Optimal investment with time-varying stochastic endowments

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This paper considers a utility maximization and optimal asset allocation problem in the presence of a stochastic endowment that cannot be fully hedged through trading in the financial market. We rely on the dynamic programming approach to solve the optimization problem. The properties of the value function, particularly the homogeneity, are used to reduce the HJB equation by one dimension. Furthermore, the optimal strategy is derived, and its asymptotic behavior is discussed.

Keywords. Utility maximization; Hamilton-Jacobi-Bellman equation; stochastic endowment; viscosity solution.

In the present paper, we analyze the utility maximization or optimal investment problem of an economic agent under stochastic endowments for a finite time period. We deal with an incomplete market setting because the endowment risk is not perfectly correlated with the traded assets in the market. A specific example to motivate our approach would be an economic agent who receives random salaries during her working life and invests her initial wealth and a fixed proportion of the salary, optimizing the utility of her wealth at the retirement age. Another example could be a pension fund which receives random contributions from the members of the pension fund and invests to generate cash flows. In fact, the original idea of the paper arises from the study of defined contribution (DC) pension plans. This kind of pension schemes has become more popular and are substituting defined benefit pension plans.

The problem of maximizing the expected utility of an economic agent by investment and/or consumption dates back to Merton (1969) and Merton (1971) and is further studied e.g. in Cuoco (1997); Duffie et al. (1997); El Karoui and Jeanblanc-Picqué (1998); Koo (1998), just to quote a few. Such problems have originally been solved by the dynamic programming approach which

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requires the assumption of Markovianity on the state process and leads to a Hamilton-Jacobi-Bellman (HJB) equation. In the literature, this approach is called primal approach. In the 1980s, researchers developed an alternative approach, the so called dual approach where the assumption of Markovian asset prices can be relaxed to solve the optimal investment problem. In a complete market setting the dual method has been studied e.g. in Cox and Huang (1989); Pliska (1986) and in an incomplete market setting e.g. by He and Pearson (1991a) and He and Pearson (1991b).

Our paper considers an optimal asset allocation problem under power utility and solves it via the classical HJB approach, taking account of exogenous stochastic endowments. Focusing on a special class of strictly positive endowments, namely the ones following a (possibly) time-inhomogeneous geometric Brownian motion, we exploit the properties of the value function of the studied control problem, particularly the homogeneity, to reduce its dimension. The reduction in the dimension of the problem will e.g. make finite difference methods easier to apply in order to numerically compute the value function and the corresponding optimal strategy. Furthermore, we do not assume that the value function belongs to the class $C^{1,2}$ and study the viscosity solutions to the HJB equation associated to our control problem. Relying on the theory of viscosity solutions, we are also able to study the asymptotic behavior of the value function and of the optimal strategy when the initial capital invested grows to infinity. The optimal strategy turns out to converge to the famous Merton ratio, i.e. to the optimal asset holding when there is only an initial wealth and no additional endowments over time.

The idea of incorporating a stochastic endowment in the classical utility maximization problem is indeed not new. Both dual and primal approaches (sometimes combined with Backward Stochastic Differential Equation (BSDE) techniques) are adopted to solve this optimization problem. Most of the papers in this field restrict themselves however to the analytically more nicely treatable case of exponential utility functions (among several others Davis (2006) and Hu et al. (2005)). Some authors deal also with the problem of maximizing the expected power utility from the terminal wealth in the presence of exogenous endowments. In Cvitanić et al. (2001), the problem is solved via duality for a broad class of utility functions under the assumption that the random endowment is bounded. The existence and uniqueness of an optimal control are proven, but an explicit representation of the optimal strategy is subordinated to the decomposition of the elements of $(\mathrm{L}^\infty)^*$ into a regular and a singular part, which is hard to characterize explicitly. The authors of Hugonnier and Kramkov (2004) overcome the problem and relax the hypothesis of boundedness, by introducing the number of random endowments as a new control variable. Horst et al. (2014) extends the approach of Hu et al. (2005) to the case of power utility functions, based on the martingale optimality principle combined with BSDE methods. They reduce the problem to the solution of a fully-coupled forward-backward stochastic differential equation, which is still not easy to solve. The most closely related to our paper is Duffie et al. (1997), in which the expected HARA utility from consumption is optimized in an infinite-time horizon and it yields an elliptic HJB equation, not depending on time. We adopt their techniques to our different setting in which the expected utility of the terminal wealth is optimized. We adjust their methods, developed in the elliptic case (infinite-time horizon), to the parabolic case (finite-time horizon) and allow for a slightly broader class of random endowments, allowing for (possibly) time-varying coefficients.

The remainder of the paper is organized as follows. Section 1 describes the model setup and particularly describes the endowment process. In Section 2, we develop some dimension-reduction methods for the value function. In Section 3, we study the asymptotic behavior of the optimal strategy, as the initial investment approaches infinity. Section 4 concludes the paper and provides some perspectives for future research.

1 The model

On a fixed filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, satisfying the usual hypotheses, consider a financial market consisting of a riskless and a risky asset. From now on let T be a fixed finite time point.
(S^0, S^1) will denote respectively the savings account and the risky asset, and we assume that the two assets follow a Black-Scholes model:

$$\begin{aligned} \mathrm{d}S_t^0 &= rS_t^0\,\mathrm{d}t,\\ \mathrm{d}S_t^1 &= \mu S_t^1\,\mathrm{d}t + \sigma S_t^1\,\mathrm{d}W_t^1, \end{aligned}$$

where $\mu,\ r \in \mathbb{R},\ \sigma > 0$ and W^1 is a Brownian motion on the above mentioned filtered space.
Assume that there is another process c with stochastic dynamics driven by another Brownian motion W^C on the same space, correlated with W^1 with a correlation coefficient $\rho \in (-1, 1)$. So, there is a Brownian motion W^2, independent of W^1 such that $W^C = \rho W^1 + \sqrt{1-\rho^2}W^2$.
The process c_t describes a random endowment or income, with the following dynamics:

$$\mathrm{d}c_t = \mu_C(t)c_t\,\mathrm{d}t + \sigma_C(t)c_t\,\mathrm{d}W_t^C, \tag{1.1}$$

where $\mu_C : [0, T] \to \mathbb{R}$, $\sigma_C : [0, T] \to \mathbb{R}_+$ are deterministic càdlàg functions.

Remark 1.1. Time-homogeneous geometric Brownian motions are sometimes chosen in the literature to model the behavior of a stochastic income (see Sundaresan and Zapatero (1997)). Thinking of the example of a DC pension scheme, we can interpret the process c as a diffusion income, or rather a proportion of it possibly changing over time, which is paid continuously into the pension fund.
From an analytical point of view, the fundamental feature of this family is that the value function of the optimization problem we are going to introduce turns out to be homogenous in the spatial variables. This will be the crucial property to achieve a reduction in the dimension of the problem.

We assume that the agent invests at any time t a proportion π_t of the wealth in the stock S^1 and $1-\pi_t$ in the bond S^0 with interest rate r. In addition, the random income is paid continuously to the account at rate c_t.
We will often talk about a "strategy" meaning by this the pair $(1-\pi, \pi)$.

The wealth process corresponding to the strategy π, $\{A_t^\pi\}_{t\in[0,t]}$, has the following dynamics:

$$\mathrm{d}A_t^\pi = \frac{A_t^\pi \pi_t}{S_t^1}\,\mathrm{d}S_t^1 + \frac{A_t^\pi(1-\pi_t)}{S_t^0}\,\mathrm{d}S_t^0 + c_t\,\mathrm{d}t, \tag{1.2}$$

and hence, defining $\theta := \frac{\mu-r}{\sigma}$, for an initial wealth $x \in \mathbb{R}_+$ we get

$$\begin{cases} \mathrm{d}A_t^\pi &= \left[A_t^\pi\left(\pi_t\sigma\theta + r\right) + c_t\right]\,\mathrm{d}t + A_t^\pi\pi_t\sigma\,\mathrm{d}W_t^1, \\ A_0^\pi &= x. \end{cases} \tag{1.3}$$

This definition reflects the fact that the only allowed additional cash injections to the fund are due to the continuous payments at rate c_t.

