Target date funds: Marketing or Finance?

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Abstract

Since Target Date Funds (TDFs) became one of the default investment strategies for the 401(k) defined contribution (DC) beneficiaries, they have developed rapidly. Usually they are structured according to the principle "young people should invest more in equities". Is this really a good recommendation for DC beneficiaries to manage their investment risk? The present paper relies on dynamic asset allocation to investigate how to optimally structure TDFs by realistically modelling the contributions made to 401(k) plans. We show that stochastic contributions *can* play an essential role in the determination of optimal investment strategies. Depending on the correlation of the contribution process with the market's stock, we find that an age-increasing equity holding can be optimal too. This result highly depends on how the contribution rule is defined.

Keywords: Utility theory, Optimal asset allocation, Defined contribution, Target date fund

1. Introduction

Target date funds (TDF) are investment funds with a prespecified maturity (target date). Because of their structure, these funds place themselves in the category of "life-cycle" funds, rather than in the category of "life-style" funds where the target is the risk profile of the investor. TDFs have developed very rapidly, particularly after they became one of the default investment strategies of a 401(k) defined contribution

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(DC) beneficiary.[1] According to Morningstar Fund Research (2012), assets in the TDFs have grown from 71 billion US dollars at the end of 2005 to approximately 378 billion dollars at the end of 2011. These funds are directly coupled with the retirement year of the DC plan investors and have the advantage that the investors do not have to choose a number of investments, but only a single fund. The main mechanism behind these TDFs is: those who retire later shall invest more in equity, while those who retire earlier shall invest less in equity. In other words, equity holding in TDFs shall decrease in age. Therefore, TDFs are usually identified by practitioners with "glide paths", i.e. the *decreasing* curve of the equity holding (as a fraction of wealth) over time.

But is this shaping of target date funds really a good recommendation for DC beneficiaries to manage the investment risks? Shall every DC beneficiary who retires in 2050 take the same target date fund, independent of his income, and risk preference? Is the popular financial advice just anecdotal evidence? Or can it be justified by rigorous theory?

There is few literature aiming to find an optimal equity holding which justifies the target date fund.[2] At first sight, target date funds are inconsistent with Merton's portfolio (c.f. Merton (1969) and Merton (1971)), which has sometimes been considered as an economic puzzle. For an investor with a constant relative risk aversion preference, Merton's optimal portfolio prescribes to invest a constant proportion of wealth in equity (constant-mix strategy is optimal), i.e. the optimal portfolio does not depend on time/age. One of the most famous rigorous economic justifications for the age-dependent (particularly age-decreasing) investment behavior is given in Jagannathan & Kocherlakota (1996). In their paper, by using economic reasonings,

[1] DC beneficiaries need to bear the entire investment risks and management of their pension plans. In the US, they can manage it by so called "Individual Retirement Accounts", or more frequently by making contributions to 401(k) plans, where the amount of contributions mainly depends on the development of an employee's salary (income).

[2] In the academic literature, the study of target date funds has been mostly based on simulation studies: taking several prevailing strategies (target date fund and constant mix strategy), compare them and find the best strategy among them. For instance, Spitzer & Singh (2008) compare the target date funds with constant-mix strategy (50-50%) by examining the ruin probability via bootstrap simulation and rolling period analysis and show that a constant mix strategy outperforms the TDF all the time.

the robustness of the arguments justifying glide paths is called into question. They discuss the three prevailing hypotheses supporting it and accept *human capital*, in form of present value of expected future earnings, as the only valid reason to solve this apparent puzzle. Using a simplified model and some qualitative arguments, they also show the validity of the argument "younger people shall invest more in stocks because younger people have more labor income ahead" *under certain conditions*, e.g. if the correlation of labor income with stock returns is not too high.

In a DC pension plan, the beneficiary makes contributions whose amount mainly depends on the development of her salary (income). Based on this fact and also motivated by Jagannathan & Kocherlakota (1996), we use an optimal dynamic asset allocation approach in continuous time to investigate the role of labor income in the form of stochastic contributions in the utility maximization of the wealth of a DC beneficiary. We find that stochastic contributions *can* play an essential role in the determination of the optimal equity holding. More interestingly, it depends much on how the contribution rule is set. We mainly discuss two models as representatives of the following general forms:

1) Contributions are adjusted as a varying proportion of the fund value, where the proportion is allowed to be stochastic but just depends on the salary process.

2) Contributions depend only on the salary process.

We show that in the first case, correlation between the asset and salary risk plays the dominant role in the asset allocation, which decides whether the equity proportion is equal to, higher or lower than Merton's portfolio. The effect of human capital, interpreted as discounted value of future wages, is rather secondary and influences the magnitude of the equity holding when there is some correlation between the two risks.

In the second case, the effect of human capital becomes more relevant. The resulting equity holding is a time-dependent and, in most -but not all- cases, time-decreasing proportion of wealth which suggests a higher equity holding than in Merton's portfolio. We will see that the correlation between the asset and salary risk is still a deciding factor in the equity holding. However, uncorrelated risks do not imply an equity holding identical to Merton's portfolio. Through our analysis, we show that the optimality of glide paths (age-decreasing equity holding) can be in most cases verified in the presence of stochastic contributions. However, in some extreme cases (e.g. the asset and income risks are highly positively correlated), the optimal equity holding could recommend that an *older beneficiary* shall invest *more* in stocks than a younger one. In other words, it is not necessarily true that one should use a glide path as reference, as already conjectured in Jagannathan & Kocherlakota (1996). Let us mention here that similar conclusions are also drawn by Dybvig & Liu (2010). However, the main focus in their paper is on the impact of retirement flexibility and borrowing constraints in determining the optimal consumption and investment for the problem of maximizing the utility from consumption and bequest. Our model neglects on the one side the fact that the retirement time could be random and the optimal stopping problem deriving from that, on the other side better fits the structure of TDFs. Moreover, from a technical point of view, Dybvig & Liu (2010) are able to reduce the optimization problem to a time-independent ODE, whereas in our case of maximizing the utility of terminal wealth complex, non-linear parabolic equations need to be considered.

The mathematical foundation of the current paper is optimal dynamic asset allocation (in an incomplete financial market). There exists a stream of literature on optimal dynamic asset allocation applied for a DC pension scheme with diverse financial market settings and different preferences.[3] In the present paper, we incorporate untradable salary risk in the contribution process and analyze the optimal asset allocation for a target date fund in the context of defined contribution schemes.

[3]Gao (2008) studies this problem under stochastic interest rates. Boulier, Huang & Taillard (2001) solve it under the constraint that a guaranteed amount is provided to the beneficiary in a stochastic interest rate framework. Blake, Wright & Zhang (2013) investigate this problem with a loss-averse preference (instead of using a conventional utility function) and study so called target-driven investing. Incorporating salary (income) risk by modelling it as a geometric Brownian motion, Zhang, Korn & Ewald (2007) focus on the influence of inflation on pension products. Giacinto, Federico, Gozzi & Vigna (2014) emphasize the possibility that the retirees can postpone annuity purchasing after retirement, i.e. they are provided with an income drawdown option. Cairns, Blake & Dowd (2006) consider this optimization problem under a utility function which uses the plan member's salary as a numeraire. Our formulation differs from Cairns, Blake & Dowd (2006): In Cairns, Blake & Dowd (2006), the pension beneficiary maximizes the expected utility of the terminal wealth divided by his terminal salary, while in our paper we maximize the expected utility of the terminal wealth. This objective is the traditional one considered in the literature, and appears to us to be more natural.

