Central limit theorems for stationary random fields under weak dependence with application to ambit and mixed moving average fields

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Abstract

We obtain central limit theorems for stationary random fields which are based on the use of a novel measure of dependence called θ-lex weak dependence. We discuss hereditary properties for θ-lex and η-weak dependence and illustrate the possible applications of the weak dependence notions to the study of the asymptotic properties of stationary random fields. Our general results are applied to mixed moving average fields (MMAF in short) and ambit fields. We show general conditions such that MMAF and ambit fields, with the volatility field being an MMAF or a p-dependent random field, are weakly dependent. For all the aforementioned models, we give a complete characterization of their weak dependence coefficients and sufficient conditions to obtain asymptotic normality of their sample moments. Finally, we give explicit computations of the weak dependence coefficients in the case of MSTOU and CARMA fields.

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1 Introduction

Many modern statistical applications consider the modeling of phenomena evolving in time and/or space with either a countable or uncountable index set. To this end, we can employ random fields on $\mathbb{Z}^m$ or $\mathbb{R}^m$ which are defined, for example, as solutions of recurrence equations, e.g. in [34], or stochastic partial differential equations [16, 25, 51]. Noticeable examples of the latter come from the class of ambit and mixed moving average fields.

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The mixed moving average fields, MMAF in short, are defined as

$$X_t = \int_S \int_{\mathbb{R}^m} f(A, t-s)\, \Lambda(dA, ds), \quad t \in \mathbb{R}^m, \tag{1.1}$$

where A is a random parameter with values in a Polish space S, f a deterministic function called kernel and Λ a Lévy basis. The above model encompasses Gaussian and non-Gaussian random fields by choosing the Lévy basis Λ. Ambit fields are defined by considering an additional multiplicative random function in the integrand of (1.1) which is called volatility or intermittency field. However, an ambit field is typically defined without allowing the presence of the random parameter A in its kernel function. We refer the reader to [6] for a comprehensive introduction to ambit fields which provide a rich class of spatial-temporal models on $\mathbb{R} \times \mathbb{R}^m$. Overall, MMAF and ambit fields are used in many applications throughout different disciplines, like geophysics [38], brain imaging [40], physics [11], biology [8, 10], economics and finance [3, 5, 22, 46, 47].

The generality and flexibility of these models motivate an in-depth analysis of their properties. If we consider purely temporal ambit fields, i.e. Lévy semistationary processes, in [7, 9, 14] the authors obtain infill asymptotic results for this class of processes, that is, under the assumption that the number of observations in a given interval approaches to infinity. For ambit fields on $\mathbb{R} \times \mathbb{R}^m$ with $m \geq 1$ where Λ is of Gaussian type and the volatility field is independent of Λ, the asymptotic behavior of the lattice power variation of the field is studied in [48]. We notice that in the literature there are no asymptotic results for partial sums of ambit fields when the number of observations approaches to infinity without infill asymptotics. If we look instead at the class of MMAF, we have several results existing in this direction.

In a stationary framework, we could attempt to show that an MMAF is strongly mixing or associated. Under strong mixing conditions several central limit theorems for strictly stationary random fields are available, see [17], [24], [27], [43], [44] (note that caution has to be used when handling some of the classical concepts of mixing for random fields, see [19] and Chapter 29 in [20]). If we then look at an MMAF on $\mathbb{R}$, i.e. a mixed moving average process, several difficulties already arise in showing that it is strongly mixing, see [26]. Usually, strong mixing is established by using a Markovian representation and showing geometric ergodicity of it. In turn this often requires smoothness conditions on the driving random noise and it is well-known that even autoregressive processes of order one are not strongly mixing when the distribution of the noise is not sufficiently regular, see [1]. Since we are interested in central limit results which do not require heavy assumptions on Λ or a Markovian representation, a different measure of dependence is required. Gaussian MMAF on $\mathbb{R}^m$ for $m \geq 2$ that satisfy the conditions of Theorem 7, pg. 73 in [53] are α mixing. However, for general driving Lévy bases no results in the literature can be found regarding the strong mixing of MMAF. On the other hand, if we look at the concept of association, a powerful measure of dependence which allows sharp central limit results (see [23, 45] for a comprehensive introduction on this topic), we are capable of obtaining central limit theorems for MMAF just under restrictive conditions on the kernel function f in (1.1), see e.g. Theorem 3.27 in [23]. Moreover, association is inherited only under monotone functions which restricts the possible extension of its related asymptotic theory. At last, we mention the results in [15] where the author shows asymptotic normality of the sample mean and sample autocovariance of a moving average field, a sub-class of the MMAF. This line of proof is not directly applicable to the study of higher order sample moments and therefore we do not pursue it.

We are interested in studying asymptotics of the partial sums (and of higher sample moments) of MMAF and ambit fields in general, i.e. without imposing regularity conditions on the driving Lévy basis Λ apart from moment conditions. To do so, we define a new notion of dependence called θ-lex weak dependence. This tool can overcome the bottlenecks identified in the literature above. We want to emphasize that, although all the examples of our theory will be taken from the aforementioned model classes, we present general central limit theorem results which can be applied to different stationary random fields.

In order to introduce θ-lex weak dependence, we first present the notions of η and θ-weak dependence introduced for stochastic processes in [32] and [29], respectively. η-weak dependence is typically associated with the study of non-causal processes whereas θ-weak dependence is connected to the analysis of the causal ones. Central limit theorems for θ-weakly dependent processes hold under weaker conditions compared to results for η-weakly dependent processes (different demands on the decay rate of the η- and θ-coefficients as determined in Theorem 2.2 [35] and Theorem 2 [29]). We have that the definition of η- and θ-weak dependence can be easily extended to the field case by following Remark 2.1 [30]. However, just for η-weakly dependent random fields, asymptotics of the partial sums of the process X have been so far analyzed in [33]. We aim to determine a central limit theorem which improves the results obtained in [33]. This is achieved by defining the notion of θ-lex-weak dependence which is a modification of the original definition of θ-weak dependence. In fact, we can show that for θ-lex-weakly dependent random fields the sufficient conditions of a very powerful central limit theorem from Dedecker [27] hold. Moreover, we obtain hereditary properties for θ-lex and η-weakly dependent random fields which allow us to easily extend the asymptotic results under weak dependence to the study of sample moments estimators.

We then look at the class of MMAF. We distinguish in our theory between influenced and non-influenced MMAF, see Definition 3.8. Influenced MMAF represent a possible extension of causal mixed moving average processes (Section 3.2 [26]) to random fields. Hence, we show that influenced MMAF are θ-lex-weakly dependent and that non-influenced MMAF are η-weakly dependent with coefficients computable in terms of the kernel function f and the characteristic quadruplet of the Lévy basis Λ. From this we notice that in the case of influenced MMAF the conditions ensuring asymptotic normality of the partial sums of X are weaker– in terms of the decay rate of the weak dependence coefficients– in comparison with the one obtained for non-influenced MMAF. We then observe a parallel between our results and the one obtained for causal and non-causal mixed moving average processes [26]. Moreover, we exploit the hereditary properties of η- as well as θ-lex-weak dependence and obtain conditions for the sample moments of order p with $p \geq 1$ to be asymptotic normally distributed. Finally, we give explicit computations for mixed spatio-temporal Ornstein-Uhlenbeck processes [47], also called MSTOU processes, and Lévy-driven CARMA fields [22, 51]. In particular, our calculations in the case of the

MSTOU processes show that it is possible to determine the asymptotic normality of the generalized method of moments estimator, GMM in short, proposed in [47].

At last, we apply our theory to ambit fields. We assume that the volatility field is an MMAF or a $p-$dependent random field which is independent of the Lévy basis Λ. Under these assumptions, we show that homogeneous and stationary ambit fields are θ-lex-weakly dependent and give sufficient conditions on the θ-lex-coefficients to ensure asymptotic normality of the sample moments.

The paper is structured as follows. In Section 2, we introduce η-weak dependence and the novel θ-lex-weak dependence. In Section 2.2 we state central limit theorems for θ-lex weakly dependent random fields in an ergodic, non-ergodic and multivariate setting. Additionally, we provide some insight into possible functional extensions of the theorem. In Section 3, we discuss weak dependence properties of MMAF. We first give a comprehensive introduction to Lévy bases and its related integration theory which leads to the formal definition of an MMAF. We discuss conditions on MMAF to be η or θ-lex-weakly dependent and their related sample moment asymptotics. In Section 3.7, we apply the developed theory to MSTOU processes and give explicit conditions assuring the asymptotic normality of their sample moments under a Gamma distributed mean reversion parameter. We conclude Section 3 with an application of the theory to Lévy-driven CARMA fields. In Section 4, we discuss weak dependence properties and related limit theorems for ambit fields. Section 5 contains the detailed proofs of most of the results presented in the paper.

2 Weak dependence and central limit theorems

We assume that all random elements in this paper are defined on a given complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$. By $\mathbb{N}$ we denote the set of non-negative integers, $\mathbb{N}^*$ denotes the set of positive integers, and $\mathbb{R}^+$ the set of the non-negative real numbers. For $x \in \mathbb{R}^d$ we define $|x| = \|x\|_\infty = \max_{j=1,\ldots,d} |x^{(j)}|$ and $\|x\|$ denotes the Euclidean norm of x. For a function $F : \mathbb{R}^d \to \mathbb{R}^k$ we define $\|F\|_\infty = \sup_{t\in\mathbb{R}^d} \|F(t)\|$ and by $\|X\|_p$ for $p > 0$ we denote the L^p-norm of a random vector X. In the following Lipschitz continuousis understood to mean globally Lipschitz. For a random field $X = (X_t)_{t\in\mathbb{R}^m}$ and a finite set $\Gamma \subset \mathbb{R}^m$ with $\Gamma = (i_1, \ldots, i_u)$ we define the vector $X_\Gamma = (X_{i_1}, \ldots, X_{i_u})$. Finally, $A \subset B$ denotes a not necessarily proper subset A of a set B and $|B|$ denotes the cardinality of B.

2.1 Weak dependence properties

For $u, n \in \mathbb{N}^*$, let $\mathcal{F}_u^*$ be the class of bounded functions from $(\mathbb{R}^n)^u$ to $\mathbb{R}$ and $\mathcal{F}_u$ be the class of bounded, Lipschitz continuous functions from $(\mathbb{R}^n)^u$ to $\mathbb{R}$ with respect to the distance δ_1 on $(\mathbb{R}^n)^u$ defined by

$$\delta_1((x_1, \ldots, x_u), (y_1, \ldots, y_u)) = \sum_{i=1}^{u} \delta(x_i, y_i),$$

where δ is the Euclidean norm on $\mathbb{R}^n$ such that $\delta(x_i, y_i) = \|x_i - y_i\|$. Now define

$$\mathcal{F} = \bigcup_{u \in \mathbb{N}^*} \mathcal{F}_u, \quad \mathcal{F}^* = \bigcup_{u \in \mathbb{N}^*} \mathcal{F}_u^*,$$

and for $G \in \mathcal{F}_n^*$

$$Lip(G) = \sup_{x \neq y} \frac{|G(x) - G(y)|}{\|x_1 - y_1\| + \ldots + \|x_u - y_u\|}$$

Definition 2.1 ([30, Definition 2.2 and Remark 2.1])**.** *Let $X = (X_t)_{t \in \mathbb{R}^m}$ be an $\mathbb{R}^n$-valued random field. Then, X is called η-weakly dependent if*

$$\eta(h) = \sup_{u,v \in \mathbb{N}^*} \eta_{u,v}(h) \underset{h \to \infty}{\longrightarrow} 0,$$

where

$$\eta_{u,v}(h) = \sup \Bigg\{ \frac{|Cov(F(X_\Gamma), G(X_{\tilde{\Gamma}}))|}{u\|G\|_\infty Lip(F) + v\|F\|_\infty Lip(G)}, \\ F, G \in \mathcal{F}, \Gamma, \tilde{\Gamma} \subset \mathbb{R}^m, |\Gamma| = u, |\tilde{\Gamma}| = v, dist(\Gamma, \tilde{\Gamma}) \geq h \Bigg\},$$

with $dist(\Gamma, \tilde{\Gamma}) = \inf_{i \in \Gamma, j \in \tilde{\Gamma}} \|i - j\|_\infty$. We call $(\eta(h))_{h \in \mathbb{R}^+}$ the η-coefficients.

In the following we consider the lexicographic order on $\mathbb{R}^m$, i.e. for distinct elements $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ and $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ we say $y <_{lex} z$ if and only if $y_1 < z_1$ or $y_p < z_p$ and $y_q = z_q$ for some $p \in \{2, \ldots, m\}$ and $q = 1, \ldots, p-1$. Furthermore, we say $y \leq_{lex} z$ if $y <_{lex} z$ or $y = z$ holds. Let us define the sets $V_t = \{s \in \mathbb{R}^m : s <_{lex} t\} \cup \{t\}$ and $V_t^h = V_t \cap \{s \in \mathbb{R}^m : \|t - s\|_\infty \geq h\}$ for $h > 0$. The same definitions of the sets V_t and V_t^h are going to be used when referring to the lexicographic order on $\mathbb{Z}^m$.

Definition 2.2. *Let $X = (X_t)_{t \in \mathbb{R}^m}$ be an $\mathbb{R}^n$-valued random field. Then, X is called θ-lex-weakly dependent if*

$$\theta(h) = \sup_{u \in \mathbb{N}^*} \theta_u(h) \underset{h \to \infty}{\longrightarrow} 0,$$

where

$$\theta_u(h) = \sup \Bigg\{ \frac{|Cov(F(X_\Gamma), G(X_j))|}{\|F\|_\infty Lip(G)}, \\ F \in \mathcal{F}^*, G \in \mathcal{F}, j \in \mathbb{R}^m, \Gamma \subset V_j^h, |\Gamma| = u \Bigg\}.$$

We call $(\theta(h))_{h \in \mathbb{R}^+}$ the θ-lex-coefficients.

Remark 2.3. *Our definition of θ-lex-weak dependence differs from the θ-weak dependence definition given in Remark 2.1 [30]. In fact, instead of considering the covariance of two arbitrary finite dimensional samples X_Γ and $X_{\tilde{\Gamma}}$, for $\Gamma, \tilde{\Gamma} \in \mathbb{R}^m$, we control the covariance of a finite dimensional sample X_Γ and an arbitrary one point sample X_j. Secondly, by assuming that all points in the sampling set Γ are lexicographically smaller than j, we provide an order in the sampling scheme.*

Let $(X_t)_{t\in\mathbb{R}^m}$ be θ-lex- or η-weakly dependent and $h:\mathbb{R}^n\to\mathbb{R}^k$ be an arbitrary Lipschitz function, then the field $(h(X_t))_{t\in\mathbb{R}^m}$ is also θ-lex- or η-weakly dependent. The latter can be readily checked based on Definition 2.1 and 2.2. In the next proposition, we give conditions for hereditary properties of functions that are only locally Lipschitz continuous. The proof of the result below is analogous to Proposition 3.2 [26].

Proposition 2.4. *Let $X=(X_t)_{t\in\mathbb{R}^m}$ be an $\mathbb{R}^n$-valued stationary random field and assume that there exists a constant $C>0$ such that $E[\|X_0\|^p]\le C$, for $p>1$. Let $h:\mathbb{R}^n\to\mathbb{R}^k$ be a function such that $h(0)=0, h(x)=(h_1(x),\dots,h_k(x))$ and*

$$\|h(x)-h(y)\|\le c\|x-y\|(1+\|x\|^{a-1}+\|y\|^{a-1}),$$

for $x,y\in\mathbb{R}^n$, $c>0$ and $1\le a<p$. Define $Y=(Y_t)_{t\in\mathbb{R}^m}$ by $Y_t=h(X_t)$. If X is η or θ-lex-weakly dependent, then Y is η or θ-lex-weakly dependent respectively with coefficients

$$\eta_Y(h)\le\mathcal{C}\eta_X(h)^{\frac{p-a}{p-1}}\quad or\ \theta_Y(h)\le\mathcal{C}\theta_X(h)^{\frac{p-a}{p-1}}$$

for all $h>0$ and a constant $\mathcal{C}$ independent of h.

2.2 Central limit theorems for θ-lex-weakly dependent random fields

In the theory of stochastic processes one of the typical ways to prove central limit type results is to approximate the process of interest by a sequence of martingale differences. This approach was first introduced by Gordin [36]. However, the latter does not apply to high-dimensional random fields as successfully as to processes. This unpleasant circumstance has been known among researchers for almost 40 years, as Bolthausen [17] noted that martingale approximation appears a difficult concept to generalize to dimensions greater or equal than two.

