \left(I - \sum_{j=1}^p B_j \right)^{-1} \text{vech}(C).$$

4.3. Verification of Assumption (A2)

We will suppose throughout that (H1) holds.

Proposition 4.7. *Suppose that (H2) holds. If the spectral radius of the matrix $\sum_{j=1}^p B_j$ is less than 1, then (A2) holds.*

Proof. Let U be the open set in $(\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q$ defined as in Section 4.1. For arbitrary $y \in U$ we define the sequence $(Y_n^y)_{n \in \mathbb{N}}$ by $Y_0^y = y$ and $Y_n^y = F(Y_{n-1}^y, 0)$, $n \geq 1$.

We denote by X_n^y and $\text{vech}(\Sigma_n^y)$ the associated values of X_n and $\text{vech}(\Sigma_n)$. Since, by definition, $X_n = G(\Sigma_n)\varepsilon_n = \Sigma_n^{1/2}\varepsilon_n$, we obtain that $X_n^y = 0$ for all $n \geq 1$. Due to (2.6), Σ_n^y can be written, for every $n > q$, as

$$\text{vech}(\Sigma_n^y) = \text{vech}(C) + \sum_{j=1}^p B_j \text{vech}(\Sigma_{n-j}^y).$$

Thus, for all $n > q$ and for all $y \in U$,

$$Y_n^y = \mathcal{C} + \tilde{B}Y_{n-1}^y \quad (4.5)$$

where \tilde{B} is defined as in Section 2. Due to (4.5), the assumption (A2) is satisfied with $a = 0$ if the spectral radius of B is less than 1. This is the case, since the spectral radius of $\sum_{j=1}^p B_j$ is supposed to be less than 1 (cf. Proposition 4.5 (i)). \square

Hence, for all $y \in U$, the sequence $(Y_n^y)_{n \in \mathbb{N}}$ converges to the unique fixed point T defined by

$$T = \mathcal{C} + \tilde{B}T. \quad (4.6)$$

Using Lemma 4.2 and the fact that the spectral radius of $\sum_{j=1}^p B_j$ is assumed to be less than 1, there is $\tilde{\Sigma} \in \mathbb{S}_d^{++}$ (cf. (4.4)) such that

$$\tilde{\Sigma} = C + \sum_{j=1}^p \sum_{r=1}^{s_j} \tilde{B}_{j,r} \tilde{\Sigma} \tilde{B}_{j,r}^t.$$

It is then easy to see that T can be written as

$$T = \left(\underbrace{\text{vech}(\tilde{\Sigma})^t, \dots, \text{vech}(\tilde{\Sigma})^t}_p, \underbrace{0, \dots, 0}_{qd} \right)^t \in U. \quad (4.7)$$

We set $\mathcal{C}_1 := (\text{vech}(C)^t, 0, \dots, 0)^t \in (\mathbb{R}^{d(d+1)/2})^p$. Then (4.6) yields

$$\sigma = \mathcal{C}_1 + B\sigma \quad (4.8)$$

where $\sigma := (\text{vech}(\tilde{\Sigma})^t, \dots, \text{vech}(\tilde{\Sigma})^t)^t \in (\mathbb{R}^{d(d+1)/2})^p$.

4.4. Verification of Assumption (A3)

If (H2) is satisfied, then E contains an open set of \mathbb{R}^d and we obtain for the algebraic variety of states with the same arguments as in Section 3.2.2 that

$$W = \overline{Z_{S_T}} = \overline{Z \bigcup_{n \in \mathbb{N}^*} F^n(T, E^n)} = \overline{Z \bigcup_{n \in \mathbb{N}^*} F^n(T, (\mathbb{R}^d)^n)} = \overline{Z \bigcup_{n \in \mathbb{N}^*} \varphi^n(T, (\mathbb{R}^d)^n)}.$$

Let $n \in \mathbb{N}^*$ and consider $y(n) \in \varphi^n(T, (\mathbb{R}^d)^n)$ given by $y(n) = \varphi^n(T, x_1, \dots, x_n)$ where $x_1, \dots, x_n \in \mathbb{R}^d$. We define $x(n)$ and $\sigma(n)$ by the coordinates of $y(n)$ as follows:

$$\begin{aligned} x(n) &= (\text{vech}(x_n x_n^t), \dots, \text{vech}(x_{n-q+1} x_{n-q+1}^t))^t \\ \text{and } \sigma(n) &= (\text{vech}(\sigma_n), \dots, \text{vech}(\sigma_{n-p+1}))^t. \end{aligned}$$