We rely on dynamic programming and use the dimension-reduction technique developed in Chen, Mereu & Stelzer (2014) to solve the optimal asset allocation problem.

The remainder of the paper is organized as follows. Section 2 describes the model setup, introduces our optimization problem and particularly the two contribution rules. In the subsequent Section 3, we rely on the separation theorem and solve the optimization problem for the first case in which the contribution is a proportion of the fund value. We also provide an example of a model with mean-reverting contributions depending both on the wealth process and the salary, and obtain an analytical solution. In Section 4, we treat the second case in which the contribution is a function of salary exclusively and use Chen, Mereu & Stelzer (2014) to solve the optimization problem. Moreover, we study numerically the effects of correlation and human capital on the optimal equity holding. Finally, we provide some concluding remarks in Section 5, and exhibit a set of detailed mathematical derivations in the appendix.

2. Model setup

On a fixed filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, satisfying the usual hypotheses, consider a financial market consisting of a riskless asset and a risky asset. From now on let T be a fixed finite time point, representing the age of retirement[4].
(S^0, S^1) will denote respectively the savings account (riskless asset) and the risky asset, and we assume that the two assets follow a Black-Scholes model:

$$\begin{aligned} \mathrm{d}S_t^0 &= rS_t^0 \, \mathrm{d}t, \\ \mathrm{d}S_t^1 &= \mu S_t^1 \, \mathrm{d}t + \sigma S_t^1 \, \mathrm{d}W_t^1, \end{aligned}$$

where $\mu,\ r \in \mathbb{R},\ \sigma > 0$ and W^1 is a Brownian motion on the above mentioned filtered probability space.

The contributions of DC plan investors are usually coupled to their salary. We assume that the salary process I has stochastic dynamics driven by another

[4]We neglect the mortality risk, assume that the retirement time is deterministic and focus on the stochastic contribution risk.

Brownian motion W^I on the same space, correlated with W^1 with a correlation coefficient ρ. So, there is a Brownian motion W^2, independent of W^1 such that $W^I = \rho W^1 + \sqrt{1-\rho^2}W^2$.
The process I_t is not constant, but rather describes a time-varying salary (e.g. overtime payments, bonuses etc.):

$$\mathrm{d}I_t = \mu_I(t, I_t)\,\mathrm{d}t + \sigma_I(t, I_t)\,\mathrm{d}W_t^I, \tag{2.1}$$

where $\mu_I : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$, $\sigma_I : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ are càdlàg in time, and locally Lipschitz continuous of at most linear growth in the second variable. We assume that the fund manager invests at any time t a proportion π_t of the wealth in the stock S^1 and $1-\pi_t$ in the bond S^0 with interest rate r, and that the corresponding wealth process evolves according to a stochastic process $\{A_t^\pi\}_{t\in[0,t]}$, whose dynamics will be specified in detail later on. In addition, there are contributions continuously flowing to the plan member's individual account at rate $c_t := f(A_t^\pi, I_t)$. The *contribution rule* is a function $f : \mathbb{R}^2 \to \mathbb{R}_+$, $f \in \mathcal{C}^2$, depending on the fund's value A_t^π and on the salary process I_t. There are several special cases, among the ones that can be considered:

- $f(A_t^\pi, I_t) = \lambda(I_t) \cdot A_t^\pi$, where $\lambda \in \mathcal{C}^2(\mathbb{R})$. Contributions are here adjusted as a varying proportion of the fund value, where the proportion is allowed to be stochastic but just depends on the salary process.

- $f(A_t^\pi, I_t) = f(I_t)$, where $f \in \mathcal{C}^2$ is strictly positive and just depends on the second variable. Here the contributions are based on the wage.

We will see that the first case can be treated sometimes in an analytic way (see Appendix A), while the second one requires more efforts, but reflects more realistic contribution processes in the DC plan. For instance, $f(A_t^\pi, I_t) = \lambda I_t$ is the contribution rule most frequently used in practice, i.e. the contribution c_t is taken as a constant fraction $\lambda \in (0,1)$ of the salary process. This is just a special case of $f(A_t^\pi, I_t) = f(I_t)$.
Remark 2.1. Under the above assumptions, the contributions $c_t := f(A_t^\pi, I_t)$ are also Itô diffusions, and will have stochastic dynamics of the type:

$$\begin{cases} \mathrm{d}c_t^\pi &= \mu_C(t, A_t^\pi, I_t, \pi_t)\,\mathrm{d}t + \sigma_C^1(t, A_t^\pi, I_t, \pi_t)\,\mathrm{d}W_t^1 + \sigma_C^2(t, A_t^\pi, I_t, \pi_t)\,\mathrm{d}W_t^2, \\ c_0^\pi &= y, \end{cases} \tag{2.2}$$

where $\mu_C : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathscr{A} \to \mathbb{R}$ and $\sigma_C^i : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathscr{A} \to \mathbb{R}_+$, $i = 1, 2$, are deterministic functions obtained in terms of the first and second derivatives of f, and $\mathscr{A} \subseteq R$ is a set the strategies take values in. We will therefore sometimes give directly the contribution process as a general Itô diffusion without referring to the income dynamics.

We will often talk about a "strategy" π meaning by this the pair $(1-\pi, \pi)$. Given an investment strategy π, the *pension wealth process* obtained by trading according to the strategy π, $\{A_t^\pi\}_{t\in[0,T]}$, has the following dynamics:

$$\mathrm{d}A_t^\pi = \frac{A_t^\pi \pi_t}{S_t^1}\,\mathrm{d}S_t^1 + \frac{A_t^\pi(1-\pi_t)}{S_t^0}\,\mathrm{d}S_t^0 + c_t\,\mathrm{d}t, \tag{2.3}$$

and hence, defining $\theta := \frac{\mu - r}{\sigma}$, for an initial wealth $x \in \mathbb{R}_+$ we get

$$\begin{cases} \mathrm{d}A_t^\pi & = [A_t^\pi(\pi_t\sigma\theta + r) + c_t]\,\mathrm{d}t + A_t^\pi\pi_t\sigma\,\mathrm{d}W_t^1, \\ A_0^\pi & = x. \end{cases} \tag{2.4}$$

Note that this definition reflects the fact that the only allowed additional cash injections to the fund are due to the continuous payments at rate $c_t = f(A_t^\pi, I_t)$.
A progressively measurable process π is said to be *admissible*, if it takes values in a fixed convex subset $\mathscr{A}$ of $\mathbb{R}$ such that $\int_0^T |\pi_s|^2\,\mathrm{d}s < \infty$ a.s. and $A_t^\pi \geq 0$ for every $t \in [0,T]$.[5] We denote by $\mathcal{A}$ the set of all admissible strategies.
For simplicity, let us assume that $\mathscr{A}$ is compact.