A progressively measurable process π is said to be *admissible*, if it takes values in a fixed closed convex subset A of $\mathbb{R}$, $\int_0^T |\pi_s|^2\,\mathrm{d}s < \infty$ a.s. and $A_t^\pi \geq 0$ for every $t \in [0,T]$. We denote by $\mathcal{A}$ the set of all admissible strategies.

We assume that A is compact, as it is common in the HJB approach.

Remark 1.2. The square-integrability condition ensures the existence and uniqueness of a solution for Equation (1.3).

Remark 1.3. In the definition of admissibility, one usually has to ensure that the wealth process never becomes negative. Under our assumption, the wealth process stays positive without any extra requirement on the admissible strategies, since the contribution process is always positive. It is worth mentioning that this assumption also reflects the fact that withdrawals do not occur.

2 The stochastic optimization problem: an HJB approach

Recall that the driving Brownian motion W^C represents the uncertainty in the income, which is supposed not to be traded in the market. This makes the market incomplete and we are then facing the problem of maximizing the expected utility of an investment in an incomplete market. More precisely, we are looking for an optimal investment strategy π^* such that

$$\mathrm{E}\left[U(A_T^{\pi^*})\right] = \sup_{\pi\in\mathcal{A}} \mathrm{E}\left[U(A_T^\pi)\right],$$

where $U : \mathbb{R} \to \mathbb{R}_+$ is a CRRA power utility function and $\mathcal{A}$ is the set of admissible strategies, i.e. for a risk aversion parameter $\gamma < 1$, $\gamma \neq 0$,

$$U(x) = \frac{x^\gamma}{\gamma}.$$

The power utility is abundantly used in both theoretical and empirical research because of its nice analytical tractability. Most importantly, the use of the power utility is also well-motivated economically, since the long-run behavior of the economy suggests that the long run risk aversion cannot strongly depend on wealth, see Campbell and Viceira (2002).

The agent wants to maximize the expected utility from the terminal wealth at the final time T, and thus the *value function* of the utility maximization problem is given by

$$v(t,x,y) := \sup_{\pi\in\mathcal{A}} \mathrm{E}\left[U(A_T^{\pi,t,x,y})\right], \quad (t,x,y)\in[0,T]\times[0,+\infty)\times(0,+\infty), \tag{2.1}$$

where we are taking as controlled process the pair $X^\pi = (A^\pi, c)$, and the notation $A^{\pi,t,x,y}$ stands for the first coordinate of the process X^π starting from the point (x,y), respectively the initial wealth and the initial endowment, at time t.
Note that the process c actually does not depend on the control π, nor on the initial wealth. We will therefore write sometimes $c^{t,y}$ for $c^{\pi,t,x,y}$. Applying well-known results in stochastic control, see e.g. Pham (2009) Chapter 3, we can write down the HJB equation for the value function of our control problem:

$$-v_t = \sup_{\pi\in A}\left\{[x(\pi\sigma\theta+r)+y]\,v_x + \mu_C(t)yv_y + \frac{1}{2}(\pi\sigma x)^2 v_{xx} + \frac{1}{2}\sigma_C^2(t)y^2v_{yy} + \rho\sigma\sigma_C(t)y\pi x v_{xy}\right\}, \tag{2.2}$$

$$v(T,x,y) = U(x), \quad \forall (x,y).$$

Here, and in the following, we will denote the partial derivatives by subscripts, and often omit the argument of the function in the equations.

2.1 Some properties of the value function

Proposition 2.1. *The value function $v(t,x,y)$ is increasing, concave, and hence continuous in the interior of the domain, in the second variable.*

Proof. The proof works along standard lines (cf. e.g. Section 3.6.1 in Pham (2009)).
Fix $0 < x_1 < x_2$, $0 < t < T$ and $y > 0$. Define $B_s := A_s^{\pi,t,x_2,y} - A_s^{\pi,t,x_1,y}$ for $s > t$. The process B hence satisfies

$$\mathrm{d}B_s = B_s(\pi_s\theta\sigma + r)\,\mathrm{d}s + \pi_s B_s\sigma\,\mathrm{d}W_s^1.$$

Since $x_2 - x_1 > 0$ is the initial condition at time t, it follows that $B_s \geq 0$ a.s. and hence $A_s^{\pi,t,x_2,y} \geq A_s^{\pi,t,x_1,y}$ for every $s > t$. For the utility function U is increasing, it also holds a.s. that

$$U(A_T^{\pi,t,x_2,y}) \geq U(A_T^{\pi,t,x_1,y}),$$

and $\forall \pi\in\mathcal{A}$

$$\mathrm{E}\left[U(A_T^{\pi,t,x_2,y})\right] \geq \mathrm{E}\left[U(A_T^{\pi,t,x_1,y})\right].$$

This implies that $v(t,x_1,y) < v(t,x_2,y)$, i.e. that the value function is monotonically increasing in the second variable.
To see that it also fulfills the second property claimed, fix again $0 < t < T$ and let $x_1, x_2 > 0$ and $\pi^1, \pi^2 \in \mathcal{A}$, and denote by $A^i := A^{\pi^i,t,x_i,y}$, $i = 1,2$. For $\lambda\in[0,1]$ write $x_\lambda := \lambda x_1 + (1-\lambda)x_2$, and

$A^\lambda := \lambda A^1 + (1-\lambda)A^2$. Now, the strategy

$$\pi_s^\lambda := \frac{\lambda A_s^1 \pi_s^1 + (1-\lambda)A_s^2\pi_s^2}{A_s^\lambda}$$

is in $\mathcal{A}$ for the set A is convex.
Moreover, the dynamics of the process A^λ are given by

$$\begin{aligned} \mathrm{d}A_s^\lambda &= A_s^\lambda(\pi_s^\lambda \theta\sigma + r)\,\mathrm{d}s + c_s^{t,y}\,\mathrm{d}s + A_s^\lambda \pi_s^\lambda \sigma\,\mathrm{d}W_s^1, \\ A_t^\lambda &= x_\lambda. \end{aligned}$$

This shows that A^λ is a wealth process starting from x^λ at t and controlled by π^λ. Using the concavity of the utility function, we get that

$$U(A_T^\lambda) = U(\lambda A_T^1 + (1-\lambda)A_T^2) \geq \lambda U(A_T^1) + (1-\lambda)U(A_T^2),$$

and hence

$$v(t, \lambda x^1 + (1-\lambda)x^2, y) \geq \lambda v(t, x^1, y) + (1-\lambda)v(t, x^2, y).$$

□

Lemma 2.2. *The value function* (2.1) *is homogeneous in* (x, y) *with degree* γ*, and therefore there exists a function* $u : [0,T] \times [0, +\infty) \to \mathbb{R}$ *such that* v *can be represented in a separable form as*

$$v(t, x, y) = y^\gamma u\left(t, \frac{x}{y}\right), \quad \forall y > 0. \tag{2.3}$$

Proof. First recall that, under the assumptions of our model, for every fixed strategy π the wealth process $A := A^\pi$ has dynamics given in (1.3) by

$$\begin{cases} \mathrm{d}A_t &= [A_t(\pi_t\sigma\theta + r) + c_t]\,\mathrm{d}t + A_t\pi_t\sigma\,\mathrm{d}W_t^1, \\ A_0 &= x. \end{cases}$$

and observe that the explicit solution to this equation is given as (see, e.g., Protter (2004), Ch. V, Theorem 52):

$$A_s^{t,x,y} = \left(x + \int_t^s \frac{c_u^{t,y}}{Z_u}\,\mathrm{d}u\right) Z_s,$$

where Z is a stochastic exponential factor given as:

$$Z_s := \exp\left\{\int_t^s \left[(\pi_u\sigma\theta + r) - \frac{1}{2}(\pi_u\sigma)^2\right]\mathrm{d}u + \int_t^s \sigma\pi_u\,\mathrm{d}W_u^1\right\}.$$