That is, $y(n) = (\sigma(n)^t, x_n^t, \dots, x_{n-q+1}^t)^t$. Then

$$\sigma(n+1) = \mathcal{C}_1 + Ax(n) + B\sigma(n) \quad (4.9)$$

where \mathcal{C}_1 and B are defined in Section 4.3 and A is given by

$$A := \begin{pmatrix} A_1 & A_2 & \dots & A_q \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M_{p \frac{d(d+1)}{2} \times q \frac{d(d+1)}{2}}(\mathbb{R}).$$

Iterating (4.9) and due to $\sigma(0) = \sigma = (\text{vech}(\tilde{\Sigma})^t, \dots, \text{vech}(\tilde{\Sigma})^t)^t$ (since $y(0) = T$) we deduce

$$\sigma(n) = \underbrace{\sum_{i=0}^{n-1} B^i \mathcal{C}_1 + B^n \sigma}_{(4.8) \sigma} + \sum_{i=1}^{n-1} B^{i-1} Ax(n-i) = \sigma + \sum_{i=1}^{n-1} B^{i-1} Ax(n-i).$$

This yields $\text{vech}(\sigma_n) = \text{vech}(\tilde{\Sigma}) + \sum_{i=1}^{n-1} K_i \text{vech}(x_{n-i} x_{n-i}^t)$ where, for all $i \in \mathbb{N}^*$, K_i is defined by $K_i := [B^{i-1}A]_{1,1} + [B^{i-2}A]_{1,2} + \dots + [B^{i-q}A]_{1,q}$ with the convention $B^0 := I$, $B^i := 0$ if $i < 0$ and $[M]_{1,j}$ is the $d(d+1)/2 \times d(d+1)/2$ block from lines 1 to $d(d+1)/2$ and from columns $(j-1)d(d+1)/2 + 1$ to $jd(d+1)/2$ of M .

Thus, W is the Zariski closure of the orbit

$$\begin{aligned} S_T &= \bigcup_{n \in \mathbb{N}^*} \left\{ y(n) : x_1, \dots, x_n \in \mathbb{R}^d \right\} \\ &= \bigcup_{n \in \mathbb{N}^*} \left\{ T + \left(\sum_{i=1}^{n-1} (K_i \text{vech}(x_{n-i} x_{n-i}^t))^t, \dots, \sum_{i=1}^{n-p} (K_i \text{vech}(x_{n-p+1-i} x_{n-p+1-i}^t))^t, \right. \right. \\ &\quad \left. \left. x_n^t, x_{n-1}^t, \dots, x_{n-q+1}^t \right)^t : x_1, \dots, x_n \in \mathbb{R}^d \right\} \end{aligned}$$

where $x_{1-q} = x_{2-q} = \dots = x_0 = 0$ (since $y(0) = T = (\sigma^t, 0, \dots, 0)^t$).

In particular, this implies $\varphi(W \cap U \times \mathbb{R}^d) \subseteq W \cap U$ (cf. (F2)), because $\varphi(U \times \mathbb{R}^d) \subseteq U$ and $\varphi(S_T \times \mathbb{R}^d) \subseteq S_T$ yields

$$\varphi(W \cap U \times \mathbb{R}^d) \subseteq \varphi(\overline{S_T} \times \mathbb{R}^d) \subseteq \overline{\varphi(\overline{S_T} \times \mathbb{R}^d)} = \overline{\varphi(S_T \times \mathbb{R}^d)} \subseteq \overline{S_T} = W,$$

since φ is a regular map and thus continuous with respect to the Zariski topology.

Theorem 4.8. *Suppose that (H2) holds and that there is a strictly stationary solution $(X_n)_{n \in \mathbb{Z}}$ for the standard GARCH(p, q) model, then the process $(Y_n)_{n \in \mathbb{Z}}$ takes its values in the algebraic variety of*

states $W \cap U$. Moreover, one has

$$\text{vech}(\Sigma_n) = \left(I - \sum_{j=1}^p B_j \right)^{-1} \text{vech}(C) + \sum_{i=1}^{\infty} K_i \text{vech}(X_{n-i} X_{n-i}^t).$$

Proof. Let $(X_n)_{n \in \mathbb{Z}}$ be a strictly stationary solution of the standard GARCH(p, q) model with conditional covariance matrices Σ_n . We denote by $X(n)$ and $\Sigma(n)$ the following random vectors:

$$\begin{aligned} X(n) &= (\text{vech}(X_n X_n^t)^t, \dots, \text{vech}(X_{n-q+1} X_{n-q+1}^t)^t)^t \\ \text{and } \Sigma(n) &= (\text{vech}(\Sigma_n)^t, \dots, \text{vech}(\Sigma_{n-p+1})^t)^t. \end{aligned}$$

Since $\Sigma(n) = \mathcal{C}_1 + AX(n-1) + B\Sigma(n-1)$ (cf. (4.9)), iterating yields

$$\Sigma(n) = \sum_{i=0}^{k-1} B^i \mathcal{C}_1 + B^k \Sigma(n-k) + \sum_{i=1}^k B^{i-1} AX(n-i) \quad (4.10)$$

for all $k \in \mathbb{N}$.