Remark 2.2. In the definition of admissibility, one has to ensure that the fund process never becomes negative. The general contribution process does not necessarily ensure positivity. In our second case in which $f(A_t^\pi, I_t) = f(I_t)$ (with f strictly positive), the wealth process stays positive without any extra requirements on the admissible strategies.

Recall that the driving Brownian motion W^I represents the uncertainty in the salary, and is spanned by two Brownian motions. One of these, W^2, is assumed not to be traded in the market. This makes the market incomplete.
Assume that the DC plan investor gets a lump-sum payment at time of retirement T [6] and wants to maximize the expected utility from this terminal wealth. More

[5] The square-integrability condition ensures the existence and uniqueness of a solution for Equation (2.4), provided some regularity assumptions on the contribution rule (e.g. Lipschitz continuity in the first variable) hold, together with some assumptions on the contribution process.

[6] Most DC pension plans pay out a lump-sum instead of annuities.

precisely, we are looking for an optimal investment strategy π^* such that

$$\mathrm{E}\left[U(A_T^{\pi^*})\right] = \sup_{\pi\in\mathcal{A}} \mathrm{E}\left[U(A_T^{\pi})\right],$$

where $U:\mathbb{R}\to\mathbb{R}_+$ is a CRRA power utility function and $\mathcal{A}$ is the set of admissible strategies, i.e. for a risk aversion parameter $\gamma<1$, $\gamma\neq 0$,

$$U(x) = \frac{x^\gamma}{\gamma}.$$

The power utility is abundantly used in both theoretical and empirical research because of its nice analytical tractability. Most importantly, the use of the power utility is also motivated economically, since the long-run behavior of the economy suggests that the long run risk aversion cannot strongly depend on wealth, see Campbell & Viceira (2002).

The *value function* of the utility maximization problem is given by

$$v(t,x,y) := \sup_{\pi\in\mathcal{A}} \mathrm{E}\left[U(A_T^{\pi,t,x,y})\right], \quad (t,x,y)\in[0,T)\times(0,+\infty)\times(0,+\infty), \tag{2.5}$$

where we are taking as controlled process the pair $X^\pi=(A^\pi, I)$, and the notation $A^{\pi,t,x,y}$ stands for the first coordinate of the process X^π starting from the point (x,y), respectively the initial wealth and the initial income, at time t.
Please note that the process I actually does not depend on the control π. Applying well-known results in stochastic control, see e.g. Pham (2009) Chapter 3, in particular Theorem 3.5.2, we can write down the HJB equation for the value function of our control problem:

$$\begin{aligned}
-v_t(t,x,y) = \sup_{\pi\in\mathscr{A}} \Big\{ & \left[x(\pi\sigma\theta+r)+f(x,y)\right] v_x(t,x,y) + \mu_I(t,y)v_y(t,x,y) \\
& + \frac{1}{2}(\pi\sigma x)^2 v_{xx}(t,x,y) + \frac{1}{2}\sigma_I(t,y)^2 v_{yy}(t,x,y) + \rho\sigma\sigma_I(t,y)\pi x v_{xy}(t,x,y)\Big\},
\end{aligned} \tag{2.6}$$

$$v(T,x,y) = U(x), \quad \forall (x,y).$$

Here and in the remainder, we will denote the partial derivatives by subscripts, and often omit the argument of the function in the equations.

The solution to the HJB PDE (2.6) depends much on the *contribution rule* and cannot be solved analytically in general. In what follows, we discuss a representative model of each of the two contribution rules: $f(A_t^\pi, I_t) = \lambda(I_t) \cdot A_t^\pi$ in Section 3 and $f(A_t^\pi, I_t) = f(I_t)$ in Section 4.

3. Contribution as a fraction of the fund value

If we are ready to constrain the contribution rule to the form $f(x, y) = \lambda(y)x$, as a fraction of the fund value, where λ is exclusively a function of y, in some cases we are able to achieve analytic solutions.

Example 3.1. Take as contribution rule $f(x, y) = \lambda(y)x$, and assume that the process $\lambda(I_t)$ evolves according to an Ornstein-Uhlenbeck process[7].
We assume directly that the dynamics of the process $\Lambda_t = \lambda(I_t)$ are given as[8]:

$$\begin{cases} \mathrm{d}\Lambda_t &= \kappa_\Lambda(\mu_\Lambda - \Lambda_t)\,\mathrm{d}t + \sigma_\Lambda\,\mathrm{d}W_t^I, \\ \Lambda_0 &= y, \end{cases} \tag{3.1}$$

where the constant parameters are such that $\kappa_\Lambda > 0$ denotes the mean reversion speed, and μ_Λ the mean reversion level, whereas $\sigma_\Lambda > 0$ is the volatility. For this model, it can be seen (see Appendix A for more details) that the optimal strategy is given by

$$\begin{aligned} \pi^* =& \pi^*(t) = \left(\theta - \rho\sigma_\Lambda \frac{\gamma}{\kappa_\Lambda}\left(e^{-\kappa_\Lambda(T-t)} - 1\right)\right)\frac{1}{\sigma(1-\gamma)} \\ =& \frac{\theta}{\sigma(1-\gamma)} - \frac{1}{\sigma(1-\gamma)}\frac{\rho\sigma_\Lambda\gamma(e^{-\kappa_\Lambda(T-t)} - 1)}{\kappa_\Lambda}. \end{aligned} \tag{3.2}$$

Let us comment on the optimal strategy (3.2).

- First, it consists of two terms: the first term is the famous Merton portfolio, and the second term is an additional component which accounts for the (hedgeable) correlated contribution risk. This clear-cut decomposition of the strategy, particularly filtering out the Merton portfolio, is only possible because the contribution rule $f(x, y) = \lambda(y)x$ allows us to use the separation

[7]This function is obviously Lipschitz continuous in x, and since the Ornstein-Uhlenbeck process is predictable, there is a solution to Equation (2.4).

[8]In the case where the dynamics of the process I_t are given instead of the dynamics of the contribution process, up to an application of Itô's formula to the process $\lambda(I_t)$, we get a similar result.