For stationary random fields $X=(X_t)_{t\in\mathbb{Z}^m}$, Dedecker derived a central limit result in [27] under the projective criterion

$$\sum_{k\in V_0^1}|X_kE[X_0|\mathcal{F}_{\Gamma(k)}]|\in L^1,\ \text{ for } \mathcal{F}_{\Gamma(k)}=\sigma(X_k:k\in V_0^{|k|}).\tag{2.1}$$

This condition is weaker than martingale-type assumptions and provides optimal results for mixing random fields. We show in this section that (2.1) is also fulfilled by appropriate θ-lex-weakly dependent random fields.

In the following by stationarity we mean stationarity in the strict sense. Let Γ be a subset of $\mathbb{Z}^m$. We define $\partial\Gamma=\{i\in\Gamma:\exists j\notin\Gamma:\|i-j\|_\infty=1\}$. Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of $\mathbb{Z}^m$ such that

$$\lim_{n\to\infty}|D_n|=\infty \text{ and } \lim_{n\to\infty}\frac{|\partial D_n|}{|D_n|}=0.$$

Theorem 2.5. *Let $X = (X_t)_{t\in\mathbb{Z}^m}$ be a stationary centered real-valued random field such that $E[|X_t|^{2+\delta}] < \infty$ for some $\delta > 0$. Additionally, assume that $\theta(h) \in \mathcal{O}(h^{-\alpha})$ with $\alpha > m(1+\frac{1}{\delta})$. Define*

$$\sigma^2 = \sum_{k\in\mathbb{Z}^m} E[X_0 X_k|\mathcal{I}],$$

where $\mathcal{I}$ is the σ-algebra of shift invariant sets as defined in [27, Section 2] (see [41, Chapter 1] for an introduction to ergodic theory). Then, σ^2 is finite, non-negative and

$$\frac{1}{|D_n|^{\frac{1}{2}}} \sum_{j\in\Gamma_n} X_j \xrightarrow[n\to\infty]{d} \varepsilon\sigma, \tag{2.2}$$

where ε is a standard normally distributed random variable which is independent of σ^2.

Proof. See Section 5.1. ∎

In the following we give an ergodic multivariate extension of the previous theorem.

Corollary 2.6. *Let $X = (X_t)_{t\in\mathbb{Z}^m}$ be a stationary ergodic centered $\mathbb{R}^n$-valued random field such that $E[\|X_t\|^{2+\delta}] < \infty$ for some $\delta > 0$. Additionally, let us assume that $\theta(h) \in \mathcal{O}(h^{-\alpha})$ with $\alpha > m(1+\frac{1}{\delta})$. Then*

$$\Sigma = \sum_{k\in\mathbb{Z}^m} E[X_0 X_k'],$$

is finite, positive definite and

$$\frac{1}{|D_n|^{\frac{1}{2}}} \sum_{j\in\Gamma_n} X_j \xrightarrow[n\to\infty]{d} N(0,\Sigma),$$

where $N(0,\Sigma)$ denotes the multivariate normal distribution with mean 0 and covariance matrix Σ.

Proof. First, the univariate result follows directly from Theorem 2.5, since X is ergodic. Now let X be multivariate. Since linear functions are Lipschitz we note that for all $a \in \mathbb{R}^n$, $a'X_t$ is a θ-lex-weakly dependent field with θ-lex-coefficients smaller or equal to those of X_t. Then

$$\frac{1}{|D_n|^{\frac{1}{2}}} \sum_{j\in\Gamma_n} a'X_j \xrightarrow[n\to\infty]{d} N(0,a'\Sigma a).$$

Applying the Cramér-Wold device, the asymptotic normality of the sample mean follows immediately. ∎

Remark 2.7. *It is natural to ask for conditions on a functional extension of Theorem 2.5. As a matter of fact, results of this kind are strongly related to the following L^p-projective criterion*

$$\sum_{k \in V_0} E[|X_k E[X_0|\mathcal{F}_{V_0^{|k|}}]|^p] < \infty, \; p \in [1, \infty], \tag{2.3}$$

where $\mathcal{F}_\Gamma = \sigma(X_k, k \in \Gamma)$. If (2.3) holds for $p = 1$, then [27, Theorem 1] provides a non-functional central limit theorem for stationary random fields which yields Theorem 2.5. Now, possible functional extensions depend on the dimension of the domain of the random field.
When $m = 1$, Dedecker and Rio showed in [31, Theorem] that if (2.3) holds for $p = 1$, then a functional central limit theorem holds.
In the general case $m > 1$, Dedecker proved in [28, Theorem 1] a functional central limit theorem if (2.3) holds for $p > 1$.
Since we can establish the connection between the L^p-projective criterion (2.3) and the summability condition of the θ-lex-coefficients of X just for $p = 1$, there is no functional extension of Theorem 2.5 readily obtainable, except for $m = 1$ (see [26, Remark 4.2]).

3 Mixed moving average fields

In this section we first introduce MMAF driven by a Lévy basis. Then, we discuss weak dependence properties of such MMAF and derive sufficient conditions such that the asymptotic results of Section 2.2 apply.

3.1 Preliminaries

Let S denote a non-empty polish space, $\mathcal{B}(S)$ the Borel σ-algebra on S, π some probability measure on $(S, \mathcal{B}(S))$ and $\mathcal{B}_b(S \times \mathbb{R}^m)$ the bounded Borel sets of $S \times \mathbb{R}^m$.

Definition 3.1. *Consider a family $\Lambda = \{\Lambda(B), B \in \mathcal{B}_b(S \times \mathbb{R}^m)\}$ of $\mathbb{R}^d$-valued random variables. Then Λ is called an $\mathbb{R}^d$-valued Lévy basis or infinitely divisible independently scattered random measure on $S \times \mathbb{R}^m$ if*

(i) the distribution of $\Lambda(B)$ is infinitely divisible (ID) for all $B \in \mathcal{B}_b(S \times \mathbb{R}^m)$,

(ii) for arbitrary $n \in \mathbb{N}$ and pairwise disjoint sets $B_1, \ldots, B_n \in \mathcal{B}_b(S \times \mathbb{R}^m)$ the random variables $\Lambda(B_1), \ldots, \Lambda(B_n)$ are independent and

(iii) for any pairwise disjoint sets $B_1, B_2, \ldots \in \mathcal{B}_b(S \times \mathbb{R}^m)$ with $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}_b(S \times \mathbb{R}^m)$ we have, almost surely, $\Lambda(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \Lambda(B_n)$.

In the following we will restrict ourselves to Lévy bases which are homogeneous in space and time and factorisable, i.e. Lévy bases with characteristic function

$$\varphi_{\Lambda(B)}(u) = E\left[e^{\mathrm{i}\langle u, \Lambda(B)\rangle}\right] = e^{\Phi(u)\Pi(B)}, \tag{3.1}$$

for all $u \in \mathbb{R}^d$, $B \in \mathcal{B}_b(S \times \mathbb{R}^m)$ and $\Pi = \pi \times \lambda$ is the product measure of the probability measure π on S and the Lebesgue measure λ on $\mathbb{R}^m$. Furthermore,

$$\Phi(u) = \mathrm{i}\langle \gamma, u\rangle - \frac{1}{2}\langle u, \Sigma u\rangle + \int_{\mathbb{R}^d} \left(e^{\mathrm{i}\langle u,x\rangle} - 1 - \mathrm{i}\langle u, x\rangle \mathbb{1}_{[0,1]}(\|x\|)\right) \nu(dx), \tag{3.2}$$

is the cumulant transform of an ID distribution with characteristic triplet (γ, Σ, ν), where $\gamma \in \mathbb{R}^d$, $\Sigma \in M_{d\times d}(\mathbb{R})$ is a symmetric positive-semidefinite matrix and ν is a Lévy-measure on $\mathbb{R}^d$, i.e.

$$\nu(\{0\}) = 0 \quad \text{and} \int_{\mathbb{R}^d} \left(1 \wedge \|x\|^2\right) \nu(dx) < \infty.$$

The quadruplet $(\gamma, \Sigma, \nu, \pi)$ determines the distribution of the Lévy basis completely and therefore it is called the characteristic quadruplet. Following [50], it can be shown that a Lévy basis has a Lévy-Itô decomposition.

Theorem 3.2. *Let $\{\Lambda(B), B \in \mathcal{B}_b(S \times \mathbb{R}^m)\}$ be an $\mathbb{R}^d$-valued Lévy basis on $S \times \mathbb{R}^m$ with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$. Then there exists a modification $\tilde{\Lambda}$ of Λ which is also a Lévy basis with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$ such that there exists an $\mathbb{R}^d$-valued Lévy basis $\tilde{\Lambda}^G$ on $S \times \mathbb{R}^m$ with characteristic quadruplet $(0, \Sigma, 0, \pi)$ and an independent Poisson random measure μ on $(\mathbb{R}^d \times S \times \mathbb{R}^m, \mathcal{B}(\mathbb{R}^d \times S \times \mathbb{R}^m))$ with intensity measure $\nu \times \pi \times \lambda$ such that*

$$\begin{aligned}\tilde{\Lambda}(B) = \gamma(\pi \times \lambda)(B) + &\tilde{\Lambda}^G(B) + \int_{\|x\|\leq 1} \int_B x(\mu(dx, dA, ds) - ds\pi(dA)\nu(dx)) \\ &+ \int_{\|x\|>1} \int_B x\mu(dx, dA, ds),\end{aligned} \tag{3.3}$$

for all $B \in \mathcal{B}_b(S \times \mathbb{R}^m)$.
If the Lévy measure additionally fulfills $\int_{\|x\|\leq 1} \|x\|\nu(dx) < \infty$, it holds that

$$\tilde{\Lambda}(B) = \gamma_0(\pi \times \lambda)(B) + \tilde{\Lambda}^G(B) + \int_{\mathbb{R}^d} \int_B x\mu(dx, dA, ds), \tag{3.4}$$

for all $B \in \mathcal{B}_b(S \times \mathbb{R}^m)$ with

$$\gamma_0 := \gamma - \int_{\|x\|\leq 1} x\nu(dx). \tag{3.5}$$

Note that the integral with respect to μ exists ω-wise as a Lebesgue integral.

Proof. Analogous to [12, Theorem 2.2]. ∎

We refer the reader to [39, Section 2.1] for further details on the integration with respect to Poisson random measures. From now on we assume that any Lévy basis has a decomposition (3.3).

Let us recall the following multivariate extension of [52, Theorem 2.7]. We denote by A' the transpose of a matrix A in what follows.

Theorem 3.3. *Let $\Lambda = \{\Lambda(B), B \in \mathcal{B}_b(S \times \mathbb{R}^m)\}$ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$, $f : S \times \mathbb{R}^m \to M_{n\times d}(\mathbb{R})$ be a $\mathcal{B}(S \times \mathbb{R}^m)$-measurable function. Then f is Λ-integrable in the sense of [52], if and only if*

$$\int_S\int_{\mathbb{R}^m} \left\| f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x \left(\mathbb{1}_{[0,1]}(\|f(A,s)x\|) - \mathbb{1}_{[0,1]}(\|x\|)\right) \nu(dx) \right\| ds\pi(dA) < \infty, \tag{3.6}$$

$$\int_S \int_{\mathbb{R}^m} \|f(A,s)\Sigma f(A,s)'\| \; ds\pi(dA) < \infty \text{ and} \tag{3.7}$$

$$\int_S \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \left(1 \wedge \|f(A,s)x\|^2\right) \nu(dx) ds\pi(dA) < \infty. \tag{3.8}$$

If f is Λ-integrable, the distribution of the stochastic integral $\int_S \int_{\mathbb{R}^m} f(A,s)\Lambda(dA,ds)$ is ID with the characteristic triplet $(\gamma_{int}, \Sigma_{int}, \nu_{int})$ given by

$$\gamma_{int} = \int_S \int_{\mathbb{R}^m} \left(f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x \left(\mathbb{1}_{[0,1]}(\|f(A,s)x\|) - \mathbb{1}_{[0,1]}(\|x\|)\right)\nu(dx)\right) ds\pi(dA),$$
$$\Sigma_{int} = \int_S \int_{\mathbb{R}^m} f(A,s)\Sigma f(A,s)' ds\pi(dA) \text{ and}$$
$$\nu_{int}(B) = \int_S \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \mathbb{1}_B(f(A,s)x)\nu(dx)ds\pi(dA),$$

for all Borel sets $B \subset \mathbb{R}^n\backslash\{0\}$.

Proof. Analogous to [12, Proposition 2.3]. ∎

Implicitly, we always assume that Σ_{int} or ν_{int} are different from zero throughout the paper to rule out the deterministic case.

For $m = 1$ it is known that the Lévy-Itô decomposition simplifies if the underlying Lévy process $L_t = \Lambda(S \times (0,t])$ is of finite variation (if and only if $\Sigma = 0$ and $\int_{|x|\le 1} |x|\nu(dx) < \infty$). Extending this one-dimensional notion, we speak of the finite variation case whenever $\Sigma = 0$ and $\int_{\|x\|\le 1}\|x\|\nu(dx) < \infty$.

Corollary 3.4. *Let $\Lambda = \{\Lambda(B), B \in \mathcal{B}_b(S \times \mathbb{R}^m)\}$ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruplet $(\gamma, 0, \nu, \pi)$ satisfying $\int_{\|x\|\le 1} \|x\|\nu(dx) < \infty$, and define γ_0 as in (3.5), such that for $\Phi(u)$ in (3.1) we have $\Phi(u) = i\langle\gamma_0, u\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u,x\rangle} - 1\right) \nu(dx)$. Furthermore, let $f : S \times \mathbb{R}^m \to M_{n\times d}(\mathbb{R})$ be a $\mathcal{B}(S \times \mathbb{R}^m)$-measurable function satisfying*

$$\int_S \int_{\mathbb{R}^m} \|f(A,s)\gamma_0\| \; ds\pi(dA) < \infty \text{ and} \tag{3.9}$$

$$\int_S \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \left(1 \wedge \|f(A,s)x\|\right) \nu(dx) ds\pi(dA) < \infty. \tag{3.10}$$

Then,

$$\int_S \int_{\mathbb{R}^m} f(A,s)\Lambda(dA,ds) = \int_S \int_{\mathbb{R}^m} f(A,s)\gamma_0 \; ds\pi(dA) + \int_{\mathbb{R}^d} \int_S \int_{\mathbb{R}^m} f(A,s)x\mu(dx,dA,ds),$$

where the right hand side denotes a ω-wise Lebesgue integral. Additionally, the distribution of the stochastic integral $\int_S \int_{\mathbb{R}^m} f(A,s)\Lambda(dA,ds)$ is ID with characteristic function

$$E\left[e^{i\langle u, \int_S \int_{\mathbb{R}^m} f(A,s)\Lambda(dA,ds)\rangle}\right] = e^{i\langle u, \gamma_{int,0}\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle u,x\rangle}-1\right)\nu_{int}(dx)}, \; u \in \mathbb{R}^d,$$

where

$$\gamma_{int,0} = \int_S \int_{\mathbb{R}^m} f(A,s)\gamma_0 \; ds\pi(dA),$$
$$\nu_{int}(B) = \int_S \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \mathbb{1}_B(f(A,s)x)\nu(dx)ds\pi(dA).$$

3.2 The MMAF framework

Definition 3.5. *Let $\Lambda = \{\Lambda(B), B \in \mathcal{B}_b(S \times \mathbb{R}^m)\}$ be an $\mathbb{R}^d$-valued Lévy basis and let $f : S \times \mathbb{R}^m \to M_{n\times d}(\mathbb{R})$ be a $\mathcal{B}(S \times \mathbb{R}^m)$-measurable function satisfying the conditions (3.6), (3.7) and (3.8). Then the stochastic integral*

$$X_t := \int_S \int_{\mathbb{R}^m} f(A, t-s)\Lambda(dA, ds) \tag{3.11}$$

is stationary, well-defined for all $t \in \mathbb{R}^m$ and its distribution is ID. The random field X is called an $\mathbb{R}^n$-valued mixed moving average field (MMAF) and f its kernel function.

In the following result we give conditions ensuring finite moments of an MMAF and explicit formulas for the first- and second-order moments.