It holds

$$c_t = c_0\mathcal{E}(P)_t, \quad P_t = \int_0^t \mu_C(s)\,\mathrm{d}s + \int_0^t \sigma_C(s)\,\mathrm{d}W_s^C.$$

The formula above implies the linearity property of the endowment process with respect to the

initial data, and the process c has hence the property that $c_s^{t,ky} = kc_s^{t,y}$.
Now, it is straightforward to check homogeneity. Indeed, for every $k > 0$ we have:

$$v(t, kx, ky) = \sup_{\pi \in \mathcal{A}} \mathrm{E}\left[U(A_T^{\pi,t,kx,ky})\right] = \sup_{\pi \in \mathcal{A}} \mathrm{E}\left[\frac{(kA_T^{\pi,t,x,y})^\gamma}{\gamma}\right] = k^\gamma v(t, x, y).$$

This implies that we can define a function $u : [0, T] \times [0, +\infty) \to [0, +\infty)$ by

$$u(t, z) := v(t, z, 1),$$

and then, for every $y > 0$, we will have that

$$v(t, x, y) = y^\gamma u\left(t, \frac{x}{y}\right).$$

□

2.2 A reduced equation for the control problem

We can now use the homogeneity property to reduce the problem by one dimension. This technique is used already in Duffie et al. (1997) to deal with the problem of optimization of consumption in presence of a random income modelled as a GBM. All results in the following are basically obtained by extending the methods in Duffie et al. (1997) to the parabolic case. In that work, the problem of maximizing the utility from consumption is considered over an infinite time horizon, which yields an elliptic HJB equation, not depending on time. We consider here the problem of maximizing the utility of the terminal wealth in a finite time horizon (time-dependent HJB equation), and allow for a slightly more general class of random endowments, since we allow for time-dependent coefficients in the GBM.

The assumption that the value function belongs to the class $\mathcal{C}^{1,2}$ is too demanding in the general case. We hence look for a characterization of the value function which uses the well-known property of the value function being a viscosity solution of the HJB equation associated to the control problem (see, e.g., Chapter 4 of Pham (2009)). We refer to Crandall et al. (1992) for the definition of (and all important results on) viscosity solutions of second order non-linear PDEs. Let us recall the following fundamental classical result.

Theorem 2.3. *The value function v is a viscosity solution of the Hamilton-Jacobi-Bellman equation* (2.2) *associated to the optimization problem.*

Proof. See, e.g., Theorem 4.3.1 in Pham (2009), together with Remark 4.3.4. Note that the value function of our control problem is locally bounded on $[0, T) \times (0, +\infty) \times (0, +\infty)$, due to Proposition 2.1. □

Theorem 2.4. *1) For a fixed initial random endowment $\bar{y} > 0$, define $\bar{u}(t, z) := v(t, z, \bar{y})$. Then*

$\bar{u} : (0,T] \times \mathbb{R}^+ \to \mathbb{R}$ is a viscosity solution of the reduced PDE with parameter $\bar{y}$:

$$\begin{aligned}
&\bar{u}_t + \bar{y}\bar{u}_z + \mu_C(t)\left[\gamma\bar{u} - z\bar{u}_z\right] + \frac{1}{2}\sigma_C^2(t)\left[\gamma(\gamma-1)\bar{u} - 2(\gamma-1)z\bar{u}_z + z^2\bar{u}_{zz}\right] + \\
&\sup_{\pi\in A}\left\{(\pi\sigma\theta + r)z\bar{u}_z + \frac{1}{2}(\pi\sigma)^2 z^2\bar{u}_{zz} + \rho\sigma_C(t)\sigma\pi(\gamma-1)z\bar{u}_z - \rho\sigma\sigma_C(t)\pi z^2\bar{u}_{zz}\right\} = 0, \qquad (2.4)\\
&\bar{u}(T,z) = \frac{z^\gamma}{\gamma}, \quad \forall z \geq 0.
\end{aligned}$$

In particular, for $\bar{y} = 1$

$$\begin{aligned}
&u_t + u_z + \mu_C(t)\left[\gamma u - zu_z\right] + \frac{1}{2}\sigma_C^2(t)\left[\gamma(\gamma-1)u - 2(\gamma-1)zu_z + z^2u_{zz}\right] + \qquad (2.5)\\
&\sup_{\pi\in A}\left\{(\pi\sigma\theta + r)zu_z + \frac{1}{2}(\pi\sigma)^2 z^2u_{zz} + \rho\sigma_C(t)\sigma\pi(\gamma-1)zu_z - \rho\sigma\sigma_C(t)\pi z^2u_{zz}\right\} = 0,
\end{aligned}$$

2) The value function of the control problem (2.1) is given by

$$v(t,x,y) = y^\gamma u\left(t, \frac{x}{y}\right), \quad \forall (t,x,y) \in (0,+\infty)\times(0,+\infty)\times[0,T],$$

where $u : (0,T] \times \mathbb{R}^+ \to \mathbb{R}$ is the unique viscosity solution to Equation (2.4) for $\bar{y} = 1$ with polynomial growth at infinity.

Proof. Define the following operators:
for $t \in [0,T]$, $x \in (0,+\infty)$, $y \in (0,+\infty)$, $s \in \mathbb{R}$, $q \in \mathbb{R}$, $p \in \mathbb{R}^2$, $M \in \mathcal{S}^2$ ($\mathcal{S}^2$ being the space of the 2−dimensional symmetric matrices)

$$\begin{aligned}
F(t,x,y,s,q,p,M) = &-q + \sup_{\pi\in A}\Big\{-yp_1 - \mu_C(t)yp_2 - (\pi\sigma\theta + r)xp_1\\
&-\frac{1}{2}\sigma_C(t)^2y^2M_{22} - \frac{1}{2}(\pi\sigma x)^2M_{11} - \rho\sigma_C(t)\sigma\pi xyM_{12}\Big\};
\end{aligned}$$

for each $\bar{y} \in (0,+\infty)$ fixed, and $t \in [0,T]$, $z \in (0,+\infty)$, $s \in \mathbb{R}$, $q \in \mathbb{R}$, $p \in \mathbb{R}$, $M \in \mathbb{R}$

$$\begin{aligned}
F^{(\bar{y})}(t,z,s,q,p,M) = &-q - \bar{y}p - \mu_C(t)\left[\gamma s - zp\right] + \frac{1}{2}\sigma_C^2(t)\left[\gamma(\gamma-1)s - 2(\gamma-1)zp + z^2M\right] +\\
&\sup_{\pi\in A}\left\{(\pi\sigma\theta + r)zp + \frac{1}{2}(\pi\sigma)^2z^2M + \rho\sigma_C(t)\sigma\pi(\gamma-1)zp - \rho\sigma\sigma_C(t)\pi z^2M\right\}.
\end{aligned}$$

1) To prove 1) we have to show that the function $\bar{u}$ is both a super- and a subsolution. We just show the subsolution property, since the arguments for the supersolution property are completely analogous.
Recall that, thanks to Theorem 2.3, v is known to be a viscosity solution of Equation (2.2). In particular, v is a viscosity subsolution, which implies that for each fixed point $(t_0, z_0, \bar{y})$ and for each $\varphi \in \mathcal{C}^{1,2}([0,T]\times[0,+\infty)\times(0,+\infty))$ such that $\varphi \geq v$ on $[0,T]\times[0,+\infty)\times(0,+\infty)$ and

$\varphi(t_0, z_0, \bar{y}) = v(t_0, z_0, \bar{y})$, it holds

$$F\left(t_0, z_0, \bar{y}, v(t_0, z_0, \bar{y}), \varphi_t(t_0, z_0, \bar{y}), D\varphi(t_0, z_0, \bar{y}), D^2\varphi(t_0, z_0, \bar{y})\right) \le 0. \tag{2.6}$$