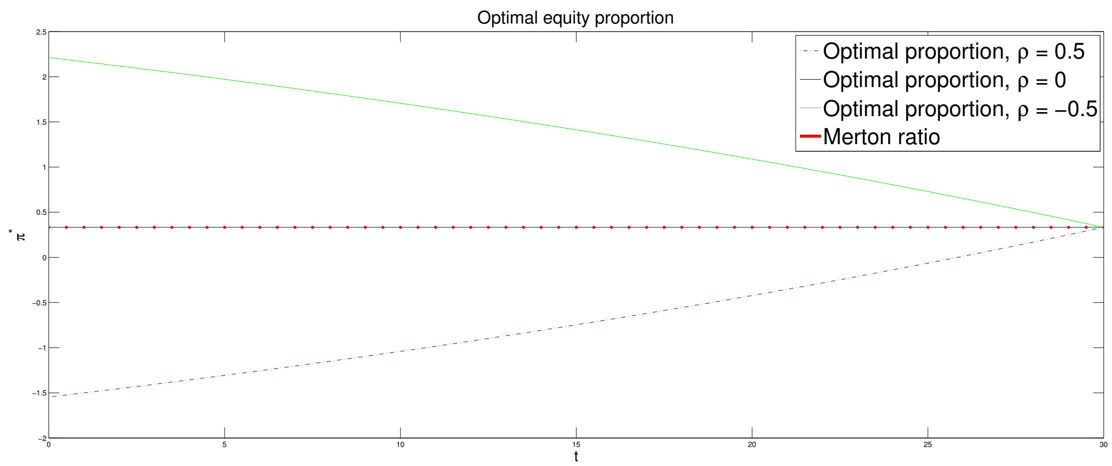


Figure 1: Optimal equity proportion against time in the model of Section 3
Parameters: $\sigma = 0.2$, $\mu = 0.06$, $r = 0.02$, $\sigma_\Lambda = 0.05$, $\kappa_\Lambda = 0.02$, $\gamma = -2$, $T = 30$.

ansatz $v(t, x, y) = \phi(t, y)U(x)$ on the value function. The explicit expression of $\phi(t, y)$ can be found in Appendix A

- Second, the correlation coefficient ρ between the tradeable and the untradeable risk plays the most important role in the magnitude of the strategy. The sign of ρ decides whether the optimal investment is identical to, higher or lower than the Merton portfolio. For $\gamma < 0$ (a relative risk aversion larger than 1), a positive correlation leads to an optimal equity holding lower than the Merton portfolio, and a negative correlation results in an optimal equity holding higher than the Merton portfolio. For $0 < \gamma < 1$ (a relative risk aversion between 0 and 1), the effect of ρ is reversed. A higher correlation means that more untradable risks can be eliminated through going short in the tradable stock.

- Third, the optimal strategy varies in time and converges towards the Merton portfolio when the time approaches maturity, as can be read from Equation (3.2) and is illustrated in Figure 1.

- Fourth, compared to the influence of the correlation coefficient ρ, the effect of income risk (contribution) is secondary. For instance, for $\rho = 0$, the resulting optimal strategy coincides with the Merton portfolio: the parameters κ_Λ and σ_Λ which drive the contribution process do not influence this result. You might argue that the effect of the income is secondary just due to our modelling, i.e.

since we use Ornstein-Uhlenbeck process to model $\lambda(I_t)$, which means that the contribution might become negative. Therefore, in this place, let us briefly mention another simpler example which helps us see the secondary effect of the contribution on the optimal investment strategy: $\lambda(I_t) = \lambda$, i.e. $f(x, y) = \lambda x$, $\lambda > 0$. In this case, we have a continuous stream of positive contributions flowing to the pension fund, but the optimal strategy is not time-dependent and it is easy to see that it coincides with Merton's portfolio. In this simple example, although the contributions are always strictly positive, the income has no effect at all.

The analysis in this section shows that adding a contribution (depending on income) in the form $f(A_t^\pi, I_t) = \lambda(I_t) \cdot A_t^\pi$ to the original Merton setting might lead to a time-dependent optimal portfolio, depending on the specification of $\lambda(I_t)$. Under certain circumstances, it is also possible to achieve a strategy which decreases in time, suggesting that younger participants shall optimally hold more equity. However, in this case, the contribution/income does not play the most important role. The correlation together with the risk aversion level and income parameter determines consequently the magnitude of the equity holding.

4. Contribution depending only on the income

In the current section, we look at the more realistic contribution process: $f(A_t^\pi, I_t) = f(I_t)$. The optimization problem becomes much more complicated. No separation ansatz seems to be adoptable to derive the value function and the optimal strategy, which has the consequence that explicitly filtering out the Merton portfolio is impossible. Some natural questions need to be answered in the next section: will the income (contribution) become now the dominant effect in the optimal strategy? In this case, do we always achieve an optimal strategy which decreases in time? Here, we model the contribution process as a process depending only on the wage, i.e. $f(x, y) = f(y)$. If $f \in \mathcal{C}^2(\mathbb{R})$ is an invertible function, the contribution process itself will be again an Itô-diffusion, whose dynamics can be derived from the dynamics of the wage process I. Slightly changing the setting with respect to the previous sections, and with a little abuse of notation, we will therefore directly model the contribution process.

In the following, we assume that the contributions have stochastic dynamics given as:

$$\begin{cases} \mathrm{d}c_t & = c_t \left(\mu_C(t) \, \mathrm{d}t + \sigma_C(t) \, \mathrm{d}W_t^I \right), \\ c_0 & = y, \end{cases} \tag{4.1}$$

where $\mu_C : [0,T] \to \mathbb{R}$ and $\sigma_C : [0,T] \to \mathbb{R}_+$ are càdlàg in time, and $y > 0$ is the initial contribution.

Unfortunately, it is very hard to make an educated guess on the solution of the PDE corresponding to this case, and so we are unable to apply classical verification methods. To solve this optimal asset allocation problem, Chen, Mereu & Stelzer (2014) reduce the HJB equation by one dimension to make the optimization problem well solvable through numerical methods. Chen, Mereu & Stelzer (2014) show that the value function reads $v(t,x,y) = y^\gamma u\left(t, \frac{x}{y}\right)$, where u is a solution of a reduced PDE (see Appendix B for more details and a precise statement of the theorem). They also show that the optimal strategy is described by $\pi_t^* = h\left(t, A_t^\Pi, c_t\right)$ where

$$h(t,x,y) := \frac{\theta}{\sigma \cdot \frac{-u_{zz}(t,\frac{x}{y})}{u_z(t,\frac{x}{y})} \frac{x}{y}} - \frac{\rho \sigma_C(t)}{\sigma} \left(\frac{1-\gamma}{\frac{-u_{zz}(t,\frac{x}{y})}{u_z(t,\frac{x}{y})} \frac{x}{y}} - 1 \right), \quad \text{if it belongs to } \mathscr{A} \text{ a.e.} \tag{4.2}$$

This also proves that the optimal strategy will depend on the current wealth x and income y only though their ratio $z = \frac{x}{y}$.

Note that the strategy in (4.2) also consists of two parts:

- The first term has a similar form as, but is not identical to, the Merton portfolio. In fact, it is now impossible to explicitly obtain the Merton portfolio as a separate summand. The difference lies in the coefficient in the denominator. Since the separation ansatz does not work in this case, the additional contribution/income influences also the relative risk aversion of the indirect utility: We now have a coefficient changing with time and the ratio of wealth over income, z. This varying RRA is given by $\frac{-u_{zz}}{u_z} z$, which does not necessarily equal $1-\gamma$. In Chen, Mereu & Stelzer (2014), the authors show that when z goes to infinity (i.e. the income is negligible compared to the wealth), this quantity converges to $1-\gamma$. In other words, asymptotically we are back to Merton's case. Due to this effect of the contribution, even when there is no correlation between asset and contribution risk, we are already obtaining a time-dependent equity holding.