Proposition 3.6. *Let X be an $\mathbb{R}^n$-valued MMAF driven by an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$ and with Λ-integrable kernel function $f : S \times \mathbb{R}^m \to M_{n\times d}(\mathbb{R})$.*

- *(i) If $\int_{\|x\|>1} \|x\|^r \,\nu(dx) < \infty$ and $f \in L^r(S\times\mathbb{R}^m, \pi\otimes\lambda)$ for $r \in [2,\infty)$, then $\mathbb{E}[\|X_t\|^r] < \infty$ for all $t \in \mathbb{R}^m$.*
- *(ii) If $\int_{\|x\|>1} \|x\|^r \,\nu(dx) < \infty$ and $f \in L^r(S\times\mathbb{R}^m, \pi\otimes\lambda) \cap L^2(S\times\mathbb{R}^m, \pi\otimes\lambda)$ for $r \in (0,2)$, then $\mathbb{E}[\|X_t\|^r] < \infty$ for all $t \in \mathbb{R}^m$.*

Consider the finite variation case, i.e. $\Sigma = 0$ and $\int_{\|x\|\leq 1}\|x\|\nu(dx) < \infty$, then the following holds

- *(i) If $\int_{\|x\|>1} \|x\|^r \,\nu(dx) < \infty$ and $f \in L^r(S\times\mathbb{R}^m, \pi\otimes\lambda)$ for $r \in [1,\infty)$, then $\mathbb{E}[\|X_t\|^r] < \infty$.*
- *(ii) If $\int_{\|x\|>1} \|x\|^r \,\nu(dx) < \infty$ and $f \in L^r(S\times\mathbb{R}^m, \pi\otimes\lambda) \cap L^1(S\times\mathbb{R}^m, \pi\otimes\lambda)$ for $r \in (0,1)$, then $\mathbb{E}[\|X_t\|^r] < \infty$.*

Proof. Analogous to [26, Proposition 2.6]. ∎

Proposition 3.7. *Let X be an $\mathbb{R}^n$-valued MMAF driven by an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$ and with Λ-integrable kernel function $f : S \times \mathbb{R}^m \to M_{n\times d}(\mathbb{R})$.*

(i) If $\int_{\|x\|>1} \|x\|\nu(dx) < \infty$ and $f \in L^1(S \times \mathbb{R}^m, \pi \times \lambda) \cap L^2(S \times \mathbb{R}^m, \pi \times \lambda)$ the first moment of X is given by

$$E[X_t] = \int_S \int_{\mathbb{R}^m} f(A, -s)\mu_\Lambda ds\pi(dA),$$

where $\mu_\Lambda = \gamma + \int_{\|x\|\geq 1} x\nu(dx)$.

(ii) If $\int_{\mathbb{R}^d} \|x\|^2\nu(dx) < \infty$ and $f \in L^2(S \times \mathbb{R}^m, \pi \times \lambda)$, then $X_t \in L^2$ and

$$Var(X_t) = \int_S \int_{\mathbb{R}^m} f(A, -s)\Sigma_\Lambda f(A, -s)' ds\pi(dA) \text{ and}$$
$$Cov(X_0, X_t) = \int_S \int_{\mathbb{R}^m} f(A, -s)\Sigma_\Lambda f(A, t - s)' ds\pi(dA),$$

where $\Sigma_\Lambda = \Sigma + \int_{\mathbb{R}^d} xx'\nu(dx)$.

(iii) Consider the finite variation case, i.e. $\Sigma = 0$ and $\int_{\|x\|\leq 1} \|x\|\nu(dx) < \infty$. If $\int_{\|x\|>1} \|x\|\nu(dx) < \infty$ and $f \in L^1(S \times \mathbb{R}^m, \pi \times \lambda)$ the first moment of X is given by

$$E[X_t] = \int_S \int_{\mathbb{R}^m} f(A, -s)\Big(\gamma_0 + \int_{\mathbb{R}^d} x\nu(dx)\Big) ds\pi(dA),$$

with γ_0 as defined in (3.5).

Proof. Immediate from [54, Section 25] and Theorem 3.3. ∎

3.3 Weak dependence properties of (A, Λ)-influenced MMAF

Since there is no natural order on $\mathbb{R}^m$ for $m > 1$ we cannot extend the definition of a natural filtration and therefore causal processes in a natural way to random fields. In the following we will propose such an extension and prove θ-lex-weak dependence for MMAF falling within this framework. Examples will be presented in Section 3.7.

Definition 3.8. *Let $X = (X_t)_{t\in\mathbb{R}^m}$ be a random field, $A = (A_t)_{t\in\mathbb{R}^m} \subset \mathbb{R}^m$ a family of Borel sets and $M = \{M(B), B \in \mathcal{B}_b(S\times\mathbb{R}^m)\}$ an independently scattered random measure. Assume that X_t is measurable with respect to $\sigma(M(B), B \in \mathcal{B}_b(S \times A_t))$. We then call A the* sphere of influence*, M the* influencer*, $(\sigma(M(B), B \in \mathcal{B}_b(S \times A_t)))_{t\in\mathbb{R}^m}$ the* filtration of influence *and X an* (A, M)-influenced random field*. If A is translation invariant, i.e. $A_t = t + A_0$, the sphere of influence is fully described by the set A_0 and we call A_0 the* initial sphere of influence.

Note that for $m = 1$, the class of causal mixed moving average processes driven by a Lévy basis Λ equals the class of (A, Λ)-influenced mixed moving average processes driven by Λ with $A_t = V_t$.

Let $A = (A_t)_{t\in\mathbb{R}^m}$ be a full dimensional, translation invariant sphere of influence with initial sphere of influence A_0. In this section we consider the filtration $(\mathcal{A}_t)_{t\in\mathbb{R}^m}$ generated by Λ, i.e. the σ-algebra generated by the set of random variables $\{\Lambda(B) : B \in \mathcal{B}(S \times A_t)\}$

with $t \in \mathbb{R}^m$.
Consider an MMAF X that is adapted to $(\mathcal{A}_t)_{t\in\mathbb{R}^m}$. Then, X is (A,Λ)-influenced and can be written as

$$X_t = \int_S \int_{\mathbb{R}^m} f(A, t-s)\Lambda(dA, ds) = \int_S \int_{A_t} f(A, t-s)\Lambda(dA, ds). \tag{3.12}$$

Note that the translation invariance of A is required to ensure stationarity of X.

In the following we discuss under which assumptions an (A,Λ)-influenced MMAF is θ-lex-weakly dependent. We start with a preliminary definition.

Definition 3.9 ([18, Definition 2.4.1])**.** *$K \subset \mathbb{R}^m$ is called a closed convex proper cone if it satisfies the following properties*

(i) $K + K \subset K$ (ensures convexity)

(ii) $\alpha K \subset K$ for all $\alpha \geq 0$ (ensures that K is a cone)

(iii) K is closed

(iv) K is pointed (i.e. $x \in K$ and $-x \in K \implies x = 0$).

We then apply a truncation technique to show that X is θ-lex-weakly dependent. Define X_j, X_Γ as in Definition 2.2 such that $j \in \mathbb{R}^m$ and $\Gamma \subset V_j^h$ (see Figure 1). We truncate X_j such that the truncation $\tilde{X}_j$ and X_Γ become independent. From our construction it will become clear that it is enough to find a truncation such that $\tilde{X}_j$ and X_i are independent for the lexicographic greatest point $i \in V_j^h$.
For a given point j, we determine the truncation of X_j by intersecting the integration set with V_j^ψ for $\psi > 0$ such that it does not intersect with A_i (see Figure 2 and 3). In the following we will describe the choice of ψ. The figures illustrate the case $m = 2$.

Let $i \in V_j^h$ be the lexicographic greatest point in V_j^h, i.e. $k \leq_{lex} i$ for all $k \in V_j^h$. In the following $dist(A,B) = \inf_{a\in A, b\in B} \|a - b\|$ denotes the Euclidean distance of the sets A and B. To ensure the existence of the above truncation, we assume that there exists an $\alpha \in \mathbb{R}^m \backslash \{0\}$ such that

$$\sup_{x\in A_0, x\neq 0} \frac{\alpha' x}{\|x\|} < 0. \tag{3.13}$$

Intuitively, (3.13) ensures that the initial sphere of influence A_0 can be covered by a closed convex proper cone. Moreover, w.l.o.g. by applying a rotation to A_0, we can always assume to work with $A_0 \subset V_0$. The following remark discuss such transformation.

Remark 3.10. *Let A_0 be a full dimensional subset of a half-space such that $A_0 \nsubseteq V_0$ Define the translation invariant sphere of influence $A = (A_t)_{t\in\mathbb{R}^m}$ by $A_t = (A_0 + t)_{t\in\mathbb{R}^m}$ and consider the (A,Λ)-influenced MMAF $X = (X_t)_{t\in\mathbb{R}^m}$ of the form $X_t = \int_S \int_{A_0+t} f(A, t - s)\Lambda(dA, ds)$. Note that if A_0 would not be full dimensional, X would be 0 since the Lebesgue measure of A_0 is zero. Define the hyperplane $D = \{x \in \mathbb{R}^m : \alpha' x = 0\}$. Using the principal axis theorem we find an orthogonal matrix O such that the axis of the first* *coordinate is orthogonal to the rotated hyperplane OD. Since O is orthogonal it holds that $|Det(D\varphi)(u)| = |Det(O)| = 1$, where $D\varphi$ denotes the Jacobian matrix of the function $\varphi : u \mapsto Ou$. Additionally, for the rotated initial set OA_0 it holds that $OA_0 \backslash V_0 \subset \{0\} \times [0,\infty)^{m-1}$, such that $\lambda(\{0\} \times [0,\infty)^{m-1}) = 0$. By substitution for multiple variables we get for $\tilde{t} = Ot$.*

$$\begin{aligned} X_t &= \int_S \int_{\mathbb{R}^m} f(A, t-s) \mathbb{1}_{A_0+t}(s) \Lambda(dA, ds) \\ &= \int_S \int_{\mathbb{R}^m} f(A, O^{-1}(Ot - Os)) \mathbb{1}_{OA_0+Ot}(Os) \Lambda(dA, ds) \\ &= \int_S \int_{OA_0+\tilde{t}} f(A, O^{-1}(\tilde{t}-\tilde{s})) \Lambda(dA, d\tilde{s}) = \int_S \int_{OA_0 \cap V_0 + \tilde{t}} f(A, O^{-1}(\tilde{t}-\tilde{s})) \Lambda(dA, d\tilde{s}) \\ &= \int_S \int_{V_{\tilde{t}}} \tilde{f}^O(A, \tilde{t}-\tilde{s}) \Lambda(dA, d\tilde{s}) = \tilde{X}_{\tilde{t}}, \end{aligned} \tag{3.14}$$

with $\tilde{f}^O(A, t-s) = f(A, O^{-1}(t-s)) \mathbb{1}_{\{s \in OA_0+t\}}$.

In Figure 4, it is pictured the smallest closed convex proper cone covering A_i which is called K. Note that all conditions can be formulated in terms of A_0 since the sphere of influence A is translation invariant.

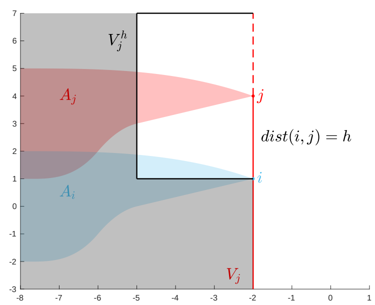

Figure 1: Integration sets A_j and A_i of X_j and X_i

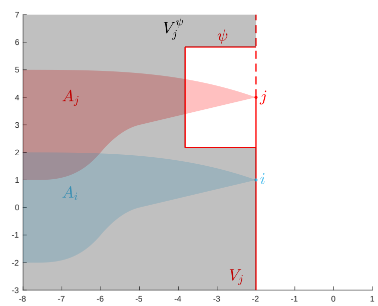

Figure 2: A_j and A_i together with V_j^ψ

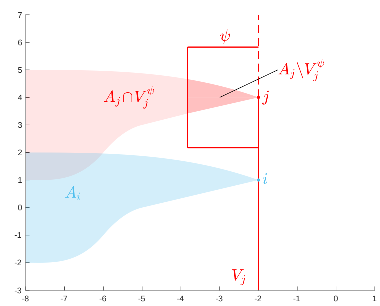

Figure 3: Integration sets A_i and $A_j \backslash V_j^\psi$ of X_i and $\tilde{X}_j$

In order to choose ψ we first define

$$b = \sup_{\substack{x \in A_0 \\ \|x\|=1}} \frac{\alpha' x}{\|\alpha\|} < 0 \text{ and } \tilde{K} = \left\{ x \in \mathbb{R}^m : \frac{\alpha' x}{\|x\|} \leq b \right\}, \tag{3.15}$$

where the last inequality follows from (3.13) (see Figure 4). It holds $-1 \leq b < 0$. For $x_1, x_2 \in \tilde{K}$ it holds

$$\frac{\alpha'(x_1+x_2)}{\|x_1+x_2\|} \leq \frac{\alpha' x_1}{\|x_1+x_2\|} + \frac{\alpha' x_2}{\|x_1+x_2\|} \leq b \frac{\|x_1\| + \|x_2\|}{\|x_1+x_2\|} \leq b$$

such that $\tilde{K}$ is a closed convex proper cone. It can be interpreted as the smallest equiangular closed convex proper cone that contains A_0. Then, $\cos(\beta + \frac{\pi}{2}) = b$ such that

$\beta = \arcsin(-b) \in [0, \frac{\pi}{2})$ (see Figure 5) and $dist(j, \tilde{K}) \geq \sin(\beta)h = -bh$ (see Figure 6). We choose ψ as

$$\psi(h) = \frac{-bh}{\sqrt{m}}. \tag{3.16}$$

In particular we have $\psi(h) = \mathcal{O}(h)$.

Let $l \in V_j^h$ be now an arbitrary point. From the given choice of ψ and i it holds $dist(l,j) \geq dist(i,j)$, $A_i \cap (A_j \backslash V_j^{\psi}) = \emptyset$, $A_i = i + A_0 \subset i + \tilde{K}$ and $A_l = l + A_0 \subset l + \tilde{K}$. Since $\tilde{K}$ is an equiangular closed convex proper cone we get $A_l \cap (A_j \backslash V_j^{\psi}) = \emptyset$.

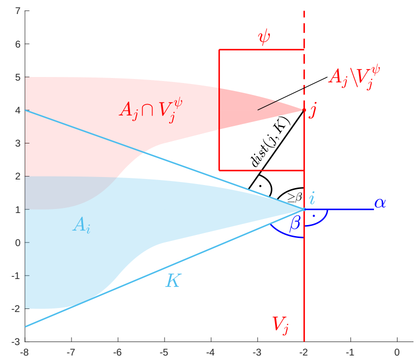

Figure 4: Choice of α and β

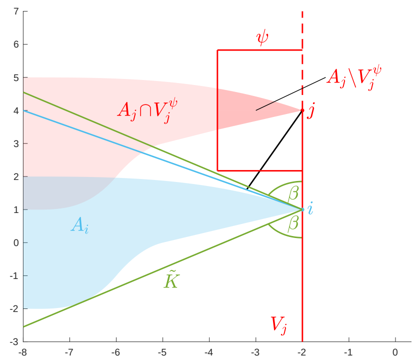

Figure 5: Choice of $\tilde{K}$

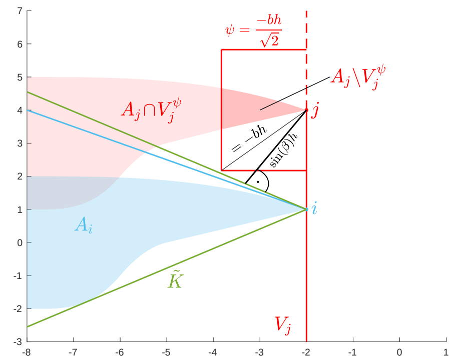

Figure 6: Construction of ψ

Hence, the conditions below, which are expressed in terms of the kernel function f and the characteristic quadruplet of the driving Lévy basis, are sufficient to show that an (A, Λ)-influenced MMAF is θ-lex-weakly dependent.

Proposition 3.11. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$ and $f : S \times \mathbb{R}^m \to M_{n \times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R}^m)$-measurable function. Consider the (A, Λ)-influenced MMAF*

$$X_t = \int_S \int_{A_t} f(A, t-s) \Lambda(dA, ds), \;\; t \in \mathbb{R}^m,$$

with translation invariant sphere of influence A such that (3.13) holds.