We want to show that the function $\bar{u}$ is also a viscosity subsolution of the reduced form $F^{(\bar{y})}$, i.e. that for every $(t_0, y_0) \in [0,T] \times (0,+\infty)$ and $\psi \in \mathcal{C}^{1,2}([0,T] \times (0,+\infty))$ such that $\psi(t_0, z_0) = \bar{u}(t_0, z_0)$ and $\psi \le \bar{u}$,

$$F^{(\bar{y})}(t_0, z_0, \bar{u}(t_0, z_0), \psi_t(t_0, z_0), D\psi(t_0, z_0), D^2\psi(t_0, z_0)) \le 0.$$

Let hence $\psi \in \mathcal{C}^{1,2}([0,T] \times (0,+\infty))$ be such that $\psi(t_0, z_0) = \bar{u}(t_0, z_0)$ and $\psi \le \bar{u}$ on $[0,T] \times (0,+\infty)$. We will denote by (t,z) the variables of the function ψ and by ψ_t and ψ_z its partial derivatives with respect to the first and (resp.) the second variable.
Define

$$\varphi(t,x,y) := \left(\frac{y}{\bar{y}}\right)^{\gamma} \psi\left(t, \bar{y}\frac{x}{y}\right).$$

Then at the point $(t_0, z_0, \bar{y})$

$$\varphi(t_0, z_0, \bar{y}) = \left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma} \psi\left(t_0, z_0\right) = \bar{u}(t_0, z_0) = v(t_0, z_0, \bar{y})$$

and for every (t,x,y)

$$\varphi(t,x,y) := \left(\frac{y}{\bar{y}}\right)^{\gamma} \psi\left(t, \bar{y}\frac{x}{y}\right) \ge \left(\frac{y}{\bar{y}}\right)^{\gamma} \bar{u}\left(t, \bar{y}\frac{x}{y}\right) = v(t,x,y),$$

where the last equality follows from the homogeneity property of the function v and the definition of the function $\bar{u}$. Moreover, the function φ is of class $\mathcal{C}^{1,2}([0,T] \times (0,+\infty) \times (0,+\infty))$ and the viscosity subsolution property of the value function yields (2.6). We can write the partial derivatives of ψ in terms of the partial derivatives of φ, getting:

$$\begin{aligned}
&\varphi_t = \left(\tfrac{y}{\bar{y}}\right)^{\gamma} \psi_t, && \varphi_{xy} = (\gamma-1)\left(\tfrac{y}{\bar{y}}\right)^{\gamma-2} \tfrac{1}{\bar{y}}\psi_z - \left(\tfrac{y}{\bar{y}}\right)^{\gamma-2} \tfrac{x}{y}\psi_{zz},\\
&\varphi_x = \left(\tfrac{y}{\bar{y}}\right)^{\gamma-1} \psi_z, && \varphi_y = \gamma\left(\tfrac{y}{\bar{y}}\right)^{\gamma-1} \tfrac{1}{\bar{y}}\psi - \left(\tfrac{y}{\bar{y}}\right)^{\gamma-1} \tfrac{x}{y}\psi_z,\\
&\varphi_{xx} = \left(\tfrac{y}{\bar{y}}\right)^{\gamma-2} \psi_{zz}, && \varphi_{yy} = \gamma(\gamma-1)\left(\tfrac{y}{\bar{y}}\right)^{\gamma-2} \tfrac{1}{\bar{y}^2}\psi - 2(\gamma-1)\left(\tfrac{y}{\bar{y}}\right)^{\gamma-2} \tfrac{1}{\bar{y}}\tfrac{x}{y}\psi_z + \left(\tfrac{y}{\bar{y}}\right)^{\gamma-2}\left(\tfrac{x}{y}\right)^2 \psi_{zz}.
\end{aligned}$$

The form above hence becomes, at the point $(t_0, z_0, \bar{y})$,

$$
\begin{aligned}
0 \leq & - F\left(t_0, z_0, \bar{y}, v(t_0, z_0, \bar{y}), \varphi_t(t_0, z_0, \bar{y}), D\varphi(t_0, z_0, \bar{y}), D^2\varphi(t_0, z_0, \bar{y})\right) = \\
& \left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma} \psi_t + \left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma} \bar{y}\psi_z + \mu_C(t_0)\left[\gamma\left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma}\psi - \left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma} z_0\psi_z\right] + \\
& \frac{1}{2}\sigma_C^2(t_0)\left[\gamma(\gamma-1)\left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma}\psi - 2(\gamma-1)\left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma} z_0\psi_z + \left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma} z_0^2\psi_{zz}\right] + \\
& \sup_{\pi\in A}\left\{(\pi\sigma\theta + r)\left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma} z_0\psi_z + \frac{1}{2}(\pi\sigma)^2 z_0^2\left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma}\psi_{zz} + \right. \\
& \left. \rho\sigma_C(t_0)\sigma\pi(\gamma-1)\left(\frac{\bar{y}}{\bar{y}}\right)^{\gamma} z_0\psi_z - \rho\sigma\sigma_C(t_0)\pi z_0^2\psi_{zz}\right\} \\
= & - F^{(\bar{y})}(t_0, z_0, \bar{u}(t_0, z_0), \psi_t(t_0, z_0), D\psi(t_0, z_0), D^2\psi(t_0, z_0)).
\end{aligned}
$$

This implies that, at the point (t_0, z_0)

$$
F^{(\bar{y})}(t_0, z_0, \bar{u}(t_0, z_0), \psi_t(t_0, z_0), D\psi(t_0, z_0), D^2\psi(t_0, z_0)) \leq 0,
$$

which gives the stated viscosity subsolution property.

2) We will first show that the function u has polynomial growth in z at infinity.
As the behavior of the power utility function at infinity is rather different for $0 < \gamma < 1$ and for $\gamma < 0$, we will consider the two cases separately.
Let hence $\gamma > 0$. Using standard moment estimation arguments (see e.g. Lemma 4.5.3 in Kunita (1997)) and the compactness of the constraint set A, we can show that $u(t, z) := v(t, z, 1)$ has polynomial growth. From the definition of the value function v we have:

$$
v(t, z, 1) := \sup_{\pi\in\mathcal{A}} \mathrm{E}\left[\frac{(A_T^{\pi})^{\gamma}}{\gamma}\Big| A_t^{\pi} = z, c_t = 1\right],
$$

and hence the above mentioned Lemma yields

$$
\begin{aligned}
|u(t, z)| &= \left|\sup_{\pi\in\mathcal{A}} \mathrm{E}\left[\frac{(A_T^{\pi})^{\gamma}}{\gamma}\Big| A_t^{\pi} = z, c_t = 1\right]\right| \\
&\leq \sup_{\pi\in\mathcal{A}}\left|\mathrm{E}\left[\frac{(A_T^{\pi})^{\gamma}}{\gamma}\Big| A_t^{\pi} = z, c_t = 1\right]\right| \\
&\leq \sup_{\pi\in\mathcal{A}}\left\{C(\gamma, \pi)\left(1 + |z|^{\frac{\gamma}{2}}\right)^2\right\} \leq K(\gamma, A)\left(1 + |z|^{\frac{\gamma}{2}}\right)^2,
\end{aligned}
$$

where C and K denote some constants independent of z. In the last inequality we use the compactness of the constraint set A together with the fact that the coefficients of the SDE describing the process A^{π} depend linearly on the strategy π.
The case $\gamma < 0$ is actually straightforward. Indeed, in this case the function $u(t, z)$ is bounded from above by 0 and increasing in z. We can therefore bound $|u(t, z)|$ uniformly on each set of the form $(0, T] \times (\bar{z}, +\infty)$, for every $\bar{z} > 0$ fixed.