- The second term is again the hedging component which accounts for the (hedgeable) correlated untradable contribution risk. It disappears when $\rho = 0$ *or the relative risk aversion of the indirect utility* $\frac{-u_{zz}}{u_z}z$ *converges to* $1-\gamma$, *for* $z \uparrow \infty$. In this case, since the comparison between the two magnitudes ($\frac{-u_{zz}}{u_z}z$ and $1-\gamma$) depends on specific parameter choices, we can identify the unambiguous impact of ρ.

Following the above two observations, we notice that there are two main factors driving the choice of the investor:

1. the correlation between the random endowment and the risky asset, which allows to hedge partially away risks from the random future contribution by choosing the investment strategy accordingly;

2. the presence of a strictly positive random endowment, which corresponds to the availability of future incomes, and corresponds to an extra - non financial - wealth, called "human capital".

In the following two subsections, we will analyze these two effects in detail.

4.1. The impact of correlation

With the contribution rule studied in Section 3, the effect of correlation is very important and the value of the correlation for the qualitative change of the strategy is $\rho = 0$. However, in this second contribution rule $\rho = 0$ is not crucial anymore. Numerical studies seem to suggest that also in the second contribution rule there is a value of the correlation determining whether the optimal proportion lies above or below the Merton ratio, but that this critical value will be now always strictly bigger than 0 (see later Figure 2). The following result aims at understanding this issue a little deeper.

Proposition 4.1. *In the model of Section 4, assume that* σ_C *is constant over time and* $\mu > r$. *If the parameters are such that* $1-\gamma \geq \frac{\mu-r}{\sigma\sigma_C}$, *then for the value*

$$\rho^* = \frac{\mu - r}{\sigma\sigma_C(1-\gamma)} > 0 \tag{4.3}$$

the corresponding optimal strategy is constant and coincides with the Merton ratio, namely

$$h(t,x,y) \equiv \frac{\mu - r}{\sigma^2(1-\gamma)},$$

if it belongs to $\mathscr{A}$.

Proof. To show the claim, we look for conditions on the correlation such that the optimal policy in (4.2) coincides with the Merton ratio. To this end, define for $z = \frac{x}{y}$

$$\delta(t,z) := -\frac{u_{zz}(t,z)}{u_z(t,z)} z - (1-\gamma).$$

We basically have to look for a ρ^* such that

$$\frac{\mu - r}{\sigma^2(1-\gamma)} = \frac{\mu - r}{\sigma^2(1-\gamma+\delta)} - \rho^* \frac{\sigma_C}{\sigma}\left(\frac{1-\gamma}{1-\gamma+\delta} - 1\right).$$

Rearranging the terms, this holds if and only if

$$\frac{\mu - r}{\sigma}\left(\frac{1}{1-\gamma} - \frac{1}{1-\gamma+\delta}\right) = -\rho^* \sigma_C (1-\gamma)\left(\frac{1}{1-\gamma+\delta} - \frac{1}{1-\gamma}\right).$$

One can then see that this equation is satisfied for $\rho^* = \frac{\mu - r}{\sigma \sigma_C (1-\gamma)}$.
That ρ^* is strictly positive follows from Equation (4.3) since $\mu > r$. Moreover, ρ^* is a correlation coefficient thanks to the assumptions on the parameters. □

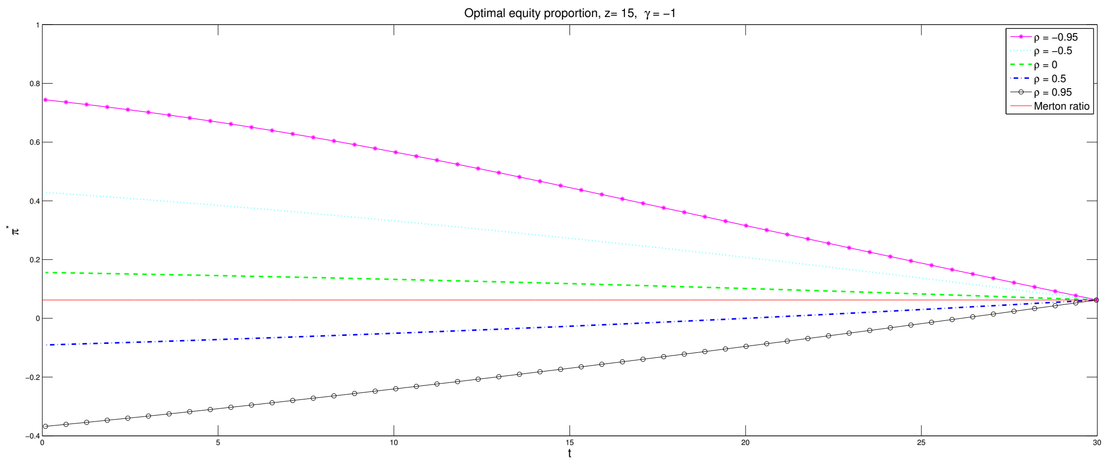
Remark 4.2. Note that also for $\delta(t,z) = 0 \ \forall (t,z)$ we would have that the optimal proportion corresponds to the Merton ratio. It seems however hard to prove whether and for which parameters this holds.
From the upcoming numerical results, we can actually conjecture that the value ρ^* in Equation (4.3) is critical in the following sense:

- if $\rho > \rho^*$, then $\pi^* < \frac{\mu - r}{\sigma^2(1-\gamma)}$ for all $t \in [0, T)$
- if $\rho < \rho^*$, then $\pi^* > \frac{\mu - r}{\sigma^2(1-\gamma)}$ for all $t \in [0, T)$.

Moreover, for $1-\gamma \leq \frac{\mu - r}{\sigma \sigma_C}$, i.e. for a sufficiently small relative risk aversion parameter (sufficiently big γ), all values of the correlation should yield optimal strategies lying above the Merton ratio.

Figure 2 plots the optimal equity proportion as a function of time for 5 different values of ρ. For the given parameters, the critical correlation coefficient is $\rho^* \approx 0.1923$. Therefore, for all $\rho < 0.1923$ (i.e. here $\rho = -0.95$, -0.5, 0), the optimal equity holding is higher than than the Merton portfolio. Furthermore, it demonstrates a glide path, a time-decreasing equity-holding. On the contrary, for all $\rho > 0.1923$ (i.e. here $\rho = 0.5$, 0.95), we observe a time-increasing equity holding which is overall lower than the Merton portfolio. This in particular shows that in this model the optimal strategy is not necessarily given as a glide path, but can be

Figure 2: Optimal equity proportion in the model of Section 4 for different values of the correlation parameter ρ.
Parameters: $T = 30$ years, and $\sigma = 0.4$, $\mu = 0.04$, $\sigma_C = 0.13$, $\mu_C = 0.02$, $r = 0.02$, $\gamma = -1$, $z = 15$.