(i) If $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty$, $\gamma + \int_{\|x\|>1} x \nu(dx) = 0$ and $f \in L^2(S \times \mathbb{R}^m, \pi \otimes \lambda)$, then X is θ-lex-weakly dependent with θ-lex-coefficients satisfying

$$\theta_X(h) \leq 2 \Big(\int_S \int_{A_0 \cap V_0^{\psi(h)}} \mathrm{tr}(f(A,-s) \Sigma_\Lambda f(A,-s)') ds \pi(dA) \Big)^{\frac{1}{2}} = \hat{\theta}_X^{(i)}(h). \tag{3.17}$$

(ii) If $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty$ and $f \in L^2(S \times \mathbb{R}^m, \pi \otimes \lambda) \cap L^1(S \times \mathbb{R}^m, \pi \otimes \lambda)$, then X is θ-lex-weakly dependent with θ-lex-coefficients satisfying

$$\begin{aligned} \theta_X(h) \leq 2 \Big(& \int_S \int_{A_0 \cap V_0^{\psi(h)}} \mathrm{tr}(f(A,-s) \Sigma_\Lambda f(A,-s)') ds \pi(dA) \\ & + \Big\| \int_S \int_{A_0 \cap V_0^{\psi(h)}} f(A,-s) \mu_\Lambda ds \pi(dA) \Big\|^2 \Big)^{\frac{1}{2}} = \hat{\theta}_X^{(ii)}(h). \end{aligned} \tag{3.18}$$

(iii) *If $\int_{\mathbb{R}^d}\|x\|\nu(dx)<\infty$, $\Sigma=0$ and $f\in L^1(S\times\mathbb{R}^m,\pi\otimes\lambda)$ with γ_0 as in (3.5), then X is θ-lex-weakly dependent with θ-lex-coefficients satisfying*

$$\begin{aligned}\theta_X(h)\leq 2\bigg(&\int_S\int_{A_0\cap V_0^{\psi(h)}}\|f(A,-s)\gamma_0\|ds\pi(dA)\\&+\int_S\int_{A_0\cap V_0^{\psi(h)}}\int_{\mathbb{R}^d}\|f(A,-s)x\|\nu(dx)ds\pi(dA)\bigg)=\hat{\theta}_X^{(iii)}(h).\end{aligned}\tag{3.19}$$

(iv) *If $\int_{\|x\|>1}\|x\|\nu(dx)<\infty$ and $f\in L^1(S\times\mathbb{R}^m,\pi\otimes\lambda)\cap L^2(S\times\mathbb{R}^m,\pi\otimes\lambda)$, then X is θ-lex-weakly dependent with θ-lex-coefficients satisfying*

$$\begin{aligned}\theta_X(h)\leq 2\bigg(&\int_S\int_{A_0\cap V_0^{\psi(h)}}\mathrm{tr}(f(A,-s)\Sigma_\Lambda f(A,-s)')ds\pi(dA)\\&+\bigg\|\int_S\int_{A_0\cap V_0^{\psi(h)}}f(A,-s)\gamma ds\pi(dA)\bigg\|^2\bigg)^{\frac{1}{2}}\\&+2\int_S\int_{A_0\cap V_0^{\psi(h)}}\int_{\|x\|>1}\|f(A,-s)x\|\nu(dx)ds\pi(dA)=\hat{\theta}_X^{(iv)}(h).\end{aligned}$$

for all $h>0$, with ψ as defined in (3.16)*. Furthermore, $\Sigma_\Lambda=\Sigma+\int_{\mathbb{R}^d}xx'\nu(dx)$ and $\mu_\Lambda=\gamma-\int_{\|x\|\geq 1}x\nu(dx)$.*

Proof. See Section 5.2. ∎

In the next proposition we consider a vector of a shifted real-valued (A,Λ)-influenced MMAF and we show that it is θ-lex weakly dependent. This result is necessary to analyze, for example, the asymptotic behavior of the sample autocovariances. Define the set of possible shifts

$$S_k=\{(a,b)'\in\{0,\dots,k\}\times\{-k,\dots,k\}^{m-1}\},\ \ k\in\mathbb{N}\tag{3.20}$$

and consider the enumeration $\{s_1,\dots,s_{|S_k|}\}$ of S_k, where $|S_k|=(k+1)(2k+1)^{m-1}$. Besides the hereditary properties from Proposition 2.4 we show that weak dependence properties are inherited by the field

$$Z_t=(X_t,X_{t+s_1},X_{t+s_2},\dots,X_{t+s_{|S_k|}}).\tag{3.21}$$

Proposition 3.12. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruplet (γ,Σ,ν,π) and $f:S\times\mathbb{R}^m\to M_{1\times d}(\mathbb{R})$ be a Λ-integrable, $\mathcal{B}(S\times\mathbb{R}^m)$-measurable function. Consider the (A,Λ)-influenced MMAF*

$$X_t=\int_S\int_{A_t}f(A,t-s)\Lambda(dA,ds),\ \ t\in\mathbb{R}^m,$$

with translation invariant sphere of influence A such that (3.13) holds. Then

$$Z_t:=\int_S\int_{A_t}g(A,t-s)\Lambda(dA,ds),\ \ t\in\mathbb{R}^m,$$

where $g(A,s) = (f(A,s), f(A,s-s_1), \ldots, f(A,s-s_{|S_k|}))'$ is a $\mathcal{B}(S\times\mathbb{R}^m)$-measurable function with values in $M_{(k+1)(2k+1)^{m-1}\times d}(\mathbb{R})$ for $k\in\mathbb{N}$, is an (A,Λ)-influenced MMAF. If X additionally satisfies the conditions of Proposition 3.11 (i), (ii), (iii) or (iv) then Z is θ-lex-weakly dependent with coefficients respectively given by

$$\begin{aligned} \theta_Z^{(i)}(h) &\leq \mathcal{D}\hat{\theta}_X^{(i)}(h-\psi^{-1}(k)), & \theta_Z^{(ii)}(h) &\leq \mathcal{D}\hat{\theta}_X^{(ii)}(h-\psi^{-1}(k)), \\ \theta_Z^{(iii)}(h) &\leq \mathcal{C}\hat{\theta}_X^{(iii)}(h-\psi^{-1}(k)) \text{ and } & \theta_Z^{(iv)}(h) &\leq \mathcal{C}\hat{\theta}_X^{(iv)}(h-\psi^{-1}(k)), \end{aligned} \tag{3.22}$$

where $\mathcal{D} = |S_k|^{m/2}$, $\mathcal{C} = |S_k|^m$ for $\psi(h) > k$ with the corresponding $\hat{\theta}^{(\cdot)}(h)$ from Proposition 3.11.

Proof. See Section 5.2. ∎

3.4 Sample moments of (A,Λ)-influenced MMAF

Let us consider an $\mathbb{R}^n$-valued (A,Λ)-influenced MMAF

$$X = (X_u)_{u\in\mathbb{Z}^m} \text{ with } X_u = \int_S \int_{A_u} f(A, u-s)\Lambda(dA, ds), \tag{3.23}$$

with full-dimensional translation invariant sphere of influence A and initial sphere of influence $A_0 \subset V_0$ such that (3.13) holds. We assume that we observe X on the finite sampling sets $D_n \subset \mathbb{Z}^m$, such that

$$\lim_{n\to\infty} |D_n| = \infty \text{ and } \lim_{n\to\infty} \frac{|D_n|}{|\partial D_n|} = 0. \tag{3.24}$$

We note that this includes in particular the equidistant sampling

$$E_n = (0,n]^m \cap \mathbb{Z}^m \text{ such that } |E_n| = n^m, n\in\mathbb{N}. \tag{3.25}$$

The sample mean of the random field X is then defined as

$$\frac{1}{|D_n|}\sum_{u\in D_n} X_u. \tag{3.26}$$

If $\int_{\|x\|>1}\|x\|\nu(dx) < \infty$ we define the centered MMAF $\tilde{X}_u = X_u - E[X_u]$ and the sample autocovariance on E_n at lag $k\in\mathbb{N}\times\mathbb{Z}^{m-1}$

$$\frac{1}{|E_{n-\tilde{k}}|}\sum_{u\in E_{n-\tilde{k}}} \tilde{X}_u\tilde{X}_{u+k}, \ k\in\mathbb{N}\times\mathbb{Z}^{m-1}, \tag{3.27}$$

where $\tilde{k} = \|k\|_\infty$. Let us start by analyzing the asymptotic properties of the sample mean (3.26) for a centered (A,Λ)-influenced MMAF.

Theorem 3.13. *Let $X = (X_u)_{u\in\mathbb{Z}^m}$ be an (A,Λ)-influenced MMAF as defined in (3.23) such that $\int_{\|x\|>1}\|x\|^{2+\delta}\nu(dx) < \infty$, $\gamma + \int_{\|x\|>1} x\nu(dx) = 0$ and $f \in L^2(S\times\mathbb{R}^m, \pi\otimes\lambda) \cap L^{2+\delta}(S\times\mathbb{R}^m, \pi\otimes\lambda)$ for some $\delta > 0$. Assume that X has θ-lex-coefficients satisfying $\theta_X(h) = \mathcal{O}(h^{-\alpha})$, where $\alpha > m(1+\frac{1}{\delta})$. Then*

$$\Sigma = \sum_{k\in\mathbb{Z}^m} E[X_0 X_k'],$$

is finite, positive semidefinite and

$$\frac{1}{|D_n|^{\frac{1}{2}}}\sum_{j\in D_n} X_j \xrightarrow[n\to\infty]{d} N(0,\Sigma). \tag{3.28}$$

Proof. By [49, Theorem 3.6], it follows that an MMAF is ergodic.
Then, the result follows from Corollary 2.6. ∎

In the theorem above, the initial sphere of influence A_0 must satisfy (3.13). Additionally, we observe a trade-off between moment conditions on X and the decay rate of the θ-lex coefficients. However, one can derive similar results for the sample mean of a MMAF by relaxing condition (3.13) and exploiting the second order moment structure of a MMAF. On the other hand, the following technique does not carry over to higher moments.

Theorem 3.14. *Let $X = (X_u)_{u\in\mathbb{Z}^m}$ be an (A,Λ)-influenced MMAF defined by*

$$X_u = \int_S\int_{A_u} f(A, u-s)\Lambda(dA, ds),$$

with full dimensional translation invariant sphere of influence A and initial sphere of influence $A_0 \subset V_0$ such that $\gamma + \int_{\|x\|>1} x\nu(dx) = 0$ and $E[\|X_0\|^2] < \infty$. Assume that X has θ-lex-coefficients satisfying $\theta_X(h) = \mathcal{O}(h^{-\alpha})$, where $\alpha > m$. Then

$$\Sigma = \sum_{k\in\mathbb{Z}^m} E[X_0 X_k'],$$

is finite, positive definite and

$$\frac{1}{|D_N|^{\frac{1}{2}}}\sum_{j\in D_N} X_j \xrightarrow[N\to\infty]{d} N(0,\Sigma). \tag{3.29}$$

Proof. See Section 5.3. ∎

To lighten notation in the following we assume that X is real-valued and centered, i.e. $E[X_0] = 0$. In order to derive asymptotic properties for the distribution of (3.27) we need to show weak dependence properties of the random field $Y = (Y_{j,k})_{j\in\mathbb{Z}^m}$ defined as

$$Y_{j,k} = X_j X_{j+k} - R(k), \;\; k \in \mathbb{N}\times\mathbb{Z}^{m-1}, \tag{3.30}$$

where

$$R(k) = Cov(X_0, X_k) = E[X_0 X_k'] = \int_S \int_{A_0 \cap A_k} f(A, -s)\Sigma_\Lambda f(A, k-s)' ds \pi(dA), \ k \in \mathbb{N} \times \mathbb{Z}^{m-1},$$

with $\Sigma_\Lambda = \Sigma + \int_{\mathbb{R}^d} xx'\nu(dx)$ for an (A,Λ)-influenced MMAF X with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$. The last equality follows from Proposition 3.7.

Proposition 3.15. *Let $X = (X_u)_{u \in \mathbb{Z}^m}$ be a real-valued (A,Λ)-influenced MMAF as defined in (3.23) such that $E[X_0] = 0$ and $E[\|X_0\|^{2+\delta}] < \infty$ for some $\delta > 0$ with θ-lex-coefficients θ_X. Then, $(Y_{j,k})_{j \in \mathbb{Z}^m}$, $k \in \mathbb{N} \times \mathbb{Z}^{m-1}$ as defined in (3.30) is θ-lex-weakly dependent with coefficients*

$$\theta_Y(h) \leq \mathcal{C} \left(\sqrt{2}\hat{\theta}_X^{(i)} \left(h - \psi^{-1}(\|k\|_\infty)\right)\right)^{\frac{\delta}{1+\delta}},$$

where $\mathcal{C}$ is a constant, independent of h, $\hat{\theta}_X^{(i)}$ from Proposition 3.12, and ψ as defined in (3.16). *Furthermore, in the finite variation case it holds*

$$\theta_Y(h) \leq \mathcal{C} \left(2\hat{\theta}_X^{(i)} \left(h - \psi^{-1}(\|k\|_\infty)\right)\right)^{\frac{\delta}{1+\delta}}.$$

Proof. Consider the 2-dimensional process $Z = (X_j, X_{j+k})_{j \in \mathbb{Z}^m}$ with $k \in \mathbb{N} \times \mathbb{Z}^{m-1}$. Proposition 3.12 implies that Z is θ-lex-weakly dependent and from the proof we obtain the coefficients

$$\theta_Z(h) \leq \sqrt{2}\hat{\theta}_X^{(i)}(h - \psi^{-1}(\|k\|_\infty)) \text{ for } \psi(h) > \|k\|_\infty.$$

Consider the function $h : \mathbb{R}^2 \to \mathbb{R}$ such that $h(x_1, x_2) = x_1 x_2$. The function h satisfies the assumptions of Proposition 2.4 for $p = 2+\delta$, $c = 1$ and $a = 2$. Considering $h(Z) = X_j X_{j+k}$ we obtain the θ-lex-coefficients of $(Y_{j,k})_{j \in \mathbb{Z}^m}$

$$\theta_Y(h) \leq \mathcal{C}(\sqrt{2}\hat{\theta}_X^{(i)}(h - \psi^{-1}(\|k\|_\infty)))^{\frac{\delta}{1+\delta}} \text{ for } \psi(h) > \|k\|_\infty.$$

The coefficients for the finite variation case can be obtained from Proposition 2.4 and (3.22). ∎

The next corollary gives asymptotic properties of the sample autocovariances (3.27) for (A,Λ)-influenced MMAF, i.e. we can give a distributional limit theorem for the process $(Y_{j,k})_{j \in \mathbb{Z}^m}$ by determining the asymptotic distribution of

$$\frac{1}{|E_{n-\tilde{k}}|^{\frac{1}{2}}} \sum_{j \in E_{n-\tilde{k}}} Y_{j,k}, \ k \in \mathbb{N} \times \mathbb{Z}^{m-1},$$

where $\tilde{k} = \|k\|_\infty$.

Corollary 3.16. *Let $X = (X_u)_{u\in\mathbb{Z}^m}$ be a real-valued (A,Λ)-influenced MMAF as defined in (3.23) such that $E[X_0] = 0$ and $E[\|X_0\|^{4+\delta}] < \infty$, for some $\delta > 0$. If $\hat{\theta}_X^{(i)}(h) = \mathcal{O}(h^{-\alpha})$, with $\hat{\theta}_X^{(i)}$ from Proposition 3.12 and $\alpha > m\left(1+\frac{1}{\delta}\right)\left(\frac{3+\delta}{2+\delta}\right)$, then*

$$\Sigma = \sum_{l\in\mathbb{Z}^m} Cov\left(\left(\begin{array}{c} Y_{0,0} \\ \vdots \\ Y_{0,k}\end{array}\right), \left(\begin{array}{c} Y_{l,0} \\ \vdots \\ Y_{l,k}\end{array}\right)'\right) = \sum_{l\in\mathbb{Z}^m} Cov\left(\left(\begin{array}{c} X_0X_0 \\ \vdots \\ X_0X_k\end{array}\right), \left(\begin{array}{c} X_lX_l \\ \vdots \\ X_lX_{l+k}\end{array}\right)'\right),$$

is finite, positive semidefinite and

$$\frac{1}{|E_{n-\tilde{k}}|^{\frac{1}{2}}} \sum_{j\in E_{n-\tilde{k}}} \left(\begin{array}{c} Y_{j,0} \\ \vdots \\ Y_{j,k}\end{array}\right) \xrightarrow[N\to\infty]{d} N(0,\Sigma),$$

where $\tilde{k} = \|k\|_\infty$.

Proof. Analogous to Theorem 3.13 we obtain the stated convergence using Proposition 3.15. ∎

Corollary 3.17. *Let $X = (X_u)_{u\in\mathbb{Z}^m}$ be a real-valued (A,Λ)-influenced MMAF as defined in (3.23) and $p \geq 1$ such that $E[|X_0|^{2p+\delta}] < \infty$ for some $\delta > 0$. If $\hat{\theta}_X^{(i)}(h) = \mathcal{O}(h^{-\alpha})$, with $\hat{\theta}_X^{(i)}$ from Proposition 3.12 and $\alpha > m\left(1+\frac{1}{\delta}\right)\left(\frac{2p-1+\delta}{p+\delta}\right)$, then*

$$\Sigma = \sum_{k\in\mathbb{Z}^m} Cov(X_0^p, X_k^p),$$

is finite, positive semidefinite and

$$\frac{1}{|E_{n-\tilde{k}}|^{\frac{1}{2}}} \sum_{j\in E_{n-\tilde{k}}} (X_j^p - E[X_0^p]) \xrightarrow[N\to\infty]{d} N(0,\Sigma),$$

where $\tilde{k} = \|k\|_\infty$.