This shows that the function u has polynomial growth at infinity which, combined with the strong comparison principle for viscosity solutions (Theorem 4.4.5 and Remark 4.3.5 in Pham (2009)), yields uniqueness. Notice that Theorem 4.4.5 requires the coefficient of the function u in the HJB equation to be constant. The arguments in the proof stay nevertheless true in the case where its coefficient is time dependent, and the results still apply in this case (see Appendix A).

The statement follows hence from the results above.

□

Remark 2.5. The reason we are allowed to just ignore the boundary conditions at zero is due to the structure of the equation. Indeed, one can apply the arguments in Oleinik and Radkevich (1971) to see that the behavior of the function at the part of the boundary characterized by $\{z = 0\}$ can be ignored, being the corresponding Fichera function strictly positive in that region.

If we denote the coefficients of u_{zz}, and u_z respectively by

$$a(z,t,\pi) := \frac{1}{2}\sigma_C^2(t)z^2 + \frac{1}{2}(\pi\sigma)^2 z^2 - \rho\sigma\sigma_C(t)\pi z^2; \tag{2.7}$$

$$b(z,t,\pi) := 1 - \mu_C(t)z + \sigma_C^2(t)(1-\gamma)z + (\pi\sigma\theta + r)z + \rho\sigma_C(t)\sigma\pi(\gamma - 1)z, \tag{2.8}$$

$$c(z,t,\pi) := c(t) = -\gamma\left(\mu_C(t) - \frac{1}{2}\sigma_C^2(t)(1-\gamma)\right), \tag{2.9}$$

and denote by $\mathcal{L}^\pi$ the operator

$$\mathcal{L}^\pi u := a(z,t,\pi)u_{zz} + b(z,t,\pi)u_z - c(z,t,\pi)u,$$

our equation reads

$$-u_t = \sup_{\pi\in\mathcal{A}} \mathcal{L}^\pi u.$$

Now, if we take the limit of the Fichera function, we obtain for every $t \in [0,T]$ and every π

$$\lim_{z\to 0} b(z,t,\pi) - a_z(z,t,\pi) = 1 > 0,$$

which implies that no boundary condition is required at $\{z = 0\}$.

2.3 The optimal strategy: a verification argument for the regular case

We have already shown how to reduce the HJB equation by one dimension, in order to simplify the optimization problem. Due to this, we can then solve (possibly numerically) the reduced problem and get the solution of the original one. But what about the optimal strategy? Can we derive an optimal strategy for the original problem given an optimal strategy for the reduced one? This will be done by means of a verification theorem. Verification results are available in this setting (see e.g. Pham (2009) Theorem 3.5.2). The arguments used to prove optimality mostly rely on the regularity of the value function.

Proposition 2.6. *Assume that $u \in \mathcal{C}^{1,2}$ is a classical solution to Equation* (2.5)*.*
Let $\pi^* := h(t, x, y)$, where

$$h(t,x,y) = \underset{\pi \in A}{\operatorname{argmax}} \left\{ (\pi\sigma\theta + r)\frac{x}{y}u_z(t, x/y) + \frac{1}{2}(\pi\sigma)^2 \left(\frac{x}{y}\right)^2 u_{zz}(t, x/y) - \right.$$

$$\left. \rho\sigma_C(t)\sigma\pi(1-\gamma)\left(\frac{x}{y}\right) u_z(t, x/y) - \rho\sigma_C(t)\sigma\pi \left(\frac{x}{y}\right)^2 u_{zz}(t, x/y) \right\}.$$

Then the optimal strategy is Markovian and given by

$$\Pi_t = h\left(t, A_t^{\Pi}, c_t\right). \tag{2.10}$$

Remark 2.7. One can actually compute the supremum analytically. It is given by

$$\pi^* = -\frac{\theta - \rho\sigma_C(t)(1-\gamma)}{\sigma}\frac{yu_z}{xu_{zz}} + \frac{\rho\sigma_C(t)}{\sigma}, \quad \text{if it belongs to } A$$

and is attained at some point on the boundary of A otherwise, since the function

$$A \ni \pi \mapsto (\pi\sigma\theta + r)zu_z + \frac{1}{2}(\pi\sigma)^2 z^2 u_{zz} - \rho\sigma_C(t)\sigma\pi(1-\gamma)zu_z - \rho\sigma_C(t)\sigma\pi z^2 u_{zz}$$

is strictly concave on the closed convex set A.

Proof. The proof works along standard lines (cf. e.g. Section 3.5 in Pham (2009)), and strongly relies on the regularity of the value function.
Fix $t \in [0, T]$ and let hence $\{A_t\}_{s\in[t,T]}$ be the solution to

$$\begin{cases} \mathrm{d}A_s &= \left[A_s\left(\Pi_s\sigma\theta + r\right) + c_s\right]\mathrm{d}s + A_s\Pi_s\sigma\,\mathrm{d}W_s^1, \quad \forall s \in (t, T] \\ A_t &= x, \\ \mathrm{d}c_s &= c_s\left(\mu_C(s)\,\mathrm{d}s + \sigma_C(s)\,\mathrm{d}W_s^C\right), \quad \forall s \in (t, T] \\ c_t &= y, \end{cases}$$

with Π as in Equation 2.10.
First observe that, since $Z_s := \frac{A_s}{c_s}$ is a well defined semimartingale, and $u(s, \cdot)$ is a concave function, the process $U_s := u(s, Z_s)$ is also a well-defined semimartingale. This is a consequence of an application of the Itô formula, which yields:

- for the process Z:

$$\begin{aligned}\mathrm{d}Z_s =& Z_s\left\{-\mu_C(s) + \sigma_C^2(s) - \Pi_s\rho\sigma\sigma_C(s) + \Pi_s\theta\sigma + r\right\}\mathrm{d}s + \mathrm{d}s + \\ & Z_s\left\{\Pi_s\sigma\,\mathrm{d}W_s^1 - \sigma_C(s)\,\mathrm{d}W_s^C\right\}.\end{aligned}$$

- for the process U_s, under the assumption that $u \in \mathcal{C}^{1,2}$,

$$
\begin{aligned}
\mathrm{d}U_s =& \Big\{ u_s(s,Z_s) + \frac{1}{2} u_{zz}(s,Z_s) Z_s^2 (\Pi_s \sigma - \rho\sigma_C(s))^2 + \frac{1}{2} u_{zz}(s,Z_s) Z_s^2 (1-\rho^2)\sigma_C^2(s) + u_z(s,Z_s) \\
&+ u_z(s,Z_s) Z_s \left(-\mu_C(s) + \sigma_C^2(s) - \Pi_s \rho\sigma\sigma_C(s) + \Pi_s \theta\sigma + r \right) \Big\} \mathrm{d}s + \\
& u_z(s,Z_s) Z_s \left\{ \Pi_s \sigma \,\mathrm{d}W_s^1 - \sigma_C(s) \,\mathrm{d}W_s^C \right\}.
\end{aligned}
$$

- finally, for the process $V_s := c_s^\gamma U_s$ we get again by the partial integration formula

$$
\begin{aligned}
\mathrm{d}V_s =& c_s^\gamma \Big\{ u_s(s,Z_s) + \frac{1}{2} u_{zz}(s,Z_s) Z_s^2 (\Pi_s \sigma)^2 + \frac{1}{2} u_{zz}(s,Z_s) Z_s^2 \sigma_C^2(s) - u_{zz}(s,Z_s) Z_s^2 \Pi_s \rho\sigma\sigma_C(s) \\
&+ u_z(s,Z_s) + u_z(s,Z_s) Z_s \left(-\mu_C(s) + (1-\gamma)\sigma_C^2(s) - (1-\gamma)\Pi_s \rho\sigma\sigma_C(s) + \Pi_s \theta\sigma + r \right) \\
&+ \gamma u(s,Z_s) \left(\mu_C(s) + \frac{\gamma-1}{2} \sigma_C^2(s) \right) \Big\} \mathrm{d}s \\
&+ (\text{local martingale terms}).
\end{aligned}
$$

It is then immediate to see that, after a localization argument to make the local martingale term vanish, and due to the fact that u solves Equation (2.5) for every value of c_s,

$$
v(t,x,y) = \mathrm{E}\left[U(A_T^{\Pi^*}) \right],
$$

which yields the optimality of Π, as conjectured.