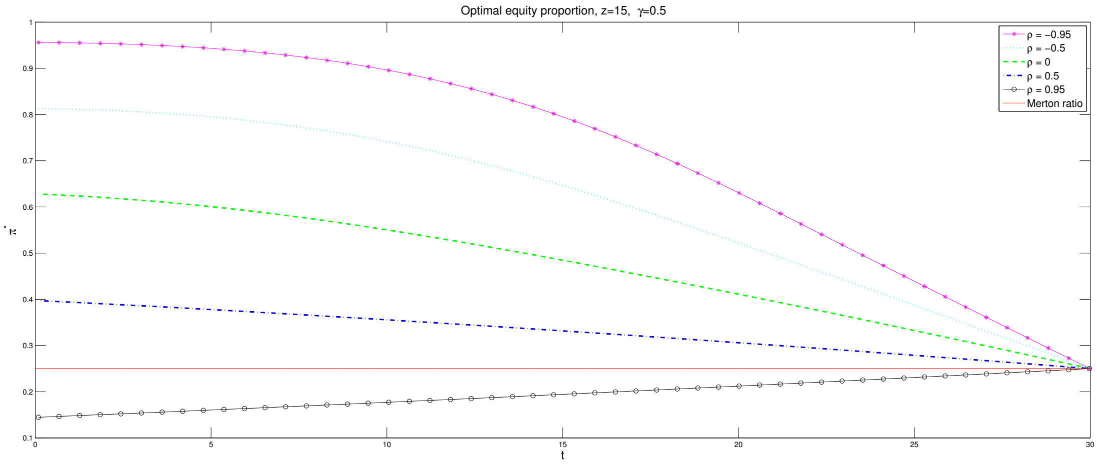
also increasing in time in some cases, depending on the correlation of the contribution process with the market assets.

If we compare now Figure 2 and Figure 3 for $\gamma = -1$ and $\gamma = 0.5$ respectively (i.e. RRA $= 2$ and 0.5, respectively), the typical glide path structure of TDFs can be better justified for less risk-averse agents, because there is a bigger region of ρ's where a time-decreasing equity holding results as the optimal solution ($\rho^* \approx 0.1923$ in the case of Figure 2 and $\rho^* \approx 0.7692$ in the case of Figure 3). This is a direct consequence of Proposition 4.1, since the critical value of the correlation for which the optimal strategy concides with the Merton ratio is decreasing in the RRA (cf. Equation (4.3)).
Also, the volatility of the stock plays a fundamental role: for a stock with a higher volatility, we observe a conservative behavior for a risk-averse investor (cf. Figure 4) already for smaller values of the correlation.

4.2. Human capital

Let us now briefly analyze the effect of human capital on the investment choices. Usually, human capital is understood as discounted value of future wages, social security, and other benefits. The total wealth of a person will then include both his

Figure 3: Optimal equity proportion in the model of Section 4 for different values of the correlation parameter ρ and positive γ.
Parameters: $T = 30$ years, and $\sigma = 0.4$, $\mu = 0.04$, $\sigma_C = 0.13$, $\mu_C = 0.02$, $r = 0.02$, $\gamma = 0.5$, $z = 15$.



current financial assets and his human capital.

In the following, we compare the behavior of a young and an old investor with the same (relative) risk aversion and the same working category (in the sense that their salary has the same correlation with the market assets). We distinguish between the two investors by choosing a different initial value $z = x/y$ and a different contract duration T. A higher z means a higher initial capital compared to the initial contribution. The young investor still needs to work for another 30 years until retirement and starts with a lower initial $z = 5$, while the old investor only needs to work for another 10 years and has an initial $z = 20$. We can consider the scenario where the young investor chooses the Target Date Fund 2045 and the old one the Target Date Fund 2025. The resulting optimal equity holdings are plotted in Figures 4 and 5 respectively.

Comparing the two blue curves (corresponding to $\sigma = 0.2$) in these two graphics, we obtain some "expected" observations: Young investors have a longer time to work and have not had much time to accumulate wealth. The ability to work (human capital) is therefore their largest asset. Older investors have already converted most of their human capital to financial capital. In this sense, young investors can

Figure 4: Optimal equity proportion in the model of Section 4 for a young investor.
Here **z=5, T=30** years, $\rho = 0.25$ and $\sigma = 0.2, 0.4$, $\mu = 0.04$, $\sigma_C = 0.13$, $\mu_C = 0.02$, $r = 0.02$, $\gamma = -1$.

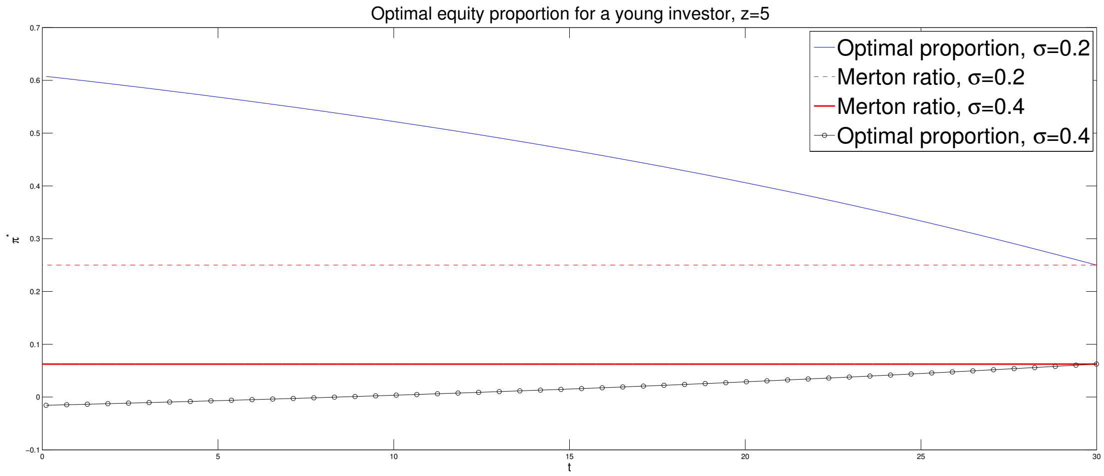
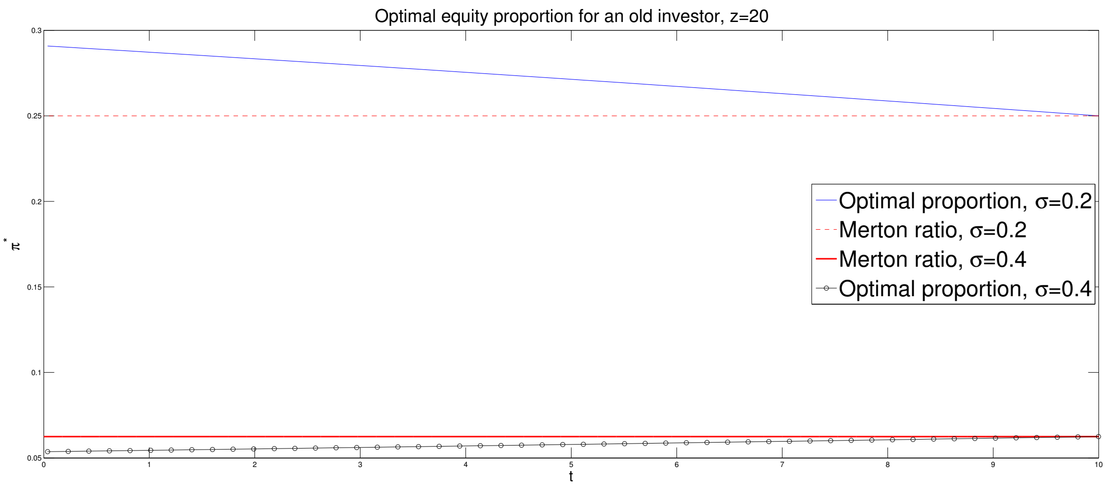