Remark 3.18. *The theory developed in this section is an important first step in showing asymptotic normality of parametric estimators based on moment functions as the generalized method of moments (for a comprehensive introduction see [37]). An example of the application of the weak dependence properties and related central limit theorems to the study of GMM estimators can be found in [26, Section 6.1], where the authors analyze parametric estimators of the supOU process.*

3.5 Weak dependence properties of non-influenced MMAF

In this subsection we consider a general MMAF $X = (X_t)_{t\in\mathbb{R}^m}$ as defined in (3.11), i.e.

$$X_t = \int_S \int_{\mathbb{R}^m} f(A, t-s)\Lambda(dA, ds),\ t \in \mathbb{R}^m,$$

and discuss under which assumptions a non-influenced MMAF is η-weakly dependent. Note that we do not demand any additional assumption on the structure of X as assumed in Section 3.2 and 3.3.

Proposition 3.19. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$ and $f : S \times \mathbb{R}^m \to M_{n\times d}(\mathbb{R})$ a $\mathcal{B}(S \times \mathbb{R}^m)$-measurable function. Consider the MMAF $X = (X_t)_{t\in\mathbb{R}^m}$ with*

$$X_t = \int_S \int_{\mathbb{R}^m} f(A, t-s)\Lambda(dA, ds), \;\; t \in \mathbb{R}^m.$$

(i) If $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty$, $\gamma + \int_{\|x\|>1} x\nu(dx) = 0$ and $f \in L^2(S \times \mathbb{R}^m, \pi \otimes \lambda)$, then X is η-weakly dependent with η-coefficients satisfying

$$\eta_X(h) \leq \left(\int_S \int_{((-\frac{h}{2},\frac{h}{2})^m)^c} \mathrm{tr}(f(A,-s)\Sigma_\Lambda f(A,-s)')ds\pi(dA) \right)^{\frac{1}{2}} = \hat{\eta}_X^{(i)}(h).$$

(ii) If $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty$ and $f \in L^2(S \times \mathbb{R}^m, \pi \otimes \lambda) \cap L^1(S \times \mathbb{R}^m, \pi \otimes \lambda)$, then X is η-weakly dependent with η-coefficients satisfying

$$\begin{aligned}\eta_X(h) \leq \Bigg(&\int_S \int_{((-\frac{h}{2},\frac{h}{2})^m)^c} \mathrm{tr}(f(A,-s)\Sigma_\Lambda f(A,-s)')ds\pi(dA) \\ &+ \left\| \int_S \int_{((-\frac{h}{2},\frac{h}{2})^m)^c} f(A,-s)\mu_\Lambda ds\pi(dA) \right\|^2 \Bigg)^{\frac{1}{2}} = \hat{\eta}_X^{(ii)}(h).\end{aligned}$$

(iii) If $\int_{\mathbb{R}^d} \|x\| \nu(dx) < \infty$, $\Sigma = 0$ and $f \in L^1(S \times \mathbb{R}^m, \pi \otimes \lambda)$ with γ_0 as in (3.5), then X is η-weakly dependent with η-coefficients satisfying

$$\begin{aligned}\eta_X(h) \leq &\int_S \int_{((-\frac{h}{2},\frac{h}{2})^m)^c} \|f(A,-s)\gamma_0\| ds\pi(dA) \\ &+ \int_S \int_{((-\frac{h}{2},\frac{h}{2})^m)^c} \int_{\mathbb{R}^d} \|f(A,-s)x\| \nu(dx)ds\pi(dA) = \hat{\eta}_X^{(iii)}(h).\end{aligned}$$

(iv) If $\int_{\|x\|>1} \|x\| \nu(dx) < \infty$ and $f \in L^1(S \times \mathbb{R}^m, \pi \otimes \lambda) \cap L^2(S \times \mathbb{R}^m, \pi \otimes \lambda)$, then X is η-weakly dependent with η-coefficients satisfying

$$\begin{aligned}\eta_X(h) \leq \Bigg(&\int_S \int_{((-\frac{h}{2},\frac{h}{2})^m)^c} \mathrm{tr}(f(A,-s)\Sigma_\Lambda f(A,-s)')ds\pi(dA) \\ &+ \left\| \int_S \int_{((-\frac{h}{2},\frac{h}{2})^m)^c} f(A,-s)\mu_\Lambda ds\pi(dA) \right\|^2 \Bigg)^{\frac{1}{2}} \\ &+ \int_S \int_{((-\frac{h}{2},\frac{h}{2})^m)^c} \int_{\|x\|>1} \|f(A,-s)x\| \nu(dx)ds\pi(dA) = \hat{\eta}_X^{(iv)}(h),\end{aligned}$$

for all $h > 0$, where $\Sigma_\Lambda = \Sigma + \int_{\mathbb{R}^d} xx'\nu(dx)$ and $\mu_\Lambda = \gamma - \int_{\|x\|\geq 1} x\nu(dx)$.

Proof. See Section 5.4. ∎

Analogous to Proposition 3.12 we obtain the following result.

Proposition 3.20. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$ and $f : S \times \mathbb{R}^m \to M_{1\times d}(\mathbb{R})$ be a Λ-integrable, $\mathcal{B}(S \times \mathbb{R}^m)$-measurable function. Consider the real-valued MMAF*

$$X_t = \int_S \int_{\mathbb{R}^m} f(A, t-s)\Lambda(dA, ds), \ t \in \mathbb{R}^m.$$

Then

$$Z_t := \int_S \int_{\mathbb{R}^m} g(A, t-s)\Lambda(dA, ds), \ t \in \mathbb{R}^m,$$

where $g(A, s) = (f(A, s), f(A, s-s_1), \ldots, f(A, s-s_{|S_k|}))'$ is a $\mathcal{B}(S \times \mathbb{R}^m)$-measurable function with values in $M_{(k+1)(2k+1)^{m-1}\times d}(\mathbb{R})$ for $k \in \mathbb{N}$, is an MMAF.
If X additionally satisfies the conditions of Proposition 3.19 (i), (ii), (iii) or (iv), then Z is η-weakly dependent with coefficients respectively given by

$$\begin{aligned} \eta_Z^{(i)}(h) \leq& \mathcal{D}\hat{\eta}_X^{(i)}(h-2k), & \eta_Z^{(ii)}(h) \leq& \ \mathcal{D}\hat{\eta}_X^{(ii)}(h-2k), \\ \eta_Z^{(iii)}(h) \leq& \mathcal{C}\hat{\eta}_X^{(iii)}(h-2k) \ \text{and} & \eta_Z^{(iv)}(h) \leq& \ \mathcal{C}\hat{\eta}_X^{(iv)}(h-2k), \end{aligned} \tag{3.31}$$

where $\mathcal{D} = |S_k|^{m/2}$, $\mathcal{C} = |S_k|^m$ for $h > 2k$ with the corresponding $\hat{\eta}^{(\cdot)}(h)$ from Proposition 3.19.

Proof. Analogous to Proposition 3.12. ∎

3.6 Sample moments of non-influenced MMAF

Let us consider an $\mathbb{R}^n$-valued MMAF

$$X = (X_u)_{u\in\mathbb{Z}^m} \text{ with } X_u = \int_S \int_{\mathbb{R}^m} f(A, u-s)\Lambda(dA, ds). \tag{3.32}$$

As in Section 3.2 we assume that we observe X on a sequence of finite sampling sets $D_n \subset \mathbb{Z}^m$, such that (3.24) holds.

Theorem 3.21. *Let $(X_u)_{u\in\mathbb{Z}^m}$ be an MMAF as defined in (3.32) such that $E[X_0] = 0$ and $E[\|X_0\|^{2+\delta}] < \infty$ for some $\delta > 0$. Assume that X has η-coefficients satisfying $\eta_X(h) = \mathcal{O}(h^{-\beta})$, where $\beta > m \max\left(2, \left(1 + \frac{1}{\delta}\right)\right)$. Then*

$$\Sigma = \sum_{u\in\mathbb{Z}^m} Cov(X_0, X_u) = \sum_{u\in\mathbb{Z}^m} E[X_0 X_u'], \tag{3.33}$$

is finite, positive semidefinite and

$$\frac{1}{|D_n|^{\frac{1}{2}}} \sum_{u\in D_n} X_u \xrightarrow[n\to\infty]{d} N(0, \Sigma). \tag{3.34}$$

Proof. Let us consider the notations and assumptions stated in Definition 2.1. Then, X is λ-weakly dependent, see definition in [33, Definition 1]. Finally [33, Theorem 2] implies the summability of σ^2 and the result stated in (3.34). The multivariate extension follows analogously to Corollary 2.6 by the Cramér-Wold device. ∎

Remark 3.22. *Theorem 3.21 can be formulated as a functional central limit theorem, following [33, Theorem 3]. Set $S_n(t) = \sum_{j \in tE_n} X_j$, $t \in \mathbb{R}^m$ with E_n as defined in (3.25) and the additional assumption that $S_n(t) = 0$ if one coordinate of t equals zero. Then, under the assumptions of Theorem 3.21 it holds that*

$$\frac{1}{n^{\frac{m}{2}}} S_n(t) \xrightarrow[n\to\infty]{\mathcal{D}([0,1])} \sigma W(t), \tag{3.35}$$

where W denotes a Brownian sheet and $\xrightarrow[n\to\infty]{\mathcal{D}([0,1])}$ denotes the convergence in the Skorokhod space.

Analogous to Proposition 3.15 we show the following result.

Proposition 3.23. *Let $(X_u)_{u\in\mathbb{Z}^m}$ be a real-valued MMAF as defined in (3.32) such that $E[X_0] = 0$ and $E[\|X_0\|^{2+\delta}] < \infty$ for some $\delta > 0$. Then, $(Y_{j,k})_{j\in\mathbb{Z}^m}$, $k \in \mathbb{N}\times\mathbb{Z}^{m-1}$ as defined in (3.30) is η-weakly dependent with coefficients*

$$\eta_Y(h) \leq \mathcal{C}(\sqrt{2}\hat{\eta}_X^{(i)}(h - 2\|k\|_\infty))^{\frac{\delta}{1+\delta}},$$

where $\mathcal{C}$ is a constant, independent of h and $\hat{\eta}_X^{(i)}$ from Proposition 3.20. Furthermore, in the finite variation case it holds

$$\eta_Y(h) \leq \mathcal{C}(2\hat{\eta}_X^{(iii)}(h - 2\|k\|_\infty))^{\frac{\delta}{1+\delta}}.$$

In the following we give asymptotic properties of the sample autocovariances (3.27).

Corollary 3.24. *Let $(X_u)_{u\in\mathbb{Z}^m}$ be a real-valued MMAF as defined in (3.32) such that $\int_{\|x\|>1}\|x\|^{4+\delta}\nu(dx) < \infty$, $\gamma + \int_{\|x\|>1} x\nu(dx) = 0$ and $f : S\times\mathbb{R}^m \to M_{1\times d}(\mathbb{R})$ satisfies $f \in L^2(S\times\mathbb{R}^m, \pi\otimes\lambda) \cap L^{4+\delta}(S\times\mathbb{R}^m, \pi\otimes\lambda)$ for some $\delta > 0$. If $\hat{\eta}_X^{(i)}(h) = \mathcal{O}(h^{-\beta})$, with $\hat{\eta}_X^{(i)}$ from Proposition 3.20 and $\beta > m\max\left(2, \left(1+\frac{1}{\delta}\right)\right)\left(\frac{3+\delta}{2+\delta}\right)$, then*

$$\Sigma = \sum_{l\in\mathbb{Z}^m} Cov\left(\begin{pmatrix} Y_{0,0} \\ \vdots \\ Y_{0,k}\end{pmatrix}, \begin{pmatrix} Y_{l,0} \\ \vdots \\ Y_{l,k}\end{pmatrix}'\right) = \sum_{l\in\mathbb{Z}^m} Cov\left(\begin{pmatrix} X_0X_0 \\ \vdots \\ X_0X_k\end{pmatrix}, \begin{pmatrix} X_lX_l \\ \vdots \\ X_lX_{l+k}\end{pmatrix}'\right),$$

is finite, positive semidefinite and

$$\frac{1}{|E_{n-\tilde{k}}|^{\frac{1}{2}}} \sum_{j\in E_{n-\tilde{k}}} \begin{pmatrix} Y_{j,0} \\ \vdots \\ Y_{j,k}\end{pmatrix} \xrightarrow[N\to\infty]{d} N(0,\Sigma),$$

where $\tilde{k} = \|k\|_\infty$.

Proof. Analogous to Theorem 3.21 we obtain the stated convergence using Proposition 3.23. ∎

Remark 3.25. *Note that for $m = 1$ Theorem 3.21 improves the only existing central limit theorem for MMA processes based on η-weak dependence (see [26, Theorem 4.1]) by reducing the necessary decay of the η-coefficients from $\beta > 4 + \frac{2}{\delta}$ to $\beta > \max\left(2, \left(1 + \frac{1}{\delta}\right)\right)$.*

Remark 3.26. *Let X be an (A, Λ)-influenced MMAF satisfying the conditions of Proposition 3.11 (i). Then X is θ-lex- and η-weakly dependent with the same weak dependence coefficients and both the asymptotic results in Section 3.4 and 3.6 can be applied.*
Note that the asymptotic results in Section 3.4 hold under weaker decay demands for the weak dependence coefficients compared to the results in Section 3.6.

3.7 Example of (A, Λ)-influenced MMAF: MSTOU processes

We apply the developed asymptotic theory to mixed spatio-temporal Ornstein-Uhlenbeck (MSTOU) processes. MSTOU processes were introduced in [47] and extend spatio-temporal Ornstein-Uhlenbeck (STOU) processes (see [11],[46]) by additionally mixing the mean reversion parameter. Moreover, this extension can cover short-range as well as long-range dependence structures in space-time.

In the following we will treat the temporal and spatial domain separately.

MSTOU processes are an example of (A, Λ)-influenced MMAF where the sphere of influence is a family of ambit sets, i.e. $A_t(x) \subset \mathbb{R} \times \mathbb{R}^m$ such that

$$\begin{cases} A_t(x) = A_0(0) + (t, x), \text{ (Translation invariant)} \\ A_s(x) \subset A_t(x), \\ A_t(x) \cap ((t, \infty)) \times \mathbb{R}^m = \emptyset. \text{ (Non-anticipative).} \end{cases} \tag{3.36}$$

Proposition 3.27. *Let Λ be a real-valued Lévy basis on $(0, \infty) \times \mathbb{R} \times \mathbb{R}^m$ with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$ such that $\int_{|x|>1} x^2 \nu(dx) < \infty$ and $f(\lambda)$ be the density function of π (i.e. the mean reversion parameter λ) with respect to the Lebesgue measure. Furthermore, let $A = (A_t(x))_{(t,x) \in \mathbb{R} \times \mathbb{R}^m}$ be an ambit set. If*

$$\int_0^\infty \int_{A_t(x)} \exp(-\lambda(t - s))\, ds d\xi f(\lambda) d\lambda < \infty,$$

then the (A, Λ)-influenced MMAF

$$Y_t(x) = \int_0^\infty \int_{A_t(x)} \exp(-\lambda(t - s))\, \Lambda(d\lambda, ds, d\xi), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

is well defined and we call $Y_t(x)$ a mixed spatio-temporal Ornstein-Uhlenbeck (MSTOU) process.

Proof. Follows immediately from [47, Corollary 1]. ∎

In order to calculate explicit conditions for the asymptotic results of Section 3.3, it becomes necessary to specify a family of ambit sets. In the following we will consider c-class MSTOU processes which are a sub-class of the g-class MSTOU processes defined in [47, Definition 9].

Definition 3.28. *Let $Y_t(x)$ be a MSTOU process as in Proposition 3.27. If, for a constant $c > 0$,*

$$A_t(x) = \{(s, \xi) : s \leq t, \|x - \xi\| \leq c|t - s|\},$$

then $Y_t(x)$ is called a c-class MSTOU process. A c-class MSTOU process is well defined if

$$\int_0^\infty \frac{1}{\lambda^{d+1}} f(\lambda) d\lambda < \infty. \tag{3.37}$$

The next theorem expresses the θ-lex coefficients of c-class MSTOU processes in terms of the characteristic quadruplet of the driving Lévy basis. We note that $A_0(0)$ is a full dimensional closed convex proper cone satisfying (3.13). From (3.16) it follows that $\psi(h) = \frac{1}{\sqrt{c^2+1}} \frac{h}{\sqrt{d+1}}$.