□

2.4 The optimal strategy: general case

The verification argument used in the regular case does not apply in general, since the value function can be even non-differentiable.

One important advantage of the viscosity solution property of the value function is that it can be used as a verification criterion. In order to verify that a given function is indeed the value function of the studied control problem, it suffices to verify that the function is a viscosity solution of the corresponding HJB equation. For this method to work, it is necessary that we know that the value function is the *unique* viscosity solution, which is already the case in our setting, due to Theorem 2.4. To do this, we need a guess of the solution, as it was the case in the previous paragraph.

Let now u be the (unique) viscosity solution to Equation (2.5). We cannot give meaning to the expression in Equation (2.7) anymore, since the second order derivatives need not exist in general. We therefore appeal to the concavity of the function u in the variable z, proven in Proposition 2.1. Thanks to a classical result by Alexandrov, this property guarantees (see Evans and Gariepy (1991), pp. 239-245) the existence a.e. of a second derivative.

Let hence $\mathcal{N}$ be the set where u_{zz} is not defined.

Consider the set $\mathcal{M}$ defined by

$$\mathcal{M} := \left\{(t,x,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ : \ \frac{x}{y} \in \mathcal{N}\right\}.$$

$\mathcal{M}$ is a null set in $\mathbb{R}^3$. We can exploit the product structure of the Lebesgue measure to see that it is sufficient to show that the set

$$\widetilde{\mathcal{M}} := \left\{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ : \ \frac{x}{y} \in \mathcal{N}\right\}$$

is a null set in $\mathbb{R}^2$.
By disintegration we can indeed see that, if λ^2 denotes the Lebesgue measure in $\mathbb{R}^2$ and λ the Lebesgue measure in $\mathbb{R}$,

$$\lambda^2(\widetilde{\mathcal{M}}) = \int_0^\infty \int_0^\infty \mathbb{1}_{\{\mathcal{S}_y\}}(x)\,\mathrm{d}y\,\mathrm{d}x = \int_0^\infty \int_0^\infty \mathbb{1}_{\{\mathcal{S}_y\}}(x)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

where $\mathcal{S}_y := \{x > 0 : \ x \in y\mathcal{N}\}$ has Lebesgue measure zero (due to invariance by linear transformations), and we can apply Tonelli's theorem as the argument is non-negative. Here, for a set A, $\mathbb{1}_{\{A\}}$ stems for the indicator function of the set A, taking the value 1 if the argument belongs to the set, and 0 otherwise.

Define a function

$$h(t,x,y) := \begin{cases} -\frac{\theta-\rho\sigma_C(t)(1-\gamma)}{\sigma}\frac{yu_z\left(t,\frac{x}{y}\right)}{xu_{zz}\left(t,\frac{x}{y}\right)} + \frac{\rho\sigma_C(t)}{\sigma} & \text{if it belongs to } A \text{ and } (t,x,y) \notin \mathcal{M} \\ \text{some fixed point at the boundary of A,} & \text{otherwise.} \end{cases} \tag{2.11}$$

The strategy Π_t defined as before

$$\Pi_t = h\left(t, A_t^\Pi, c_t\right)$$

is therefore again admissible, and will be the intuitive candidate for the optimal strategy.

To prove optimality, one should use the fact that the value function is the unique viscosity solution to the HJB equation. More precisely, one has to show that the function

$$\bar{v}(t,x,y) := \mathrm{E}\left[A_T^{\Pi,t,x,y}\right]$$

is also a viscosity solution to the HJB equation for v with boundary condition $\bar{v}(T,x,y) = U(x)$. The argument above works, for example, as a verification for the strategies approximated numerically according to (2.11). The reason why this argument applies is that, for a concave function, it is sufficient to fulfill the equation a.e. pointwise. It is a consequence of the maximum principle which for semiconvex/semiconcave functions goes by the name of Jensen's lemma (see e.g. Crandall et al. (2000)).

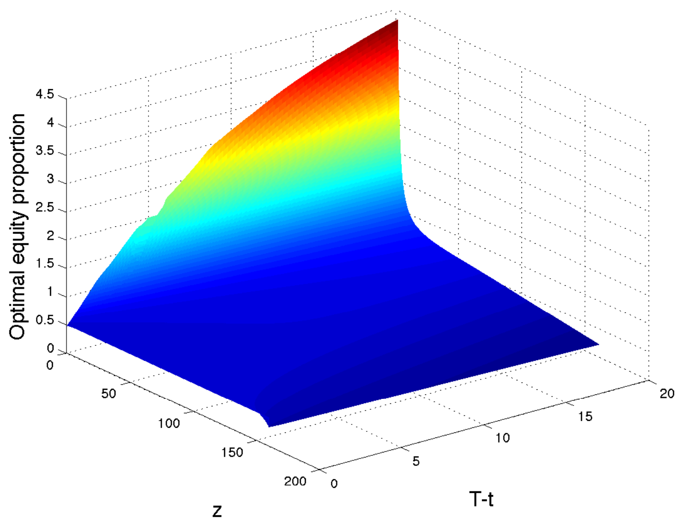
2.5 A numerical example

We want now to apply some finite difference methods to solve Equation (2.5).
Finite difference methods are well-known to work for finding viscosity solutions of non-linear PDEs, and the literature on the topic is rich. We will apply the methods developed in Wang and Forsyth (2008), and refer to this paper for a more detailed description of the algorithms and a survey of the literature on the topic. If we specify the model for $T = 20$ years, by taking the following parameters

$\sigma = 0.2$	$\mu = 0.04$	$\sigma_C = 0.1$	$\mu_C = 0.02$
$r = 0$		$\rho = 0.5$	$\gamma = -1$

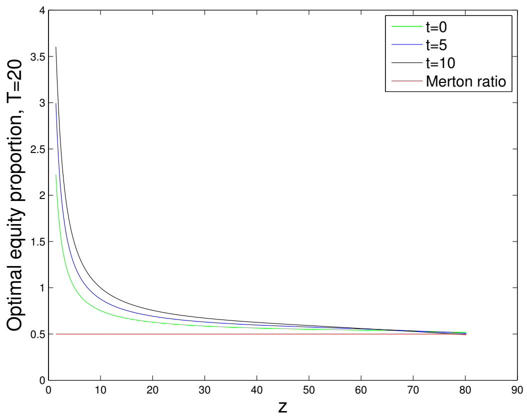
we obtain that the optimal proportion to be invested in the stock is given as in Figure 1. The choice of the parameters is purely academic, but it is roughly based on the empirical estimations in Topel and Ward (1992), since we have in mind the optimization problem of a defined contribution pension scheme, where the random endowment will be (a portion of) the related salary process.

Figure 1: Optimal equity proportion as a function of the ratio wealth to income $z = x/y$ and of the time to maturity $T - t$.



The numerical results, and in particular Figure 2, show that the optimal strategy is bounded from below by the Merton ratio, and converges towards it as the initial wealth significantly exceeds the initial endowment, or as $T - t \to 0$. This is actually perfectly intuitive, since the amount available to the investor is always higher compared to the case without endowments, and a riskier behavior is allowed due to the future endowments. Moreover, as the endowments become negligible with respect to the wealth, the problem resembles the classical one considered by Merton.
These phenomena can be actually rigorously shown, as we will do in the following section.

Figure 2: Optimal equity proportion at different time points.



3 Asymptotics

In this section, we examine the asymptotic behavior of the value function and the optimal policy as the initial capital converges to infinity.