Now for any $M = (\text{vech}(M_1)^t, \dots, \text{vech}(M_p)^t)^t$ and $N = (\text{vech}(N_1)^t, \dots, \text{vech}(N_p)^t)^t$ in $(\mathbb{R}^{d(d+1)/2})^p$, let us denote $M \geq N$ if and only if $M_1 \geq N_1, \dots, M_p \geq N_p$ (where, for all $M_i, N_i \in \mathbb{S}_d$, $M_i \geq N_i \Leftrightarrow M_i - N_i \geq 0 \Leftrightarrow M_i - N_i$ positive semi-definite). This defines a partial order on $(\mathbb{R}^{d(d+1)/2})^p$.

Then (4.10) yields $\Sigma(n) \geq \sum_{i=0}^{k-1} B^i \mathcal{C}_1$. Since $\Sigma(n)$ is finite the series $\sum_{i=0}^{k-1} B^i \mathcal{C}_1$ converges as $k \rightarrow \infty$ (see for instance [33] for further details concerning partially ordered topological spaces; in particular the Corollary after Lemma 5 proves that our series must converge). Setting $\tilde{\sigma} := \sum_{i=0}^{\infty} B^i \mathcal{C}_1$, it is easy to see that $\tilde{\sigma} = \mathcal{C}_1 + B\tilde{\sigma}$. Using the definitions of B and \mathcal{C}_1 , we obtain that $\tilde{\sigma} = (\sigma_1^t, \sigma_1^t, \dots, \sigma_1^t)^t$ for some $\sigma_1 \in \mathbb{R}^{d(d+1)/2}$ which fulfils $\sigma_1 = \text{vech}(C) + \sum_{j=1}^p B_j \sigma_1$. One may then verify that $\sigma_1 = \text{vech}(\Sigma_1)$ for some $\Sigma_1 \in \mathbb{S}_d^{++}$ and hence that the spectral radius of $\sum_{j=1}^p B_j$ is less than 1 (cf. Proposition 4.3). Due to Proposition 4.5 (i) we obtain that the spectral radius of B is also less than 1. Thus $\tilde{\sigma} = \sigma$.

Next, since the spectral radius of B is less than 1, the sequence $(B^k)_{k \in \mathbb{N}}$ converges to zero as $k \rightarrow \infty$. The random vectors $(\Sigma(n-k))_{k \in \mathbb{N}}$ have a constant law because $(X_n)_{n \in \mathbb{Z}}$ is supposed to be a strictly stationary solution of the GARCH model. Thus $B^k \Sigma(n-k)$ converges to zero in probability when $k \rightarrow \infty$.

With an analog argument as for $\sum_{i=0}^{k-1} B^i \mathcal{C}_1$ one can see that $\sum_{i=1}^k B^{i-1} AX(n-i)$ converges almost surely as $k \rightarrow \infty$. Hence, taking the limit of (4.10) yields

$$\Sigma(n) = \sigma + \sum_{i=1}^{\infty} B^{i-1} AX(n-i) \quad \text{a.s.}$$

Using the matrices K_i , defined during the investigation of the variety of states W , we obtain

$$\text{vech}(\Sigma_n) = \text{vech}(\tilde{\Sigma}) + \sum_{i=1}^{\infty} K_i \text{vech}(X_{n-i} X_{n-i}^t) \quad \text{a.s.}$$

This shows that $(Y_n)_{n \in \mathbb{Z}}$ takes its values in the variety W and hence in $W \cap U$. Note that the strictly stationary solution is causal. To finish the proof we refer to Remark 4.6 from which we obtain $\text{vech}(\tilde{\Sigma}) = (I - \sum_{j=1}^p B_j)^{-1} \text{vech}(C)$. \square

4.5. Foster - Lyapunov Condition (FL)

We now derive a function V satisfying the Foster-Lyapunov-condition provided that the spectral radius of $\sum_{i=1}^q A_i + \sum_{j=1}^p B_j$ is less than 1. That is, we prove the following theorem:

Theorem 4.9. *Suppose that the spectral radius of the matrix $\sum_{i=1}^q A_i + \sum_{j=1}^p B_j$ is less than 1. Then there exist a function $V : U \rightarrow [1, \infty)$ and positive constants $\alpha < 1$, $b < \infty$ as well as a Borel set K in $W \cap U$ such that the (FL) - condition is satisfied, i.e. there are positive constants $\alpha < 1$, $b < \infty$ such that*

$$PV(x) \leq \alpha \cdot V(x) + b \cdot 1_K(x) \forall x \in W \cap U.$$

Proof. For notational convenience we suppose that in the BEKK representation (2.4) $l_i = s_j = 1$ for all $i = 1, \dots, q$ and $j = 1, \dots, p$, since the extension to general $l_i, s_j \neq 1$ is obvious and trivial. We set $\bar{A}_i := \bar{A}_{i,1}$ and $\bar{B}_j := \bar{B}_{j,1}$. That is, we have

$$\Sigma_n = C + \sum_{i=1}^q \bar{A}_i X_{n-i} X_{n-i}^t \bar{A}_i^t + \sum_{j=1}^p \bar{B}_j \Sigma_{n-j} \bar{B}_j^t. \quad (4.11)$$

If the spectral radius of the matrix $\sum_{i=1}^q A_i + \sum_{j=1}^p B_j$ is less than 1, then, due to Proposition 4.3 and Remark 4.4, there exists $\Sigma \in \mathbb{S}_d^{++}$ such that $\Sigma = C + \sum_{i=1}^q \bar{A}_i^t \Sigma \bar{A}_i + \sum_{j=1}^p \bar{B}_j^t \Sigma \bar{B}_j$.

We define the map $V : U \rightarrow [1, \infty)$ by

$$V(Y_n) := \text{tr}(V_1 \Sigma_n) + \dots + \text{tr}(V_p \Sigma_{n-p+1}) + X_n^t V_{p+1} X_n + \dots + X_{n-q+1}^t V_{p+q} X_{n-q+1} + 1$$

where $\text{tr}(\cdot)$ denotes the trace of a matrix and the $d \times d$ matrices $(V_i)_{1 \leq i \leq p+q}$ are given by

$$V_k := \frac{p-k+1}{p+q} C + \sum_{j=k}^p \bar{B}_j^t \Sigma \bar{B}_j, \quad 1 \leq k \leq p$$

$$V_{p+k} := \frac{q-k+1}{p+q} C + \sum_{i=k}^q \bar{A}_i^t \Sigma \bar{A}_i, \quad 1 \leq k \leq q.$$

Setting $y = (\text{vech}(\Sigma_{n-1})^t, \dots, \text{vech}(\Sigma_{n-p})^t, X_{n-1}^t, \dots, X_{n-q}^t)^t \in U$ we obtain

$$\begin{aligned} \mathbb{E}[V(Y_n) | Y_{n-1} = y] &= \mathbb{E}[\text{tr}(V_1 \Sigma_n) + X_n^t V_{p+1} X_n | Y_{n-1} = y] + \text{tr}(V_2 \Sigma_{n-1}) + \dots + \text{tr}(V_p \Sigma_{n-p+1}) \\ &\quad + X_{n-1}^t V_{p+2} X_{n-1} + \dots + X_{n-q+1}^t V_{p+q} X_{n-q+1} + 1. \end{aligned} \quad (4.12)$$

Using (4.11) for Σ_n , we deduce for the first term at the right hand side

$$\begin{aligned} &\mathbb{E}[\text{tr}(V_1 \Sigma_n) + X_n^t V_{p+1} X_n | Y_{n-1} = y] \\ &= \mathbb{E}[X_n^t V_{p+1} X_n | Y_{n-1} = y] + \text{tr}(V_1 C) + \text{tr}(V_1 \bar{A}_1 X_{n-1} X_{n-1}^t \bar{A}_1^t) + \dots + \text{tr}(V_1 \bar{A}_q X_{n-q} X_{n-q}^t \bar{A}_q^t) \\ &\quad + \text{tr}(V_1 \bar{B}_1 \Sigma_{n-1} \bar{B}_1^t) + \dots + \text{tr}(V_1 \bar{B}_p \Sigma_{n-p} \bar{B}_p^t) \\ &= \mathbb{E}[X_n^t V_{p+1} X_n | Y_{n-1} = y] + \text{tr}(V_1 C) + X_{n-1}^t \bar{A}_1^t V_1 \bar{A}_1 X_{n-1} + \dots + X_{n-q}^t \bar{A}_q^t V_1 \bar{A}_q X_{n-q} \\ &\quad + \text{tr}(\bar{B}_1^t V_1 \bar{B}_1 \Sigma_{n-1}) + \dots + \text{tr}(\bar{B}_p^t V_1 \bar{B}_p \Sigma_{n-p}). \end{aligned}$$