Theorem 3.29. *Let $(Y_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ be a c-class MSTOU process and $(\gamma, \Sigma, \nu, \pi)$ the characteristic quadruplet of its driving Lévy basis. Moreover, let $f(\lambda)$ be the density of π with respect to the Lebesgue measure.*

(i) If $\int_{|x|>1} x^2 \nu(dx) < \infty$ and $\gamma + \int_{|x|>1} x\, \nu(dx) = 0$, then $Y_t(x)$ is θ-lex-weakly dependent. For $c \in (0, 1]$, $\theta_Y(h)$ satisfies

$$m = 1 : \theta_Y(h) \leq \left(2c\Sigma_\Lambda \int_0^\infty \frac{(2\lambda\psi(h) + 1)}{\lambda^2} e^{-2\lambda\psi(h)} f(\lambda) d\lambda \right)^{\frac{1}{2}},$$

$$m \geq 2 : \theta_Y(h) \leq 2 \left(V_m(c)\Sigma_\Lambda \int_0^\infty \frac{m! \sum_{k=0}^m \frac{1}{k!}(2\lambda\psi(h))^k}{(2\lambda)^{d+1}} e^{-2\lambda\psi(h)} f(\lambda) d\lambda \right)^{\frac{1}{2}},$$

and for $c > 1$

$$m = 1 : \theta_Y(r) \leq 2 \Big(c\Sigma_\Lambda \int_0^\infty \frac{(2\lambda\frac{\psi(h)}{c} + 1)}{2\lambda^2} e^{-2\lambda\frac{\psi(h)}{c}} ds f(\lambda) d\lambda \Big)^{\frac{1}{2}},$$

$$m \geq 2 : \theta_Y(h) \leq 2 \left(V_m(c)\Sigma_\Lambda \int_0^\infty \frac{m! \sum_{k=0}^m \frac{1}{k!}(2\frac{\lambda\psi(h)}{c})^k}{(2\lambda)^{d+1}} e^{-2\frac{\lambda\psi(h)}{c}} f(\lambda) d\lambda \right)^{\frac{1}{2}}.$$

(ii) If $\int_\mathbb{R} |x|\, \nu(dx) < \infty$, $\Sigma = 0$ and γ_0 as defined in (3.5), then $Y_t(x)$ is θ-lex-weakly dependent. For $c \in (0, 1]$

$$m \in \mathbb{N} : \theta_Y(h) \leq 2V_m(c) \left(|\gamma_0| + \int_\mathbb{R} |x|\nu(dx) \right) \left(\int_0^\infty \frac{m! \sum_{k=0}^m \frac{1}{k!}(\lambda\psi(h))^k}{\lambda^{d+1}} e^{-\lambda\psi(h)} f(\lambda) d\lambda \right),$$

and for $c > 1$

$$m \in \mathbb{N} : \theta_Y(h) \le 2V_m(c)\left(|\gamma_0| + \int_{\mathbb{R}} |x|\nu(dx)\right)\left(\int_0^\infty \frac{m!\sum_{k=0}^m \frac{1}{k!}\left(\frac{\lambda\psi(h)}{c}\right)^k}{\lambda^{d+1}} e^{-\frac{\lambda\psi(h)}{c}} f(\lambda)d\lambda\right).$$

$V_m(c) = \frac{\left(\Gamma(\frac{1}{2})c\right)^m}{\Gamma(\frac{m}{2}+1)}$ denotes the volume of the m-dimensional ball with radius c, $\psi(h) = \frac{1}{\sqrt{c^2+1}}\frac{h}{\sqrt{m+1}}$ and $\Sigma_\Lambda = \Sigma + \int_{\mathbb{R}} x^2\nu(dx)$.

Proof. (i) Let us consider the case $m = 1$. From Proposition 3.11 we deduce

$$\theta_Y(h) \le 2\left(\Sigma_\Lambda \int_0^\infty \int_{A_0(0)\cap V_0^{\psi(h)}} \exp(2s\lambda)ds\, d\xi\, f(\lambda)d\lambda\right)^{\frac{1}{2}}. \tag{3.38}$$

As first step, one has to evaluate the truncated integration set $A_0(0)\cap V_0^{\psi(h)}$. Depending on the width of $A_0(0)$, we distinguish the two cases illustrated in the following figures. Figure 7 and 8 consider the case $c \in (0,1]$ and Figure 9 and 10 cover the case $c > 1$.

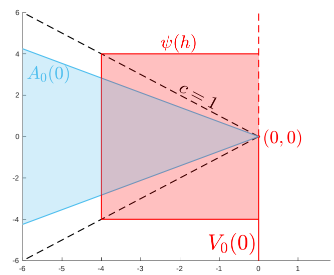

Figure 7: Integration set $A_0(0)$ with $(V^h_{(0,0)})^c$ for $c = \frac{1}{\sqrt{2}}$ and $h = 4\sqrt{3}$.

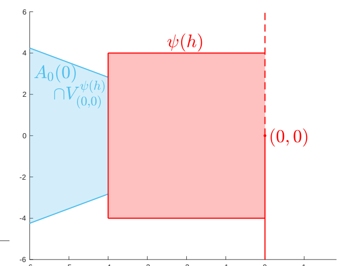

Figure 8: Truncated set $A_0(0) \cap V^{\psi(h)}_{(0,0)}$ for $c = \frac{1}{\sqrt{2}}$ and $h = 4\sqrt{3}$.

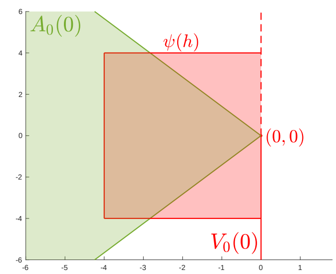

Figure 9: Integration set $A_0(0)$ with $(V^h_{(0,0)})^c$ for $c = \sqrt{2}$ and $h = 4\sqrt{6}$.

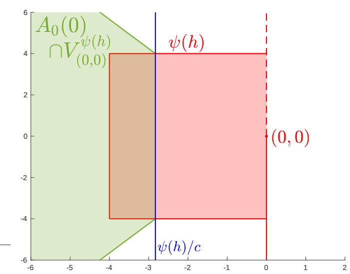

Figure 10: Truncated set $A_0(0) \cap V^{\psi(h)}_{(0,0)}$ for $c = \sqrt{2}$ and $h = 4\sqrt{6}$.

Let $c \in (0,1]$, then (3.38) is equal to

$$2\left(\Sigma_\Lambda \int_0^\infty \int_{-\infty}^{-\psi(h)} \int_{\|\xi\|\le cs} d\xi\, e^{2s\lambda} ds f(\lambda)d\lambda\right)^{\frac{1}{2}} = 2\left(\Sigma_\Lambda \int_0^\infty \int_{-\infty}^{-\psi(h)} (-2cs)e^{2s\lambda}ds f(\lambda)d\lambda\right)^{\frac{1}{2}}$$
$$= \left(2c\Sigma_\Lambda \int_0^\infty \frac{(2\lambda\psi(h)+1)}{\lambda^2} e^{-2\lambda\psi(h)} f(\lambda)d\lambda\right)^{\frac{1}{2}}.$$

The integral $\int_{\|\xi\|\le cs} d\xi$ is the volume of an m-dimensional ball of radius cs which for $m = 1$ is equal to $-2cs$.

For $c > 1$ we can bound (3.38) by

$$2\left(\Sigma_\Lambda \int_0^\infty \int_{-\infty}^{-\frac{\psi(h)}{c}} (-2cs)e^{2s\lambda}ds f(\lambda)d\lambda\right)^{\frac{1}{2}} = 2\left(c\Sigma_\Lambda \int_0^\infty \frac{(2\lambda\frac{\psi(h)}{c}+1)}{2\lambda^2} e^{-2\lambda\frac{\psi(h)}{c}} ds f(\lambda)d\lambda\right)^{\frac{1}{2}}.$$

In a similar way, one can derive the θ-lex coefficients for $m \ge 2$.

(ii) Analogous to (i).

∎

We now give explicit computations of the θ-lex-coefficients of a c-class MSTOU process in the case in which the mean reverting parameter λ is gamma distributed. For a $Gamma(\alpha,\beta)$ distributed mean reversion parameter λ, i.e. $f(\lambda)=\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\,\mathbb{1}_{[0,\infty)}(\lambda)$, the c-class MSTOU process is well defined if $\alpha>m+1$ and $\beta>0$ due to condition (3.37).

Theorem 3.30. *Let $(Y_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ be a c-class MSTOU process and (γ,Σ,ν,π) the characteristic quadruplet of its driving Lévy basis. Moreover, let the mean reversion parameter λ be $Gamma(\alpha,\beta)$ distributed with $\alpha>m+1$ and $\beta>0$.*

(i) If $\int_{|x|>1}x^2\,\nu(dx)<\infty$, $\gamma+\int_{|x|>1}x\nu(dx)=0$, then $Y_t(x)$ is θ-lex-weakly dependent. For $c\in[0,1]$,

$$m=1:\theta_Y(h)\leq 2\left(\frac{c\Sigma_\Lambda\beta^\alpha}{2\Gamma(\alpha)}\left(\frac{\Gamma(\alpha-2)}{(2\psi(h)+\beta)^{\alpha-2}}+\frac{2\psi(h)\Gamma(\alpha-1)}{(2\psi(h)+\beta)^{\alpha-1}}\right)\right)^{\frac{1}{2}},$$

$$m\geq 2:\theta_Y(h)\leq 2\left(V_m(c)\frac{m!\Sigma_\Lambda\beta^\alpha}{2^{m+1}}\sum_{k=0}^{m}\frac{(2\psi(h))^k}{k!(2\psi(h)+\beta)^{\alpha-m-1+k}}\frac{\Gamma(\alpha-m-1+k)}{\Gamma(\alpha)}\right)^{\frac{1}{2}},$$

and for $c>1$

$$m\in\mathbb{N}:\theta_Y(h)\leq 2\left(V_m(c)\frac{m!\Sigma_\Lambda\beta^\alpha}{2^{m+1}}\sum_{k=0}^{m}\frac{\left(\frac{2\psi(h)}{c}\right)^k}{k!\left(\frac{2\psi(h)}{c}+\beta\right)^{\alpha-m-1+k}}\frac{\Gamma(\alpha-m-1+k)}{\Gamma(\alpha)}\right)^{\frac{1}{2}},$$

such that $\theta_Y(h)=\mathcal{O}(h^{\frac{(m+1)-\alpha}{2}})$.

(ii) If $\int_{\mathbb{R}}|x|\,\nu(dx)<\infty$, $\Sigma=0$ and γ_0 as defined in (3.5), then $Y_t(x)$ is θ-lex-weakly dependent. For $c\in(0,1]$,

$$m\in\mathbb{N}:\theta_Y(h)\leq 2V_m(c)m!\beta^\alpha\left(|\gamma_0|+\int_{\mathbb{R}}|x|\nu(dx)\right)\sum_{k=0}^{m}\frac{\psi(h)^k}{k!(\psi(h)+\beta)^{\alpha-m-1+k}}\frac{\Gamma(\alpha-m-1+k)}{\Gamma(\alpha)},$$

and for $c>1$

$$m\in\mathbb{N}:\theta_Y(h)\leq 2V_m(c)d!\beta^\alpha\left(|\gamma_0|+\int_{\mathbb{R}}|x|\nu(dx)\right)\sum_{k=0}^{m}\frac{\left(\frac{\psi(h)}{c}\right)^k}{k!\left(\frac{\psi(h)}{c}+\beta\right)^{\alpha-m-1+k}}\frac{\Gamma(\alpha-m-1+k)}{\Gamma(\alpha)},$$

such that $\theta_Y(h)=\mathcal{O}(h^{(m+1)-\alpha})$.

This implies the following sufficient conditions for the asymptotic normality of the sample mean and the sample autocovariance function.

Corollary 3.31. *Let $(Y_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ be a c-class MSTOU process and (γ,Σ,ν,π) the characteristic quadruplet of its driving Lévy basis. Moreover, let the mean reversion parameter λ be Gamma(α,β) distributed with $\alpha>m+1$ and $\beta>0$.*

(i) If $\gamma+\int_{|x|>1}x\nu(dx)=0$, $\int_{|x|>1}|x|^{2+\delta}\nu(dx)<\infty$ for some $\delta>0$ and $\alpha>(m+1)\left(3+\frac{2}{\delta}\right)$, then the sample mean of $Y_t(x)$ as defined in (3.26) is asymptotically normal.

(ii) If $\gamma+\int_{|x|>1}x\nu(dx)=0$, $\int_{|x|>1}|x|^{4+\delta}\nu(dx)<\infty$ for some $\delta>0$ and $\alpha>(m+1)\left(\frac{3+\delta}{2+\delta}\right)\left(3+\frac{2}{\delta}\right)$, then the sample autocovariances as defined in (3.27) are asymptotically normal.

Corollary 3.32. *Let $(Y_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ be a c-class MSTOU process and (γ,Σ,ν,π) the characteristic quadruplet of its driving Lévy basis. Moreover, let the mean reversion parameter λ be Gamma(α,β) distributed such that $\alpha>d+1$ and $\beta>0$.*

(i) If $\int_{\mathbb{R}}|x|\nu(dx)<\infty$, $\Sigma=0$, γ_0 as defined in (3.5) and $\alpha>(m+1)\left(2+\frac{2}{\delta}\right)$, then the sample mean of $Y_t(x)$ as defined in (3.26) is asymptotically normal.

(ii) If $\int_{\mathbb{R}}|x|\nu(dx)<\infty$, $\Sigma=0$, γ_0 as defined in (3.5), $\int_{|x|>1}|x|^{4+\delta}\nu(dx)<\infty$ for some $\delta>0$ and $\alpha>(m+1)\left(\frac{3+\delta}{2+\delta}\right)\left(2+\frac{2}{\delta}\right)$, then the sample autocovariances as defined in (3.27) are asymptotically normal.

Remark 3.33. *Since the c-class MSTOU processes satisfy the assumptions of Theorem 3.14, we can derive asymptotic normality of the sample mean for this fields under the weaker assumptions $E[Y_t(x)^2]<\infty$ and $\alpha>3(d+1)$.*

We conclude with some remarks regarding the short and long range dependence of an MSTOU process.

Definition 3.34. *A stationary random field $Y=(Y_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ is said to have temporal short-range dependence if*

$$\int_0^\infty Cov(Y_t(x),Y_{t+\tau}(x))d\tau<\infty,$$

and temporal long-range dependence if the integral is infinite.
If $Cov(Y_t(x),Y_t(x+m_x))=C(|m_x|)$ for all $m_x\in\mathbb{R}^m$ and a positive definite function C the random field Y is called isotropic. Now, an isotropic random field is said to have spatial short-range dependence if

$$\int_0^\infty C(r)dr<\infty.$$

We have that an MSTOU is a stationary and isotropic random field, see Theorem 5 [47]. By assuming a *Gamma*(α,β)-distributed random parameter λ, we have the following results, as shown in Section 6 [26] and Section 3.3 [47]:

(i) For $m = 0$, we have that Y is a supOU process which is well-defined for $\alpha > 1$ and $\beta > 0$. Thus, we obtain a long-memory process for $1 < \alpha \leq 2$ and a short memory one for $\alpha > 2$.

(ii) For $m = 1$, Y is well-defined if $\alpha > 2$ and $\beta > 0$. Y exhibits temporal as well as spatial long-range dependence for $2 < \alpha \leq 3$. If $\alpha > 3$ we observe temporal and spatial short-range dependence.

(iii) For $m = 3$, Y is well-defined if $\alpha > 4$ and $\beta > 0$. Y exhibits temporal as well as spatial long-range dependence for $4 < \alpha \leq 5$. If $\alpha > 5$ we observe temporal and spatial short-range dependence.

It is then easy to see that the assumptions in the Corollaries 3.31 and 3.32 imply that we are in the realm of short-range dependence.

Remark 3.35. *(GMM estimator)*

For $m = 0$, a consistent GMM estimator for the supOU process is defined in [55]. In [26], the authors show asymptotic normality of the estimator and that if the underlying Lévy process is of finite variation and all moments exist, then the GMM estimator is asymptotic normally distributed for $\alpha > 2$.

For $m \geq 1$, a consistent GMM estimator for a c-class MSTOU process is introduced in [47]. The results in Corollaries 3.31 and 3.32 should pave the way for an analysis of the asymptotic normality of the GMM estimator defined in [47] using arguments similar to [26]. For example, when $m = 1$, in the finite variation case and when all moments exist, we can apply our results to short-range dependent MSTOU processes with $\alpha > 4$.

3.8 Example of non-influenced MMAF: Lévy-driven CARMA fields

We conclude the section by applying our developed asymptotic theory to the class of Lévy-driven CARMA fields defined on $\mathbb{R}^m$.

CARMA (continuous autoregressive moving average) fields are an extension of the well-known CARMA processes (see e.g. [21] for a comprehensive introduction) and have been introduced in [16, 22, 42, 51].

In [22], the authors define CARMA fields as isotropic random fields

$$Y(t) = \int_{\mathbb{R}^m} g(t-s) dL(s), \quad t \in \mathbb{R}^m, \tag{3.39}$$

where g is a radially symmetric kernel and L a real-valued Lévy basis on $\mathbb{R}^m$. When the Lévy basis L has a finite second order structure, the CARMA fields generates a rich family of isotropic covariance functions on $\mathbb{R}^m$ which are not necessarily non-negative or monotone.