Proposition 3.1. *Let $\varphi : [0, T] \to \mathbb{R}_+$, be the non-negative solution of the following linear ODE*

$$\begin{cases} \varphi_t &= \varphi(r - \mu_C(t) + \theta\rho\sigma_C(t)) - 1 \\ \varphi(T) &= 0. \end{cases} \tag{3.1}$$

It holds

$$\exp\{K(T-t)\}U(x) \leq v(t, x, y) \leq \exp\{K(T-t)\}U(x + \varphi(t)y), \tag{3.2}$$

where K is constant and given by

$$K := \frac{\theta^2}{2}\frac{\gamma}{1-\gamma} + r\gamma.$$

Proof. To prove the two inequalities in (3.2) we will use the comparison results for viscosity solutions (see e.g. Pham (2009), Section 4.4). It will be therefore sufficient to prove that

(i) $f(t, x, y) = f(t, x) := \exp\{K(T-t)\}U(x)$ is a subsolution to (2.2)

(ii) $g(t, x, y) := \exp\{K(T-t)\}U(x + \phi(t, y))$ is a supersolution to (2.2), for $\phi(t, y) = \varphi(t)y$.

(i) First observe that at the terminal time $t = T$, $f(T, x, y) = U(x)$ and the terminal condition of (2.2) is satisfied.
Computing the derivatives with respect to t and x we obtain

$$f_t(t, x) = -K\exp\{K(T-t)\}U(x),$$
$$f_x(t, x) = \exp\{K(T-t)\}U'(x),$$
$$f_{xx}(t, x) = \exp\{K(T-t)\}U''(x).$$

Substituting in (2.2) and computing the maximum in π we obtain

$$\begin{aligned}
&-\frac{K}{\gamma}\exp\{K(T-t)\}x^{\gamma}-\frac{1}{2}\frac{\theta^2\exp\{2K(T-t)\}x^{2\gamma-2}}{(\gamma-1)\exp\{K(T-t)\}x^{\gamma-2}}+y\exp\{K(T-t)\}x^{\gamma-1}+\\
&rx\exp\{K(T-t)\}x^{\gamma-1}\\
=&\exp\{K(T-t)\}\left[-\frac{x^{\gamma}}{\gamma}\left(K-\frac{1}{2}\frac{\theta^2\gamma}{1-\gamma}-r\gamma\right)+yx^{\gamma-1}\right]\\
=&x^{\gamma-1}y\exp\{K(T-t)\}\geq 0.
\end{aligned}$$

(ii) Again notice that also the function g satisfies the terminal condition in (2.2), at least with the inequality. Indeed, at the terminal time $t=T$, we have $g(T,x,y)=U(x+\phi(T,y))=U(x)$, since the function ϕ satisfies $\phi(T,y)=0$.
Denote by $\zeta=\zeta(t,x,y):=\frac{x}{\phi(t,y)}$ and $\xi(t):=\exp\{K(T-t)\}$.
To check the supersolution property, after computing again the partial derivatives of g, substituting in (2.2), and computing the supremum, we therefore obtain in terms of the new variable

$$\begin{aligned}
&K\xi(t)(\phi(t,y))^{\gamma}(1+\zeta)^{\gamma}\frac{1}{\gamma}-\xi(t)(\phi(t,y))^{\gamma-1}(1+\zeta)^{\gamma-1}\phi_t(t,y)\\
&+\frac{1}{2}\frac{(\xi(t))^2(1+\zeta)^{2\gamma-4}(\phi(t,y))^{2\gamma-4}}{\xi(t)(\gamma-1)(\phi(t,y))^{\gamma-2}(1+\zeta)^{\gamma-2}}\Big\{(\phi(t,y))^2(\zeta+1)^2\theta^2+\\
&\rho^2\sigma_C^2(t)y^2(\gamma-1)^2(\phi_y(t,y))^2+2\theta\rho\sigma_C(t)y\phi(t,y)\phi_y(t,y)(\gamma-1)(1+\zeta)\Big\}\\
&-\xi(t)(\phi(t,y))^{\gamma-1}(1+\zeta)^{\gamma-1}y-r\xi(t)(\phi(t,y))^{\gamma-1}(1+\zeta)^{\gamma-1}x\\
&-\mu_C(t)y\xi(t)(\phi(t,y))^{\gamma-1}\phi_y(t,y)(1+\zeta)^{\gamma-1}+\\
&-\frac{1}{2}\sigma_C^2(t)y^2\xi(t)\left[(\gamma-1)(\phi(t,y))^{\gamma-2}(\phi_y(t,y))^2(1+\zeta)^{\gamma-2}+(\phi(t,y))^{\gamma-1}(1+\zeta)^{\gamma-1}\phi_{yy}(t,y)\right]=\\
&\xi(t)(\phi(t,y))^{\gamma}\left\{\frac{K}{\gamma}(1+\zeta)^{\gamma}-(1+\zeta)^{\gamma-1}\frac{\phi_t(t,y)}{\phi(t,y)}+\frac{1}{2}\frac{1}{\gamma-1}\frac{\theta^2}{(\phi(t,y))^2}(\phi(t,y))^2(1+\zeta)^{\gamma}+\right.\\
&-\frac{1}{2}\frac{1}{\gamma-1}\frac{(\phi_y(t,y))^2}{(\phi(t,y))^2}(1+\zeta)^{\gamma-2}\rho^2\sigma_C^2(t)y^2(\gamma-1)^2+\frac{1}{2}\frac{1}{\gamma-1}\frac{\phi_y(t,y)}{\phi(t,y)}(\gamma-1)2\theta\rho\sigma_C(t)y(1+\zeta)^{\gamma-1}\\
&+\frac{y}{\phi(t,y)}(1+\zeta)^{\gamma-1}-r\zeta(1+\zeta)^{\gamma-1}\\
&\left.-\mu_C(t)y\frac{\phi_y(t,y)}{\phi(t,y)}(1+\zeta)^{\gamma-1}-\frac{1}{2}\frac{\sigma_C(t)y(\gamma-1)(\phi_y(t,y))^2}{(\phi(t,y))^2(1+\zeta)^{\gamma-2}}-\frac{1}{2}\frac{\sigma_C^2(t)y^2\phi_{yy}(t,y)}{\phi(t,y)}(1+\zeta)^{\gamma-1}\right\}.
\end{aligned}$$

We can rewrite the expression above by collecting the coefficients of the powers of $(1+\zeta)$, obtaining

$$
\begin{aligned}
&\xi(t)(\phi(t,y))^\gamma \cdot \\
&\Big\{(1+\zeta)^\gamma \overbrace{\left[\frac{K}{\gamma}-\frac{\theta^2}{1-\gamma}-r\right]}^{=0} \\
&(1+\zeta)^{\gamma-1}\left[r-\frac{\phi_t(t,y)}{\phi(t,y)}+\frac{\phi_y(t,y)}{\phi(t,y)}\theta\rho\sigma_C(t)y-\frac{y}{\phi(t,y)}-\frac{\phi_y(t,y)}{\phi(t,y)}\mu_C(t)y-\frac{1}{2}\frac{\phi_{yy}(t,y)}{\phi(t,y)}\sigma_C^2(t)y^2\right] \\
&(1+\zeta)^{\gamma-2}\underbrace{\left[\frac{1}{2}\frac{(\phi_y(t,y))^2}{\phi(t,y)^2}(1-\rho^2)(1-\gamma)\sigma_C^2(t)y^2\right]}_{\geq 0}\Big\}.
\end{aligned}
$$

From this representation it is then easy to see that, if $\phi(t,y)=\varphi(t)y$ and φ is a solution of (3.1), the function g will be a supersolution to (2.2), which concludes. □

Remark 3.2. Another possible way to prove the estimates in (3.2) is to use financial arguments and notice that

- the lower bound corresponds to the value function of the problem without random endowments, and can therefore be improved when random endowments are paid
- an alternative upper bound can be found by comparing the problem to the one of an artificial market model where the endowment can also be traded.

Also notice that the lower bound for the value function in the proposition above holds without any extra assumption.