Since $X_n = \Sigma_n^{1/2} \varepsilon_n$, $\Sigma_n^{1/2} \Sigma_n^{1/2} = \Sigma_n$ and $\mathbb{E}[\varepsilon_n \varepsilon_n^t] = I_d$, we obtain

$$\begin{aligned}
\mathbb{E}[X_n^t V_{p+1} X_n | Y_{n-1} = y] &= \mathbb{E}[\text{tr}(X_n (V_{p+1} X_n)^t) | Y_{n-1} = y] \\
&= \text{tr}(\mathbb{E}[X_n X_n^t V_{p+1} | Y_{n-1} = y]) = \text{tr}(\mathbb{E}[X_n X_n^t | Y_{n-1} = y] V_{p+1}) = \text{tr}(\Sigma_n V_{p+1}) \\
&= \text{tr}(V_{p+1} C) + \text{tr}(V_{p+1} \bar{A}_1 X_{n-1} X_{n-1}^t \bar{A}_1^t) + \dots + \text{tr}(V_{p+1} \bar{A}_q X_{n-q} X_{n-q}^t \bar{A}_q^t) \\
&\quad + \text{tr}(V_{p+1} \bar{B}_1 \Sigma_{n-1} \bar{B}_1^t) + \dots + \text{tr}(V_{p+1} \bar{B}_p \Sigma_{n-p} \bar{B}_p^t) \\
&= \text{tr}(V_{p+1} C) + X_{n-1}^t \bar{A}_1^t V_{p+1} \bar{A}_1 X_{n-1} + \dots + X_{n-q}^t \bar{A}_q^t V_{p+1} \bar{A}_q X_{n-q} \\
&\quad + \text{tr}(\bar{B}_1^t V_{p+1} \bar{B}_1 \Sigma_{n-1}) + \dots + \text{tr}(\bar{B}_p^t V_{p+1} \bar{B}_p \Sigma_{n-p}).
\end{aligned}$$

Hence, (4.12) can be rewritten as

$$\begin{aligned}
\mathbb{E}[V(Y_n) | Y_{n-1} = y] &= \text{tr}[(\bar{B}_1^t (V_1 + V_{p+1}) \bar{B}_1 + V_2) \Sigma_{n-1}] + \dots + \text{tr}[(\bar{B}_{p-1}^t (V_1 + V_{p+1}) \bar{B}_{p-1} + V_p) \Sigma_{n-p+1}] \\
&\quad + \text{tr}[\bar{B}_p^t (V_1 + V_{p+1}) \bar{B}_p \Sigma_{n-p}] + X_{n-1}^t (\bar{A}_1^t (V_1 + V_{p+1}) \bar{A}_1 + V_{p+2}) X_{n-1} + \dots \\
&\quad + X_{n-q+1}^t (\bar{A}_{q-1}^t (V_1 + V_{p+1}) \bar{A}_{q-1} + V_{p+q}) X_{n-q+1} + X_{n-q}^t \bar{A}_q^t (V_1 + V_{p+1}) \bar{A}_q X_{n-q} \\
&\quad + \text{tr}[(V_1 + V_{p+1}) C] + 1.
\end{aligned}$$

By definition of V_i , we deduce

$$\begin{aligned}
\bar{B}_k^t (V_1 + V_{p+1}) \bar{B}_k + V_{k+1} &= V_k - \frac{C}{p+q}, \quad 1 \leq k \leq p-1 \\
\bar{B}_p^t (V_1 + V_{p+1}) \bar{B}_p &= V_p - \frac{C}{p+q} \\
\bar{A}_k^t (V_1 + V_{p+1}) \bar{A}_k + V_{p+k+1} &= V_{p+k} - \frac{C}{p+q}, \quad 1 \leq k \leq q-1 \\
\bar{A}_q^t (V_1 + V_{p+1}) \bar{A}_q &= V_{p+q} - \frac{C}{p+q}.
\end{aligned}$$

Furthermore, V_k is symmetric positive definite for all $k = 1, \dots, p+q$ which implies that $V_k - \frac{C}{p+q}$ is symmetric positive semi-definite for all $k = 1, \dots, p+q$.

Consider the non-negative constants $(\alpha_k)_{1 \leq k \leq p+q}$ defined by

$$\alpha_k := \max \left\{ x^t \left(V_k - \frac{C}{p+q} \right) x : x \in \mathbb{R}^d, x^t V_k x = 1 \right\}.$$

Since the maximum is calculated over the unit sphere with respect to the induced norm by V_k and since this unit sphere is compact, there exists $x_k \in \mathbb{R}^d$ such that $x_k^t V_k x_k = 1$ and

$$\alpha_k = x_k^t \left(V_k - \frac{C}{p+q} \right) x_k = 1 - x_k^t \frac{C}{p+q} x_k.$$

The matrices V_k , $k = 1, \dots, p+q$, and $\frac{C}{p+q}$ are positive definite which yields $0 \leq \alpha_k < 1$ for all $k = 1, \dots, p+q$.