On the other hand, in [51], the author defines CARMA(p,q) fields based on a system of stochastic partial differential equations. For $0 \leq q < p$, the mild solution of the system is called a causal CARMA field and is given by

$$Y(t) = b^T \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_m} e^{A_1(t_1-s_1)} \cdots e^{A_m(t_m-s_m)} c \, dL(s), \quad (t_1, \ldots, t_m) \in \mathbb{R}^m, \tag{3.40}$$

where $A_1, \dots, A_m$ are companion matrices, L is a real-valued Lévy basis on $\mathbb{R}^m$, $b = (b_0, \dots, b_{p-1})^T \in \mathbb{R}^p$ with $b_q \neq 0$ and $b_i = 0$ for $i > q$ and $c = (0, \dots, 0, 1)^T \in \mathbb{R}^p$, see [51, Definition 3.3].
In [16], the author shows the existence of a mild solution for the CARMA stochastic partial differential equation, c.f. [16, equation (1.7)], in [16, Theorem 5.3]. The causal CARMA fields presented in [51] can be seen as a special case of the CARMA random fields defined in [16]. A more subtle relationship exists between the definition of CARMA field in [16] and [22] just when m is odd, see [16, Section 7].

In general, our framework can be applied to the class of CARMA fields introduced in [16] and [22] when the conditions of the below theorem are satisfied.

Theorem 3.36. *Let L be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$ such that $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty$ and $\gamma + \int_{\|x\|>1} x \nu(dx) = 0$. Let $g : \mathbb{R}^m \to M_{n\times d}(\mathbb{R})$ such that g is exponentially bounded in norm, i.e. there exists $M, K \in \mathbb{R}^+$ such that*

$$\|g(t)\|^2 \leq M e^{-K\|t\|}, \text{ for all } t \in \mathbb{R}^m. \tag{3.41}$$

Then, the moving average field $X_t = \int_{\mathbb{R}^m} g(t-s)L(ds)$, $t \in \mathbb{R}^m$ is an η-weakly dependent field with exponentially decaying η-coefficients. Due to the equivalence of norms the result does not depend on a specific choice of norms.

Proof. See Section 5. ∎

Remark 3.37. *Since the kernels in (3.39) and (3.40) satisfy equation (3.41), for example, we can show that these fields are η-weakly dependent by applying Theorem 3.36.*

4 Ambit fields

In the following we will briefly introduce stationary ambit fields. We discuss weak dependence properties of such fields and give sufficient conditions for the applicability of the results in Section 2.2.

4.1 The ambit framework

Let $A_t(x) \subset \mathbb{R}\times\mathbb{R}^m$ for $(t, x) \in \mathbb{R}\times\mathbb{R}^m$ be an ambit set as defined in (3.36). By $\mathcal{P}'$ we denote the usual predictable σ-algebra on $\mathbb{R}$, i.e. the σ-algebra generated by all left-continuous adapted processes. Then, a random field $X : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is called predictable if it is measurable with respect to the σ-algebra $\mathcal{P}$ defined by $\mathcal{P} = \mathcal{P}' \otimes \mathcal{B}(\mathbb{R}^m)$.

Definition 4.1. *Let Λ be a real-valued Lévy basis on $\mathbb{R}\times\mathbb{R}^m$ with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$, σ a predictable stationary random field on $\mathbb{R}\times\mathbb{R}^m$ independent of Λ. Furthermore, let $l : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be a measurable function and $A_t(x)$ an ambit set. We assume that $f(\xi, s) = \mathbb{1}_{A_t(x)}(\xi, s) l(\xi, s) \sigma_s(\xi)$ satisfies (3.6), (3.7) and (3.8) almost surely. Then, the random field*

$$Y_t(x) = \int_{A_t(x)} l(x-\xi, t-s) \sigma_s(\xi) \Lambda(d\xi, ds), \ (t, x) \in \mathbb{R}\times\mathbb{R}^m, \tag{4.1}$$

is called an ambit field and it is stationary (see p. 185 [6]).

Remark 4.2. *Ambit fields require us to define integrals with respect to Lévy bases where the integrand is stochastic. Although the integration theory from Rajput and Rosinski just enables us to define stochastic integrals with respect to deterministic integrands [52], one can extend this theory to stochastic integrands which are predictable and independent of the Lévy basis. In fact, we can condition on the σ-algebra generated by the field σ and use again the integration theory introduced in [52]. Then, such integrals are well defined if the kernel function satisfy the sufficient conditions (3.6), (3.7) and (3.8) almost surely. Allowing for dependence between the volatility field and the Lévy basis demands the use of a different integration theory as presented in Section 1.2.1 [2], Proposition 39 [6], Theorem 3.2 [13] and [25].*

We conclude this section by giving explicit formulas for the first and second moment of an ambit field.

Proposition 4.3. *Let Y be an ambit field as defined in (4.1) driven by a real-valued Lévy basis with characteristic quadruplet $(\gamma, \Sigma, \nu, \pi)$ and Λ-integrable kernel function $f(\xi, s) = \mathbb{1}_{A_t(x)}(\xi, s)l(\xi, s)\sigma_s(\xi)$, where σ is predictable, stationary and independent of Λ.*

(i) If $E[|Y_t(x)|] < \infty$ the first moment of Y is given by

$$E[Y_t(x)] = \mu_\Lambda E[\sigma_t(x)] \int_{A_t(x)} l(x - \xi, t - s) d\xi ds,$$

where $\mu_\Lambda = \gamma + \int_{|x|\geq 1} x\nu(dx)$.

(ii) If $E[Y_t(x)^2] < \infty$ it holds

$$\begin{aligned} Var(Y_t(x)) =& \Sigma_\Lambda E[\sigma_t(x)^2] \int_{A_t(x)} l(x - \xi, t - s)^2 d\xi ds \\ &+ \mu_\Lambda^2 \int_{A_t(x)} \int_{A_t(x)} l(x - \xi, t - s) l(x - \tilde{\xi}, t - \tilde{s}) \rho(s, \tilde{s}, \xi, \tilde{\xi}) d\xi ds d\tilde{\xi} d\tilde{s} \text{ and} \\ Cov(Y_t(x), Y_{\tilde{t}}(\tilde{x})) =& \Sigma_\Lambda E[\sigma_t(x)^2] \int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} l(x - \xi, t - s) l(\tilde{x} - \xi, \tilde{t} - s) d\xi ds \\ &+ \mu_\Lambda^2 \int_{A_t(x)} \int_{A_{\tilde{t}}(\tilde{x})} l(x - \xi, t - s) l(\tilde{x} - \tilde{\xi}, \tilde{t} - \tilde{s}) \rho(s, \tilde{s}, \xi, \tilde{\xi}) d\xi ds d\tilde{\xi} d\tilde{s}, \end{aligned}$$

where $\Sigma_\Lambda = \Sigma + \int_{\mathbb{R}} x^2 \nu(dx)$ and $\rho(s, \tilde{s}, \xi, \tilde{\xi}) = E[\sigma_s(\xi)\sigma_{\tilde{s}}(\tilde{\xi})] - E[\sigma_s(\xi)]E[\sigma_{\tilde{s}}(\tilde{\xi})]$.

Proof. Immediate from [6, Proposition 41]. ∎

4.2 Weak dependence properties of ambit fields

Let us consider a stationary ambit field $Y = (Y_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ as defined in (4.1). In order to analyze the covariance structure of Y, it becomes necessary to specify a model for σ. In

[4] the authors proposed to model σ by kernel-smoothing of a homogeneous Lévy basis, i.e. a moving average random field

$$\sigma_t(x) = \int_{A_t^\sigma(x)} j(x-\xi, t-s)\Lambda^\sigma(d\xi, ds), \tag{4.2}$$

where Λ^σ is a real valued Lévy basis independent of Λ with characteristic quadruplet $(\mu_\sigma, \Sigma_\sigma, \nu_\sigma, \pi_\sigma)$, $A^\sigma = (A_t^\sigma(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ an ambit set as defined in (3.36) and j a real valued Λ^σ-integrable function. In the following we extend this model and assume σ to be an $(A^\sigma, \Lambda^\sigma)$-influenced MMAF, i.e.

$$\sigma_t(x) = \int_S \int_{A_t^\sigma(x)} j(A, x-\xi, t-s)\Lambda^\sigma(dA, d\xi, ds). \tag{4.3}$$

Proposition 4.4. *Let $Y = (Y_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ be an ambit field as defined in (4.1) with $\sigma = (\sigma_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ being a predictable $(A^\sigma, \Lambda^\sigma)$-influenced MMAF as defined in (4.3) and such that $A_0(0)$ and $A_0^\sigma(0)$ satisfy (3.13), $j \in L^1(S\times\mathbb{R}\times\mathbb{R}^m, \pi\otimes\lambda)\cap L^2(S\times\mathbb{R}^m\times\mathbb{R}, \pi\otimes\lambda)$, where λ indicates the Lebesgue measure on $\mathbb{R}^{m+1}$, and $\int_{|x|>1}|x|^2\nu_\sigma(dx) < \infty$.*

(i) If $l \in L^2(\mathbb{R}\times\mathbb{R}^m)$, $\int_{|x|>1}|x|^2\nu(dx) < \infty$ and $\gamma + \int_{|x|>1} x\nu(dx) = 0$, then Y is θ-lex-weakly dependent with θ-lex-coefficients $\theta_Y(h)$ satisfying

$$\begin{aligned}\theta_Y(h) \leq & 2\Big(\Sigma_\Lambda E[\sigma_0(0)^2]\int_{A_0(0)\cap V_{(0,0)}^{\psi(h)}} l(-\xi,-s)^2 d\xi ds\Big)^{\frac{1}{2}} \\ & + 2\Bigg(\Bigg(\Sigma_{\Lambda^\sigma}\int_S\int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi(h)}} j(A,-\xi,-s)^2 d\xi ds \pi(dA) \\ & + \mu_{\Lambda^\sigma}^2\Bigg(\int_S\int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi(h)}} j(A,-\xi,-s)d\xi ds\pi(dA)\Bigg)^2\Bigg) \\ & \times \Sigma_\Lambda\int_{A_0(0)\setminus V_{(0,0)}^{\psi(h)}} l(-\xi,-s)^2 d\xi ds\Bigg)^{\frac{1}{2}}.\end{aligned} \tag{4.4}$$

(ii) If $l \in L^1(\mathbb{R}\times\mathbb{R}^m)\cap L^2(\mathbb{R}\times\mathbb{R}^m)$, $\int_{|x|>1}|x|^2\nu(dx) < \infty$, then Y is θ-lex-weakly dependent with θ-lex-coefficients $\theta_Y(h)$ satisfying

$$\begin{aligned}\theta_Y(h) \leq & 2\Bigg(\Big(\Sigma_\Lambda E[\sigma_0(0)^2]\int_{A_0(0)\cap V_{(0,0)}^{\psi(h)}} l(-\xi,-s)^2 d\xi ds \\ & + \mu_\Lambda E[\sigma_0(0)^2]\Big(\int_{A_0(0)\cap V_{(0,0)}^{\psi(h)}} l(-\xi,-s)d\xi ds\Big)^2\Bigg)^{\frac{1}{2}}\end{aligned} \tag{4.5}$$

$$+ 2\Bigg(\Bigg(\Sigma_{\Lambda^\sigma}\int_S\int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi(h)}} j(A,-\xi,-s)^2 d\xi ds\pi(dA)$$

$$
\begin{aligned}
&+ \mu_{\Lambda^\sigma}^2 \bigg(\int_S \int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi(h)}} j(A,-\xi,-s) d\xi ds \pi(dA) \bigg)^2 \bigg) \\
&\times \bigg(\Sigma_\Lambda \int_{A_0(0)\setminus V_{(0,0)}^{\psi(h)}} l(-\xi,-s)^2 d\xi ds \\
&\qquad + \mu_\Lambda \Big(\int_{A_0(0)\setminus V_{(0,0)}^{\psi(h)}} l(-\xi,-s) d\xi ds \Big)^2 \bigg) \bigg)^{\frac{1}{2}}.
\end{aligned}
$$

(iii) If $l \in L^1(\mathbb{R}\times\mathbb{R}^m)$, $\int_{\mathbb{R}} |x|\nu(dx) < \infty$ and $\Sigma = 0$, then Y is θ-lex-weakly dependent with θ-lex-coefficients $\theta_Y(h)$ satisfying

$$
\begin{aligned}
\theta_Y(h) \leq & 2\Sigma_\sigma \Big(|\gamma_0| + \int_{\mathbb{R}} |x|\nu(dx) \Big) \Big(\int_{A_0(0)\cap V_{(0,0)}^{\psi(h)}} |l(-\xi,-s)| d\xi ds \Big) \\
& + 2 \bigg(\Sigma_{\Lambda^\sigma} \int_S \int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi(h)}} j(A,-\xi,-s)^2 d\xi ds \pi(dA) \\
& + \mu_{\Lambda^\sigma}^2 \bigg(\int_S \int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi(h)}} j(A,-\xi,-s) d\xi ds \pi(dA) \bigg)^2 \bigg) \\
& \times \Big(|\gamma_0| + \int_{\mathbb{R}} |x|\nu(dx) \Big) \Big(\int_{A_0(0)\setminus V_{(0,0)}^{\psi(h)}} |l(-\xi,-s)| d\xi ds \Big),
\end{aligned}
\tag{4.6}
$$

for all $h > 0$, with $\psi(h) = \frac{-bh}{2\sqrt{m+1}}$ and b as defined in (3.15)*, $\mu_\Lambda = \gamma + \int_{|x|\geq 1} x\nu(dx)$, $\Sigma_\Lambda = \Sigma + \int_{\mathbb{R}} x^2\nu(dx)$, $\mu_{\Lambda^\sigma} = \gamma_\sigma + \int_{|x|\geq 1} x\nu_\sigma(dx)$ and $\Sigma_{\Lambda^\sigma} = \Sigma_\sigma + \int_{\mathbb{R}} x^2\nu_\sigma(dx)$.*

Proof. See Section 5.6. ∎

We now analyze the case in which σ is a p-dependent random field for $p \in \mathbb{N}$.

Proposition 4.5. *Let $Y = (Y_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ be an ambit field as defined in (4.1) with a predictable p-dependent stationary random field $\sigma_t(x)$ for $p \in \mathbb{N}$. Assume that $A_0(0)$ satisfies (3.13). Additionally assume that $l \in L^2(\mathbb{R}^m \times \mathbb{R})$, $\int_{|x|>1} |x|^2\nu(dx) < \infty$ and $\gamma + \int_{|x|>1} x\nu(dx) = 0$. Then, for sufficiently big h, Y is θ-lex-weakly dependent with θ-lex-coefficients $\theta_Y(h)$ satisfying*

$$
\theta_Y(h) \leq 2\Big(\Sigma_\Lambda E[\sigma_0(0)^2] \int_{A_0(0)\cap V_{(0,0)}^{\psi(h)}} l(-\xi,-s)^2 d\xi ds \Big)^{\frac{1}{2}}, \tag{4.7}
$$

with $\psi(h) = \frac{-bh}{2\sqrt{m+1}}$ and b as defined in (3.15)*, $\Sigma_\Lambda = \Sigma + \int_{\mathbb{R}} x^2\nu(dx)$.*

Proof. See Section 5.6. ∎

4.2.1 Volatility fields

If σ is a $(A^\sigma,\Lambda^\sigma)$-influenced MMAF as defined in (4.2), j is a non-negative kernel function and the following assumptions hold

$$(\mathrm{H}):\begin{cases}\text{The Lévy basis } \Lambda^\sigma \text{ has generating quadruple } (\gamma_\sigma,0,\nu_\sigma,\pi_\sigma) \text{ such that}\\ \int_{\mathbb{R}}|x|\nu_\sigma(dx)<\infty,\quad \gamma_\sigma-\int_{|x|\leq 1}x\nu_\sigma(dx)\geq 0 \text{ and } \nu_\sigma(\mathbb{R}^-)=0,\end{cases}$$

then σ has values in $\mathbb{R}^+$ and we call it volatility or intermittency field. Note that Assumption (H) imply that Λ^σ satisfies the finite variation case. This model is used in several applications of the ambit fields, see [6].

By assuming additionally that $j\in L^1(S\times\mathbb{R}\times\mathbb{R}^m,\pi\otimes\lambda)\cap L^2(S\times\mathbb{R}^m\times\mathbb{R},\pi\otimes\lambda)$ and $\int_{|x|>1}|x|^2\nu_\sigma(dx)<\infty$, the results in Proposition 4.4 (i) and (ii) hold. On the other hand, the results in Proposition 4.4(iii) can be improved.