Corollary 3.3. *For $y>0$ fixed and $x\to+\infty$ it holds*

$$
v(t,x,y) \sim \exp\{K(T-t)\}U(x). \tag{3.3}
$$

As a consequence of the previous theorem, extending again to the parabolic case the arguments in Duffie et al. (1997), we can get some asymptotics for the reduced function u, as stated in the following.

Proposition 3.4. *It holds that*

(i) For $z\to\infty$, $u(t,z)\sim\exp\{K(T-t)\}\frac{z^\gamma}{\gamma}$.

(ii) The function h defined as in (2.11) is such that, out of a negligible set, for $x/y\to\infty$

$$
h(t,x,y) \sim \frac{\theta}{\sigma(1-\gamma)},
$$

if this is in A, i.e. it converges to the Merton ratio.

Proof. (i) This is an immediate consequence of Theorem 3.3.

(ii) For $\lambda > 0$, define

$$w^{(\lambda)}(t,z) := e^{-K(T-t)}\lambda^{-\gamma}u(t,\lambda z).$$

First notice that, due to Theorem (3.3), it holds for $\lambda \to \infty$

$$w^{(\lambda)}(t,z) \to \frac{z^\gamma}{\gamma},$$

locally uniformly. Moreover, $w^{(\lambda)}$ is concave in z, which implies (see Rockafellar (1997), Theorem 25.7) that the partial derivatives with respect to z will also converge

$$w_z^{(\lambda)} \to z^{\gamma-1}.$$

From (3.1) it follows also that

$$w_t^{(\lambda)} \to 0,$$

being the convergence of the functions $w^{(\lambda)}$ and their derivatives uniform in t.
Now, since u solves (2.4) for almost every (t,z), for $w(t,z) := e^{-K(T-t)}u(t,z)$, it will hold a.e.

$$\begin{aligned}
&w_t - Kw + \left(\mu_C(t)\gamma - \frac{1}{2}\sigma_C(t)^2\gamma(1-\gamma)\right)w + \big((1-\gamma)\sigma_C(t)^2 - \mu_C(t)\big)zw_z + \\
&\frac{1}{\lambda}w_z + \frac{1}{2}\sigma_C(t)^2z^2w_{zz} + \\
&\sup_{\pi\in\mathcal{A}}\left\{(\pi\sigma\theta + r)zw_z + \frac{1}{2}(\pi\sigma)^2z^2w_{zz} - \rho\sigma_C(t)\sigma\pi(1-\gamma)zw_z - \rho\sigma_C(t)\sigma\pi z^2w_{zz}\right\} = 0,
\end{aligned}$$

If we now define by

$$\ell(z) := \lim_{z\to\infty}\frac{w_{zz}}{z^{\gamma-2}} = \lim_{z\to\infty}\frac{u_{zz}}{e^{K(T-t)}z^{\gamma-2}},$$

and take the limit for $z \to \infty$ in the above equation, we obtain a second degree polynomial for $\ell(z)$

$$\begin{aligned}
&\left(\mu_C\gamma - K - \frac{1}{2}\sigma_C^2\gamma(1-\gamma)\right)\frac{1}{\gamma} + \big(r - \mu_C + (1-\gamma)(1-\rho^2)\sigma_C^2 + \theta\rho\widetilde{\sigma}_C\big) \\
&+ \frac{1}{2}(1-\rho^2)\sigma_C^2\ell(z) - \frac{1}{2}(\theta - \rho\sigma_C(1-\gamma))^2\frac{1}{\ell(z)} = 0.
\end{aligned}$$

By the concavity of the function w in z, we know that we will have to select the negative root, which is given as

$$
\begin{aligned}
\ell(z) =& \frac{1}{(1-\rho^2)\sigma_C^2(t)}\Big\{ -\frac{1}{2}\frac{1}{1-\gamma}(k_0(t,z)-k_1(t,z)) \\
& - \sqrt{\frac{1}{4}\frac{1}{(1-\gamma)^2}(k_0(t,z)-k_1(t,z))^2 + \frac{1}{(1-\gamma)^2}k_0(t,z)k_1(t,z)}\Big\} \\
=& \frac{1}{(1-\rho^2)\sigma_C^2(t)}\left\{ -\frac{1}{2}\frac{1}{1-\gamma}(k_0(t,z)-k_1(t,z)) - \sqrt{\frac{1}{4}\frac{1}{(1-\gamma)^2}(k_0(t,z)+k_1(t,z))^2}\right\} \\
=& \frac{1}{(1-\rho^2)\sigma_C^2(t)}\left\{ -\frac{1}{2}\frac{1}{1-\gamma}(k_0(t,z)-k_1(t,z)) - \frac{1}{2}\frac{1}{1-\gamma}(k_0(t,z)+k_1(t,z))\right\} \\
=& \frac{1}{(1-\rho^2)\sigma_C^2(t)}\left\{ -\frac{1}{1-\gamma}k_0(t,z))\right\} = \gamma - 1,
\end{aligned}
$$

where the coefficients k_0 and k_1 are defined as

$$
\begin{aligned}
k_0(t,z) &:= \sigma_C(t)(1-\gamma)^2(1-\rho^2) \geq 0, \\
k_1(t,z) &:= (\theta - \rho(1-\gamma)\sigma_C(t))^2 \geq 0.
\end{aligned}
$$

This implies that

$$
\lim_{z\to\infty} w_{zz}(t,z) = (\gamma-1)z^{\gamma-2}.
$$

and therefore

$$
\lim_{z\to\infty} \frac{u_z}{zu_{zz}} = \frac{1}{\gamma-1},
$$

which, substituted in (2.10), yields the claim.

□

4 Conclusion

We consider an optimal asset allocation problem in an incomplete market, where exogenous stochastic endowments flow continuously into the portfolio according to a time-inhomogeneous geometric Brownian motion. We analyze the viscosity solution of the HJB PDE, reduce its dimension, and prove that the optimal strategy can be recovered from the optimal policy of a reduced problem. We are also able to describe the asymptotic behavior of the value function, and the strategy when the initial wealth goes to infinity.

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A Appendix

We show here that Theorem 4.4.5 of Pham (2009) can be still applied if the coefficient of the function u in the HJB equation is not assumed to be constant anymore. The arguments in the proof stay true in the case where its coefficient is time dependent.
Let $-c(t) = -c(t, y)$ be this coefficient. In our case we would have that $c(t) = -\gamma\mu_C(t) + \frac{1}{2}\sigma_C^2(t)\gamma(1-\gamma)$, but it suffices to have a locally integrable coefficient c for our argument to work. Define a transformed function $w(t, z) := e^{\int_0^t c(s)\,\mathrm{d}s} u(t, z)$. Then

$$\begin{aligned}
\partial_t w &= e^{\int_0^t c(s)\,\mathrm{d}s} \partial_t u(t, z) + e^{\int_0^t c(s)\,\mathrm{d}s} c(t) u(t, z); \\
\partial_z w &= e^{\int_0^t c(s)\,\mathrm{d}s} \partial_z u(t, z); \\
\partial_{zz} w &= e^{\int_0^t c(s)\,\mathrm{d}s} \partial_{zz} u(t, z).
\end{aligned}$$

Hence, since u satisfies Equation (2.5), which is of the type

$$u_t = -c(t)u + G(u_z, u_{zz}),$$

where G is a positive homogeneous function (of degree 1) in the two variables, we have that

$$e^{\int_0^t c(s)\,\mathrm{d}s} u_t = -e^{\int_0^t c(s)\,\mathrm{d}s} c(t)u + G(e^{\int_0^t c(s)\,\mathrm{d}s} u_z, e^{\int_0^t c(s)\,\mathrm{d}s} u_{zz}),$$

and the transformed function w satisfies

$$w_t = e^{\int_0^t c(s)\,\mathrm{d}s} u_t + e^{\int_0^t c(s)\,\mathrm{d}s} c(t)u = G(e^{\int_0^t c(s)\,\mathrm{d}s} u_z, e^{\int_0^t c(s)\,\mathrm{d}s} u_{zz}) = G(w_z, w_{zz}).$$

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