Setting $\alpha_0 := \max\{\alpha_k : k = 1, \dots, p+q\}$ we obtain $0 \leq \alpha_0 < 1$ and

$$V_k - \frac{C}{p+q} \leq \alpha_0 V_k \forall k \in \{1, \dots, p+q\}.$$

Hence, for all $M \in \mathbb{S}_d^{++}$ and all $k \in \{1, \dots, p+q\}$,

$$\text{tr} \left[\left(V_k - \frac{C}{p+q} \right) M \right] \leq \alpha_0 \text{tr}(V_k M).$$

We deduce $\mathbb{E}[V(Y_n) | Y_{n-1} = y] \leq \alpha_0 V(y) + \text{tr}(\Sigma C) + 1 - \alpha_0$.

If we choose $\alpha := (\alpha_0 + 1)/2 \in [1/2, 1)$ and $b := \text{tr}(\Sigma C) + 1 - \alpha_0 \in (0, \infty)$, then the (FL) - condition is satisfied with the set K given by

$$K := \left\{ x \in W \cap U : V(x) \leq \frac{b}{\alpha - \alpha_0} \right\}.$$

□

4.6. Proof of Theorem 2.4

Now we can prove our stationarity and ergodicity result for standard GARCH(p, q) processes. The main remaining problem is that K is not compact and, hence, it is somewhat tricky to prove that it is small.

Proof of Theorem 2.4. (i) Since due to Proposition 4.5 (ii) the spectral radius of $\sum_{j=1}^p B_j$ is also less than 1, Proposition 4.7 and Theorem 4.8 imply that (A2) and (A3) hold. Using then Proposition 3.10 we deduce that $(Y_n)_{n \in \mathbb{N}^*}$ is ψ -irreducible and aperiodic on the state space $(W \cap U, \mathcal{B}(W \cap U))$.

Define $U_C = (\text{vech}(\{x \in \mathbb{S}_d^{++} : x \geq C\}))^p \times (\mathbb{R}^d)^q$ which is a closed set and a proper subset of U . Then we have by inspecting the iteration that $Y_k \in W \cap U_C$ for all $Y_0 \in W \cap U$ and $k \in \mathbb{N}$, $k \geq p$. By the way, $T \in U_C$ by (4.7). By condition (H3), Theorem 4.9 ensures the existence of a function V which fulfils the (FL) - condition on the set K . Now we show that K is small.

Let $K_1 = K \cap U_C^c$ and $K_2 = K \setminus K_1$. Using the self-duality of the cone of positive semi-definite matrices, it is straightforward to see that V maps unbounded (with respect to norms on $(\mathbb{R}^{d(d+1)/2})^p \times (\mathbb{R}^d)^q$) subsets of U to unbounded subsets of \mathbb{R}^+ and thus K is a bounded subset of U . Inspecting the iteration defining the GARCH processes further we see that Y_p is not only in $W \cap U_C$ when $Y_0 \in K_1$, but necessarily also in a compact set $\tilde{K} \subseteq W \cap U_C$ conditional on $\|\varepsilon_i\| \leq \eta$ for $i = 1, 2, \dots, p$ and a fixed $\eta > 0$. W.l.o.g. one can assume $\tilde{K} \supseteq K_2$. This implies $P^p(x, \tilde{K}) \geq P(\|\varepsilon_1\| \leq \eta)^p =: \zeta > 0$ for all $x \in K_1$ due to (H2).

Moreover, the Markov chain $(Y_n)_{n \in \mathbb{N}^*}$ has the Feller property, as an elementary and standard dominated convergence argument shows, and $\text{supp } \psi$ has non-empty interior (see Proposition 3.10). Thus, [26, Proposition 6.2.8] shows that \tilde{K} is petite (see [25]), i.e. there is a non-degenerate measure ν on $\mathcal{B}(W \cap U)$ and a probability measure a on \mathbb{N}^* such that $\sum_{i=1}^{\infty} a(\{i\}) P^i(x, B) \geq \nu(B)$ for all $x \in \tilde{K}$ and Borel sets $B \subseteq W \cap U$. Using Chapman-Kolmogorov this implies $0.5 \sum_{i=1}^{\infty} a(\{i\}) P^i(x, B) + 0.5 \sum_{i=p+1}^{\infty} a(\{i-p\}) P^i(x, B) \geq 0.5 \zeta \nu(B)$ for all $x \in \tilde{K} \cup K_1$ and Borel sets $B \subseteq W \cap U$. Thus $\tilde{K} \cup K_1$ is petite. Since $K \subseteq \tilde{K} \cup K_1$, also K is petite and thus small by [26, Theorem 5.5.7].