Figure 5: Optimal equity proportion in the model of Section 4 for an old investor.
Here **z=20, T=10** years, $\rho = 0.25$ and $\sigma = 0.2, 0.4$, $\mu = 0.04$, $\sigma_C = 0.13$, $\mu_C = 0.02$, $r = 0.02$, $\gamma = -1$.



borrow from their future income to invest more in the risky asset, which leads to a higher equity holding. Sometimes there is another interpretation to human capital: Young and old investors with the same relative risk aversion want to achieve the same "ideal" portfolio (in our case a certain mixture of the risky and risk free asset). Since the ability to work is not affected by market risk, human capital is frequently considered more like the risk free asset. Young investors, which have a lot of human capital (i.e. risk free asset), need to invest more in equity to achieve this ideal portfolio. On the other side, old investors have little human capital and will consequently invest less in equity to achieve this ideal portfolio.

Note that the above-mentioned "expected" result can be only achieved for some specific choices of the parameters. In many situations, the validity of the argument will be violated, e.g. by choosing a higher correlation coefficient between the market and income risk or a different stock volatility. In our example, we exhibit how the investment behavior of the young and old investor will change when we move from $\sigma = 0.2$ to $\sigma = 0.4$. The resulting optimal equity holding for the young and old investor for the more volatile risky asset are shown by the dotted curves in Figures 4 and 5. Both investors' investment behavior does not indicate a glide path and suggests a lower equity holding than the Merton portfolio. In particular, the young investor rich in human capital shall optimally invest less in the risky asset than the older one. This example also shows that the optimal equity holding can demonstrate a rising glide path (dotted curves corresponding to $\sigma = 0.4$ in Figures 4 and 5) also for moderate values of the correlation. Our results go therefore beyond the conjectures of Jagannathan & Kocherlakota (1996) by stressing the dependence of this inverse behavior also on other parameters of the model, like relative risk aversion and volatility of the stock.

5. Conclusion

Given the rapidly increasing popularity of target date funds, the present paper aims to find a rigorous theoretical foundation to justify their policy, and to find out whether they can help the DC beneficiaries to effectively manage the investment risks of their pension plans. Motivated by the qualitative results provided by Jagannathan & Kocherlakota (1996) to verify the hypothesis "young people shall invest more in risky stocks", we come up with a more realistic continuous-time model setup to examine the essence of TDFs. Compared to their paper, our model extends their qualitative analysis to a more quantitative one, and allows us to examine the effect of different time horizons. We designed two different contribution rules to describe the contributions flowing to the 401(k) plans, both of which depend on salary. When the contribution process is a function of salary only, say a constant fraction of the salary, the resulting optimal strategy is in most cases a time-decreasing equity holding, i.e. the common policy used in TDFs is justified. However, in some cases, e.g. when the asset and salary are highly positively correlated, the optimal strategy could be a time-increasing equity holding, which suggests that older beneficiaries shall invest more in equity.

Appendix A. Computing the strategy 1. $f(A_t^\pi, I_t) = \lambda(I_t) \cdot A_t^\pi$

If we consider contribution rules of the form: $f(x, y) = \lambda(y)x$, where λ is exclusively a function of y, it is possible to reduce the dimension of the HJB equation. Assume indeed that the contribution rule f is of the type $f(x, y) = \lambda(y)x$, i.e. the contributions are proportional to the fund value. Thinking of the basic case of the Merton portfolio allocation problem, we could try to solve the HJB equation by making an ansatz for the value function of the type $v(t, x, y) = \phi(t, y)U(x)$ to get rid of the dependence on x. Recalling that in the power utility case

$$\frac{xU'(x)}{U(x)} = \gamma, \quad \frac{x^2U''(x)}{U(x)} = \gamma(\gamma - 1),$$

we get

$$\begin{aligned}
\phi_t(t, y) = -\sup_{\pi\in\mathscr{A}} \Big\{ &(\pi\sigma\theta + r)\gamma\phi(t, y) + \gamma\rho\sigma\sigma_I(t, y)\phi_y(t, y) + \qquad \text{(A.1)}\\
&\gamma(\gamma - 1)\frac{1}{2}(\pi\sigma)^2\phi(t, y) + \gamma\rho\sigma\sigma_I(t, y)\pi\phi_y(t, y)\Big\}\\
&-\frac{\gamma f(x, y)}{x}\phi(t, y) - \mu_I(t, y)\phi_y(t, y) - \frac{1}{2}\sigma_I(t, y)^2\phi_{yy}(t, y). \qquad \text{(A.2)}
\end{aligned}$$

This yields an independent two-dimensional PDE[9].

Now, in the setting described in Example 3.1 we can directly take Λ as the second component of the controlled process instead of I, and the associated HJB equation reads:

$$-v_t = \sup_{\pi\in\mathscr{A}}\Big\{\pi\sigma\theta x v_x + \frac{1}{2}(\pi\sigma x)^2 v_{xx} + \rho\sigma\sigma_\Lambda\pi x v_{xy}\Big\} + (r+y)xv_x + \kappa_\Lambda(\mu_\Lambda - y)v_y + \frac{1}{2}\sigma_\Lambda^2 v_{yy}.$$

The ansatz approach works in this case, and enables us to reduce the previous equation by one dimension. Deriving a reduced equation as in (A.2), it is then

[9] It is apparent that such an ansatz cannot work if the dependence on x of the function f is of a different type (e.g. constant in x).

enough to find a function $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\phi \in \mathcal{C}^{1,2}$, such that

$$- \phi_t = \sup_{\pi \in \mathscr{A}} \left\{ \pi\sigma\gamma(\theta\phi + \rho\sigma_\Lambda\phi_y) - \frac{1}{2}\pi^2\sigma^2\gamma(1-\gamma)\phi \right\} + (r+y)\gamma\phi + \kappa_\Lambda(\mu_\Lambda - y)\phi_y + \frac{1}{2}\sigma_\Lambda^2\phi_{yy}, \tag{A.3}$$

$\phi(T, y) = 1, \quad \forall y \in \mathbb{R}.$

Taking a look at the supremum in this equation, it is easy to see that if the function ϕ depends exponentially on y, the structure simplifies. This leads to the ansatz:

$$\phi(t, y) = \exp\left\{\alpha(t)y + \beta(t)\right\},$$

for $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$, $\mathcal{C}^1-$functions such that $\alpha(T) = \beta(T) = 0$. Computing the derivatives, we have

$$\begin{aligned} \phi_t(t, y) &= \phi(t, y)(\alpha'(t)y + \beta'(t)) \\ \phi_y(t, y) &= \phi(t, y)\alpha(t) \\ \phi_{yy}(t, y) &= \phi(t, y)(\alpha(t))^2. \end{aligned}$$

Substituting into the equation of ϕ the above expressions, we get for the optimal strategy

$$\pi^* = \frac{\theta\phi + \rho\sigma_\Lambda\phi_y}{\sigma(1-\gamma)\phi} = \frac{\theta}{\sigma(1-\gamma)} + \frac{\rho\sigma_\Lambda\phi_y}{\sigma(1-\gamma)\phi}. \tag{A.4}$$

Using now that the function ϕ is strictly positive, it is possible to factorize the dependence on ϕ in Equation (A.3), and a polynomial of first degree in y is obtained. For the equation to be satisfied, the coefficient of y as well as the constant term in the equation above must be zero, and this condition yields the following two (integrable) ODEs for α and β:

$$\begin{cases} -\alpha'(t) = & \gamma - \kappa_\Lambda\alpha(t), \qquad \forall t \in [0, T] \\ \alpha(T) = & 0, \end{cases}$$

$$\begin{cases} -\beta'(t) = & \frac{\gamma(\theta+\rho\sigma_\Lambda\alpha(t))^2}{2(1-\gamma)} + r\gamma + \kappa_\Lambda\mu_\Lambda\alpha(t) + \frac{1}{2}\sigma_\Lambda^2\alpha^2(t), \qquad \forall t \in [0, T] \\ \beta(T) = & 0, \end{cases}$$

where the last expression derives from substituting (A.4) in the equation for ϕ.

If we recall that their solutions are given by:

$$
\begin{aligned}
\alpha(t) =& \frac{\gamma}{\kappa_\Lambda}\left(1-e^{-\kappa_\Lambda(T-t)}\right),\\
\beta(t) =& \left[\frac{\gamma}{2(1-\gamma)}\left(\theta+\rho\sigma_\Lambda\frac{\gamma}{\kappa_\Lambda}\right)^2 + r\gamma + \mu_\Lambda\gamma + \frac{1}{2}\sigma_\Lambda^2\frac{\gamma^2}{\kappa_\Lambda^2}\right](T-t)+\\
& \left[\frac{\gamma}{1-\gamma}\left(\theta+\rho\sigma_\Lambda\frac{\gamma}{\kappa_\Lambda}\right)\rho\sigma_\Lambda\frac{\gamma}{\kappa_\Lambda} + \mu_\Lambda\gamma + \sigma_\Lambda^2\frac{\gamma^2}{\kappa_\Lambda^2}\right]\frac{e^{-\kappa_\Lambda(T-t)}-1}{\kappa_\Lambda}+\\
& \left[-\frac{\gamma}{2(1-\gamma)}\rho^2\sigma_\Lambda^2\left(\frac{\gamma}{\kappa_\Lambda}\right)^2 - \frac{1}{2}\sigma_\Lambda^2\frac{\gamma^2}{\kappa_\Lambda^2}\right]\frac{e^{-2\kappa_\Lambda(T-t)}-1}{2\kappa_\Lambda}.
\end{aligned}
$$

we can recover the optimal strategy, which is given in Equation (3.2). A standard verification argument yields then that the value function is given by $\phi(t,y)U(x)$ and the optimal strategy is as in Equation (A.4).

Appendix B. Computing the strategy 2. $f(A_t^\pi, I_t) = f(I_t)$

If we now take as controlled process the pair $X^\pi = (A^\pi, c)$, we obtain, exactly as in Appendix Appendix A, the HJB equation in terms of the parameters describing the dynamics of the contribution process:

$$
\begin{aligned}
-v_t(t,x,y) = \sup_{\pi\in\mathscr{A}} \Big\{ & [x(\pi\sigma\theta + r) + y]\, v_x(t,x,y) + \mu_C(t,y)v_y(t,x,y)\\
& + \frac{1}{2}(\pi\sigma x)^2 v_{xx}(t,x,y) + \frac{1}{2}\sigma_C(t,y)^2 v_{yy}(t,x,y) + \rho\sigma\sigma_C(t,y)\pi x v_{xy}(t,x,y)\Big\},
\end{aligned}
\tag{B.1}
$$

$$
v(T,x,y) = U(x), \quad \forall (x,y).
$$

To solve this optimal asset allocation problem, Chen, Mereu & Stelzer (2014) rely on the properties of the value function, particularly homogeneity, to reduce the HJB equation by one dimension and to make the optimization problem thus well solvable through numerical methods. More concretely, the solution to the reduced problem is

Theorem Appendix B.1 (Chen, Mereu & Stelzer (2014), Theorem 2.4)**.** *The value function of the control problem* (2.5) *is given by*

$$
v(t,x,y) = y^\gamma u\left(t, \frac{x}{y}\right), \quad \forall (t,x,y) \in [0,T) \times (0,+\infty) \times (0,+\infty),
$$

where $u : [0,T) \times \mathbb{R}^+ \to \mathbb{R}$ is the unique viscosity solution with polynomial growth at infinity to the following equation

$$
\begin{aligned}
&u_t + u_z + \mu_C(t)\left[\gamma u - z u_z\right] + \frac{1}{2}\sigma_C^2(t)\left[\gamma(\gamma-1)u - 2(\gamma-1)z u_z + z^2 u_{zz}\right] + \qquad \text{(B.2)}\\
&\sup_{\pi \in \mathscr{A}}\left\{(\pi\sigma\theta + r)z u_z + \frac{1}{2}(\pi\sigma)^2 z^2 u_{zz} + \rho\sigma_C(t)\sigma\pi(\gamma-1)z u_z - \rho\sigma\sigma_C(t)\pi z^2 u_{zz}\right\} = 0,
\end{aligned}
$$

Furthermore, the optimal strategy is given by $\Pi_t = h\left(t, A_t^{\Pi}, c_t\right)$ where

$$
h(t,x,y) := \frac{\theta}{\sigma \cdot \frac{-u_{zz}(t,\frac{x}{y})}{u_z(t,\frac{x}{y})}\frac{x}{y}} - \frac{\rho\sigma_C(t)}{\sigma}\left(\frac{1-\gamma}{\frac{-u_{zz}(t,\frac{x}{y})}{u_z(t,\frac{x}{y})}\frac{x}{y}} - 1\right), \quad \textit{if it belongs to } A \textit{ a.e.}
$$

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