Applying Theorem 3.11 and Theorem 3.13 we obtain the claimed positive Harris recurrence, geometric ergodicity as well as geometric β -mixing and $\pi(V) < \infty$ for the stationary distribution π .

Let $(X_n)_{n \in \mathbb{Z}}$ now be the unique stationary GARCH process. Then $\pi(V) < \infty$ implies

$$\mathbb{E}[X_n^t V_{p+1} X_n] \leq \mathbb{E}[V(Y_n)] = \pi(V) < \infty \forall n \in \mathbb{Z}$$

by definition of V (cf. proof of Theorem 4.9). This shows that $X_n \in L^2$ for all $n \in \mathbb{Z}$.

Since $\mathbb{E}[\Sigma_n] = \mathbb{E}[X_n X_n^t]$, we deduce $\mathbb{E}[\Sigma_n] < \infty$. Using the diagonal dominance property of a positive semi-definite matrix ($|m_{ij}| \leq 0.5(m_{ii} + m_{jj})$ for $M = (m_{ij})_{1 \leq i, j \leq d} \in \mathbb{S}_d^+$), this implies $\Sigma_n \in L^1$ for all $n \in \mathbb{Z}$.

(ii) We now assume that there is a weakly stationary solution for the standard GARCH(p, q) model. Then $\Sigma := \mathbb{E}[X_n X_n^t]$ is well-defined. Since $\Sigma = \mathbb{E}[\Sigma_n]$, taking the expectation in (2.4) on both sides yields

$$\Sigma = C + \sum_{i=1}^q \sum_{k=1}^{l_i} \bar{A}_{i,k} \Sigma \bar{A}_{i,k}^t + \sum_{j=1}^p \sum_{r=1}^{s_j} \bar{B}_{j,r} \Sigma \bar{B}_{j,r}^t.$$

Due to Proposition 4.3 the spectral radius of the matrix $(\sum_{i=1}^q A_i + \sum_{j=1}^p B_j)$ has to be less than 1. \square

Note that the proof shows that π is concentrated on $W \cap U_C$, so in the stationary regime the GARCH covariance matrices are always bigger than or equal to C .

A. Algebraic Geometry

In this appendix we summarise the necessary details of algebraic geometry to understand the statement of our main result. For more details and comprehensive treatments we refer to [4, 28].

We denote by $\mathbb{R}[X_1, \dots, X_n]$ the polynomial ring in n variables formed from the set of polynomials in the variables X_1, \dots, X_n with coefficients in the field \mathbb{R} .

Definition A.1. (i) A subset $V \subseteq \mathbb{R}^n$ is called semi-algebraic if it admits some representation of the form

$$V = \bigcup_{i=1}^s \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^n : P_{i,j}(x) \sim_{ij} 0\},$$

where, for all $i = 1, \dots, s$ and $j = 1, \dots, r_i$,

(a) $\sim_{ij} \in \{>, =, <\}$

(b) $P_{i,j}(X) \in \mathbb{R}[X]$, $X = (X_1, \dots, X_n)$.

(ii) A subset $V \subseteq \mathbb{R}^n$ is called algebraic if it can be represented as

$$V = \{x \in \mathbb{R}^n : P_1(x) = \dots = P_k(x) = 0\}$$

where $k \in \mathbb{N}^*$ and $P_i(X) \in \mathbb{R}[X_1, \dots, X_n]$ for all $i = 1, \dots, k$.

Remark A.2. Real algebraic sets can be represented by one single polynomial, namely, if $V = \{P_1 = \dots = P_k = 0\}$, then we can take $P := P_1^2 + \dots + P_k^2$.

Definition A.3 (“Zariski topology”). The topology over \mathbb{R}^n for which the algebraic sets in \mathbb{R}^n are the closed sets is called the Zariski topology.

Remark A.4. (i) The Zariski topology is not Hausdorff (i.e. it does not separate points).

(ii) Every Zariski closed set in \mathbb{R}^n is also closed in the usual topology on \mathbb{R}^n . Thus, the usual topology is finer than the Zariski topology.

(iii) We define the Zariski closure of a set A by $\overline{A}^Z := \bigcap_{\substack{B \text{ Zariski closed} \\ B \supseteq A}} B$.

Definition A.5. An algebraic set $V \subseteq \mathbb{R}^n$ is said to be irreducible if it cannot be decomposed as $V = V_1 \cup V_2$, where both V_1 and V_2 are algebraic sets and $V_1 \neq V$ and $V_2 \neq V$. If V is an irreducible algebraic set, it is also called algebraic variety.

Definition A.6. Let $V \subseteq \mathbb{R}^n$ be an algebraic variety and define the ideal of V by

$$I(V) := \{P \in \mathbb{R}[X_1, \dots, X_n] : P(x) = 0 \quad \forall x \in V\}.$$

It is an easy consequence of the Hilbert Basis Theorem (cf. for example [23]) that the ideal $I(V)$ has to be finitely generated, i.e. there exist $l \in \mathbb{N}^*$ and $Q_1, \dots, Q_l \in I(V)$ such that $I(V)$ is the ideal generated by these polynomials. We then call

$$\rho(V) := \sup_{x \in V} \text{rank} \left(\frac{\partial Q_i}{\partial x_j}(x) \right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}}$$

the rank of the ideal $I(V)$.

A point $x_0 \in V$ is said to be a regular point of V if $\rho(V) = \text{rank} \left(\frac{\partial Q_i}{\partial x_j}(x_0) \right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}}$. Otherwise x_0 is called a singular point of V . We write $\mathcal{R}(V)$ to denote the set of regular points of V and $\mathcal{S}(V)$ for the set of singular points.

A natural class of maps are those such that preimages of algebraic sets are again algebraic, i.e. maps which are continuous with respect to the Zariski topology.

Definition A.7. Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ be algebraic varieties. Then $f : V \rightarrow W$ is said to be a regular map, if all its components $(f_i)_{1 \leq i \leq m}$ are regular functions, i.e., for all $i = 1, \dots, m$, there exist $P_i, Q_i \in \mathbb{R}[X_1, \dots, X_n]$ such that

$$V \cap \{x \in \mathbb{R}^n : Q_i(x) = 0\} = \emptyset \quad \text{and} \quad f_i(x) = \frac{P_i(x)}{Q_i(x)} \quad \forall x \in V.$$

Proposition A.8. Let $V \subseteq \mathbb{R}^n$, $W \subseteq \mathbb{R}^m$ be algebraic varieties and $f : V \rightarrow W$ a regular map. Then f is continuous with respect to the Zariski topology.

B. Theory of Markov Chains

In this appendix we recall the theorems for Markov chains used in the proof of Theorem 3.12. To this end let $(X_t)_{t \in \mathbb{N}}$ be a Markov chain on the state space $(S, \mathcal{B}(S))$ with transition probability kernel P .

Theorem B.1 (cf. [26], Theorem 14.2.2).

Suppose that the non-negative functions V, f, s satisfy the relationship

$$PV(x) \leq V(x) - f(x) + s(x) \quad \forall x \in S,$$

then, for each $x \in S$ and any stopping time τ , we have

$$\mathbb{E}_x \left[\sum_{k=0}^{\tau-1} f(X_k) \right] \leq V(x) + \mathbb{E}_x \left[\sum_{k=0}^{\tau-1} s(X_k) \right].$$

Proposition B.2 (cf. [26], Proposition 9.1.7 (ii)).

Suppose that $(X_t)_{t \in \mathbb{N}}$ is ψ -irreducible. If there exists some petite set $C \in \mathcal{B}(S)$ such that $L(x, C) = 1$ for all $x \in S$, then $(X_t)_{t \in \mathbb{N}}$ is Harris recurrent.

Theorem B.3 (cf. [26], Theorem 14.0.1).

Suppose that the chain $(X_t)_{t \in \mathbb{N}}$ is ψ -irreducible and aperiodic and let $f \geq 1$ be a function on S . Then the following conditions are equivalent:

(i) The chain is positive recurrent with invariant probability measure π and

$$\pi(f) = \int_S \pi(dx) f(x) < \infty.$$

(ii) There exist some petite set $C \in \mathcal{B}(S)$, a positive constant $b < \infty$ and some extended-valued non-negative function V satisfying $V(x_0) < \infty$ for some $x_0 \in S$ and

$$\Delta V(x) := PV(x) - V(x) \leq -f(x) + b\mathbb{1}_C(x) \forall x \in S.$$

Theorem B.4 (cf. [26], Theorem 15.0.1).

Suppose that the chain $(X_t)_{t \in \mathbb{N}}$ is ψ -irreducible and aperiodic. If there exist a petite set $C \in \mathcal{B}(S)$, constants $b < \infty$, $\beta > 0$ and an extended-valued function $V \geq 1$ finite at some $x_0 \in S$ satisfying

$$\Delta V(x) \leq -\beta V(x) + b\mathbb{1}_C(x) \forall x \in S,$$

then there exist $r > 1$, $R < \infty$ such that for any $x \in \{y \in S : V(y) < \infty\}$

$$\sum_{n=1}^{\infty} r^n \|P^n(x, \cdot) - \pi\|_V \leq RV(x) \tag{B.1}$$

with $\|\mu\|_V := \sup_{g: |g| \leq V} |\mu(g)|$ for any signed measure μ defined on $(S, \mathcal{B}(S))$.

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