Corollary 4.6. *Let $Y=(Y_t(x))_{(t,x)\in\mathbb{R}\times\mathbb{R}^m}$ be an ambit field as defined in (4.1) with predictable volatility field $\sigma_t(x)$ being an $(A^\sigma,\Lambda^\sigma)$-influenced MMAF such that $A_0(0)$ and $A_0^\sigma(0)$ satisfy (3.13), $j\in L^1(S\times\mathbb{R}\times\mathbb{R}^m,\pi\otimes\lambda)$, $l\in L^1(\mathbb{R}\times\mathbb{R}^m)$ and Assumption (H) holds. Let γ_0 with respect to Λ and $\gamma_{0,\sigma}$ with respect to Λ^σ be defined as in (3.5). Then, Y is θ-lex-weakly dependent with θ-lex-coefficients $\theta_Y(h)$ satisfying*

$$\begin{aligned}\theta_Y(h)\leq &2\Big(|\gamma_{0,\sigma}|+\int_{\mathbb{R}}|x|\nu_\sigma(dx)\Big)\Big(|\gamma_0|+\int_{\mathbb{R}}|x|\nu(dx)\Big)\\ &\quad\times\Big(\int_S\int_{A_0(0)}|j(A,-\xi,-s)|d\xi ds\pi(dA)\Big)\Big(\int_{A_0(0)\cap V_{(0,0)}^{\psi(h)}}|l(-\xi,-s)|d\xi ds\Big)\\ &+2\Big(|\gamma_{0,\sigma}|+\int_{\mathbb{R}}|x|\nu_\sigma(dx)\Big)\Big(|\gamma_0|+\int_{\mathbb{R}}|x|\nu(dx)\Big)\\ &\quad\times\Big(\int_{A_0(0)\setminus V_{(0,0)}^{\psi(h)}}|l(-\xi,-s)|d\xi ds\Big)\Big(\int_S\int_{A_0(0)\cap V_{(0,0)}^{\psi(h)}}|j(A,-\xi,-s)|d\xi ds\pi(dA)\Big),\end{aligned}\tag{4.8}$$

for all $h>0$, with $\psi(h)=\frac{-bh}{2\sqrt{m+1}}$ and b as defined in (3.15).

Proof. Analogous to Proposition 4.4. ∎

4.3 Sample moments of ambit fields

In this section we study the asymptotic distribution of sample moments of Y. As in Section 3.2 we assume that we observe Y on a sequence of finite sampling sets $D_n\subset\mathbb{Z}\times\mathbb{Z}^m$, such that (3.24) holds.

Theorem 4.7. *Let $Y=(Y_t(x))_{(t,x)\in\mathbb{Z}\times\mathbb{Z}^m}$ be an ambit field as defined in (4.1) such that $E[Y_t(x)]=0$, $E[|Y_t(x)|^{2+\delta}]<\infty$ for some $\delta>0$. Additionally assume that Y is θ-lex-weakly dependent with θ-lex-coefficients satisfying $\theta_Y(h)=\mathcal{O}(h^{-\alpha})$, where $\alpha>m(1+\frac{1}{\delta})$. Then*

$$\sigma^2=\sum_{(u_t,u_x)\in\mathbb{Z}\times\mathbb{Z}^m}E[Y_0(0)Y_{u_t}(u_x)|\mathcal{I}],$$

with $\mathcal{I}$ from Theorem 2.5, is finite, non-negative and

$$\frac{1}{|D_n|^{\frac{1}{2}}} \sum_{(u_t,u_x)\in D_n} Y_{u_t}(u_x) \xrightarrow[n\to\infty]{d} \varepsilon\sigma, \tag{4.9}$$

where ε is a standard normally distributed random variable which is independent of σ^2.

Proof. The result follows from Theorem 2.5. ∎

Corollary 4.8. *Let $Y = (Y_t(x))_{(t,x)\in\mathbb{Z}\times\mathbb{Z}^m}$ be an ambit field as defined in (4.1) such that $E[|Y_t(x)|^{2p+\delta}] < \infty$ for $p \geq 1$ and some $\delta > 0$. Additionally, let us assume that Y is θ-lex-weakly dependent with θ-lex-coefficients satisfying $\theta_Y(h) = \mathcal{O}(h^{-\alpha})$, where $\alpha > m\left(1+\frac{1}{\delta}\right)\left(\frac{2p-1+\delta}{p+\delta}\right)$. Then*

$$\Sigma = \sum_{(u_t,u_x)\in\mathbb{Z}\times\mathbb{Z}^m} Cov(Y_0(0)^p, Y_{u_t}(u_x)^p|\mathcal{I}),$$

with $\mathcal{I}$ from Theorem 2.5, is finite, non-negative and

$$\frac{1}{|D_n|^{\frac{1}{2}}} \sum_{(u_t,u_x)\in D_n} Y_{u_t}(u_x)^p - E[Y_0(0)^p] \xrightarrow[n\to\infty]{d} \varepsilon\sigma, \tag{4.10}$$

where ε is a standard normally distributed random variable which is independent of σ^2.

Proof. Analogous to Corollary 3.17. ∎

Remark 4.9. *Theorem 4.7 and Corollary 4.8 are important first steps to develop statistical inference for the class of ambit fields. However, we note that the limits in (4.9) and (4.10) are of mixed Gaussian type. Conditions which ensure the ergodicity of an ambit field with a deterministic kernel can be found in Theorem 3.6 [49] whereas for the case of a non-deterministic kernel this remains an open problem.*

5 Proofs

5.1 Proofs of Section 2.2

We establish Theorem 2.5 by extending results from [29] to higher dimensions. This enables us to connect the conditions on the asymptotic result stated in [27] with our definition of the θ-lex-coefficients.

Define the space of bounded, Lipschitz continuous functions $\mathscr{L}_1 = \{g : \mathbb{R} \to \mathbb{R}$, bounded and Lipschitz continuous with $Lip(g) \leq 1\}$.

For a σ-algebra $\mathscr{M}$ and an $\mathbb{R}^n$-valued integrable random field $X = (X_t)_{t\in\mathbb{Z}^m}$ we define the following two mixingale-type measures of dependence

(i) $\gamma(\mathscr{M}, X) = ||E[X|\mathscr{M}] - E[X]||_1$ and

(ii) $\theta(\mathscr{M}, X) = \sup_{g\in\mathscr{L}_1} ||E[g(X)|\mathscr{M}] - E[g(X)]||_1$.

Using the above measures of dependence we define the following dependence coefficients

$$\begin{aligned}\gamma_h &= \sup_{j\in\mathbb{Z}^m} \gamma\Big(\mathcal{F}_{V_j^h}, X_j\Big) \text{ and}\\ \theta_h &= \sup_{j\in\mathbb{Z}^m} \theta\Big(\mathcal{F}_{V_j^h}, X_j\Big),\end{aligned} \tag{5.1}$$

for $h \in \mathbb{N}^*$. Obviously, it holds $\gamma(\mathscr{M}, X) \leq 2\|X\|_1$ and $\gamma(\mathscr{M}, X) \leq \theta(\mathscr{M}, X)$ such that $\gamma_h \leq \theta_h$ for all $h \in \mathbb{N}^*$. If X is stationary we can write γ_h and θ_h from (5.1) as

$$\begin{aligned}\gamma_h &= \gamma\Big(\mathcal{F}_{V_0^h}, X_0\Big) \text{ and}\\ \theta_h &= \theta\Big(\mathcal{F}_{V_0^h}, X_0\Big),\end{aligned} \tag{5.2}$$

$h \in \mathbb{N}^*$. First, we extend Proposition 2.3 from [30] and connect the θ-lex-coefficients $\theta(h)$ from Definition 2.2 with the mixingale-type coefficient θ_h defined above.

Lemma 5.1. *Let $X = (X_t)_{t\in\mathbb{Z}^m}$ be a real-valued random field. Then it holds that*

$$\theta(h) = \theta_h, \quad h \in \mathbb{N}^*.$$

Proof. Fix $u, h \in \mathbb{N}^*$. We first show $\theta_u(h) \leq \theta_h$. Let $F \in \mathcal{F}^*$, $G \in \mathcal{F}$, $j \in \mathbb{R}^m$, $k \leq u$ and $\Gamma = \{i_1, \ldots, i_k\}$ with $i_1, \ldots, i_k \in V_j^h$. Now

$$\begin{aligned}\left|Cov\left(\frac{F(X_\Gamma)}{||F||_\infty}, \frac{G(X_j)}{Lip(G)}\right)\right| &= \left|E\left[\frac{F(X_\Gamma)}{||F||_\infty}\frac{G(X_j)}{Lip(G)} - E\left[\frac{F(X_\Gamma)}{||F||_\infty}\right]E\left[\frac{G(X_j)}{Lip(G)}\right]\right]\right|\\ &= \left|E\left[E\left[\frac{F(X_\Gamma)}{||F||_\infty}\frac{G(X_j)}{Lip(G)}\middle|\mathcal{F}_{V_j^h}\right] - \frac{F(X_\Gamma)}{||f||_\infty}E\left[\frac{G(X_j)}{Lip(G)}\right]\right]\right|\\ \leq E\left[\left|\frac{F(X_\Gamma)}{||F||_\infty}\right|\left|E\left[\frac{G(X_j)}{Lip(G)}\middle|\mathcal{F}_{V_j^h}\right] - E\left[\frac{G(X_j)}{Lip(G)}\right]\right|\right] &\leq \left\|E\left[\frac{G(X_j)}{Lip(G)}\middle|\mathcal{F}_{V_j^h}\right] - E\left[\frac{G(X_j)}{Lip(G)}\right]\right\|_1\\ &= \theta(\mathcal{F}_{V_j^h}, X_j) \leq \theta_h.\end{aligned}$$

Taking the supremum on the left hand side we obtain $\theta_u(h) \leq \theta_h$ and finally $\theta(h) \leq \theta_h$. To prove the converse inequality, we first remark that by the martingale convergence theorem

$$\theta(\mathcal{F}_{V_j^h}, X_j) = \lim_{k\to\infty} \theta(\mathcal{F}_{V_j^h\backslash V_j^k}, X_j), \tag{5.3}$$

Now, let $G \in \mathscr{L}_1$, i.e. $G \in \mathcal{F}$ with $Lip(G) \leq 1$ and $j \in \mathbb{R}^m$. We first define $X_j^h(k) = \{X_i : i \in V_j^h\backslash V_j^k\}$ and $F(X_j^h(k)) = sign(E[G(X_j)|\mathcal{F}_{V_j^h\backslash V_j^k}] - E[g(X_j)])$ for $k > h$. Then $F \in \mathcal{F}^*$ with $\|F\|_\infty = 1$ and it holds

$$E\left[\left|E[g(X_j)|\mathcal{F}_{V_j^h\backslash V_j^k}] - E[g(X_j)]\right|\right] = E\left[\Big(E[g(X_j)|\mathcal{F}_{V_j^h\backslash V_j^k}] - E[g(X_j)]\Big)F\Big(X_j^h(k)\Big)\right]$$

$$= E\left[E\left[F\left(X_j^h(k)\right)g(X_j)|\mathcal{F}_{V_j^h\setminus V_j^k}\right] - E\left[F\left(X_j^h(k)\right)\right]E\left[g(X_j)\right]\right]$$
$$= Cov\left(F\left(X_j^h(k)\right), g(X_j)\right) \leq \theta(h).$$

Using (5.3) we can deduce the stated equality. ∎

We define Q_X as the generalized inverse of the tail function $x \mapsto P(|X| > x)$ and G_X as the inverse of $x \mapsto \int_0^x Q_X(u)du$.

Lemma 5.2. *Let $X = (X_t)_{t\in\mathbb{Z}^m}$ be a stationary centered real-valued random field such that $\|X_0\|_2 < \infty$ and assume that*

$$\int_0^{||X_0||_1} \tilde{\theta}(u) Q_{X_0} \circ G_{X_0}(u) du < \infty, \tag{5.4}$$

with Q_X and G_X as defined above and $\tilde{\theta}(u) = \sum_{k\in V_0} \mathbb{1}_{\left\{u<\theta_{|k|}\right\}}$. Then

$$\sum_{k\in V_0} |E[X_k E_{|k|}[X_0]]| < \infty, \tag{5.5}$$

where $E_{|k|}[X_0] = E[X_0|F_{V_0^{|k|}}]$.

Proof. First, let us observe that X_k is $\mathcal{F}_{V_0^{|k|}}$ measurable, since $k \in V_0^{|k|}$. Then define $\varepsilon_k = sign(E_{|k|}[X_0])$ such that

$$\sum_{k\in V_0} |E[X_k E_{|k|}[X_0]]| \leq \sum_{k\in V_0} E[|X_k||E_{|k|}[X_0]|] = \sum_{k\in V_0} E[|X_k|\varepsilon_k E_{|k|}[X_0]] = \sum_{k\in V_0} E[E_{|k|}[|X_k|\varepsilon_k X_0]]$$
$$= \sum_{k\in V_0} Cov(|X_k|\varepsilon_k, X_0).$$

We use Equation (4.2) of [29, Proposition 1] to get

$$\leq 2\sum_{k\in V_0} \int_0^{\gamma\left(\mathcal{F}_{V_0^{|k|}}, X_0\right)/2} Q_{\varepsilon_k|X_k|} \circ G_{X_0}(u)du = 2\int_0^{\|X_0\|_1} \sum_{k\in V_0} \mathbb{1}_{\left\{u<\gamma\left(\mathcal{F}_{V_0^{|k|}}, X_0\right)/2\right\}} Q_{X_k} \circ G_{X_0}(u)du$$
$$\leq 2\int_0^{\|X_0\|_1} \sum_{k\in V_0} \mathbb{1}_{\left\{u<\theta\left(\mathcal{F}_{V_0^{|k|}}, X_0\right)/2\right\}} Q_{X_k} \circ G_{X_0}(u)du \leq 2\int_0^{\|X_0\|_1} \sum_{k\in V_0} \mathbb{1}_{\left\{u<\theta_{|k|}/2\right\}} Q_{X_k} \circ G_{X_0}(u)du.$$

Now let $\tilde{\theta}(u) = \sum_{k\in V_0} \mathbb{1}_{\left\{u<\theta_{|k|}\right\}}$ and note that $Q_{X_k} = Q_{X_0}$, such that

$$\leq 2\int_0^{||X_0||_1} \tilde{\theta}(u) Q_{X_0} \circ G_{X_0}(u)du.$$

This shows that (5.5) holds if (5.4) is satisfied. ∎

We now derive sufficient criteria such that (5.4) holds similar to [29, Lemma 2].

Lemma 5.3. *Let $X = (X_t)_{t\in\mathbb{Z}^m}$ be a stationary real-valued random field and θ_h defined as above. Then (5.4) holds if $\|X\|_r < \infty$ for some $r > p > 1$ and $\sum_{h=0}^{\infty}(h+1)^{m(p-1)\frac{(r-1)}{(r-p)}-1}\theta_h < \infty$. In particular for $p = 2$ and $r = 2+\delta$ with $\delta > 0$ the above condition holds if $\theta_h \in \mathcal{O}(h^{-\alpha})$ for $\alpha > m(1+\frac{1}{\delta})$.*

Proof. As stated in [29, Proof of Lemma 2] we note that $\int_0^{\|X\|_1} Q_X^{r-1}\circ G_X(u)du = \int_0^1 Q_X^r(u)du = E[|X|^r]$. Applying Hölder's inequality with $q = \frac{r-1}{r-p}$ and $q' = \frac{r-1}{p-1}$ gives

$$\begin{aligned}
&\left(\int_0^{\|X\|_1} \tilde{\theta}(u)^{p-1} Q_X^{p-1}\circ G_X(u)du\right)^{(r-1)}\\
\le &\left(\int_0^{\|X\|_1} \tilde{\theta}(u)^{(p-1)\frac{(r-1)}{(r-p)}}du\right)^{(r-p)} \left(\int_0^{\|X\|_1} Q_X^{r-1}\circ G_X(u)du\right)^{(p-1)}\\
= &\left(\int_0^{\|X\|_1} \tilde{\theta}(u)^{(p-1)\frac{(r-1)}{(r-p)}}du\right)^{(r-p)} \|X\|_r^{(rp-r)}.
\end{aligned}$$

Let us note that θ_h as defined in (5.2) is non-increasing. Then, for any function f we have

$$\begin{aligned}
f(\tilde{\theta}(u)) = f\left(\sum_{k\in V_0} \mathbb{1}_{\{u<\theta_{|k|}\}}\right) &= \sum_{h=0}^{\infty} f\left(\sum_{k\in V_0} \mathbb{1}_{\{u<\theta_{|k|}\}}\right)\mathbb{1}_{\{\theta_{h+1}\le u<\theta_h\}}\\
&= \sum_{h=0}^{\infty} f\left(\sum_{k\in V_0:|k|\le h} 1\right)\mathbb{1}_{\{\theta_{h+1}\le u<\theta_h\}}
\end{aligned}$$

Note that $\sum_{k\in V_0:|k|\le h} 1 = \sum_{i=0}^{m-1} h(2h+1)^i = \frac{1}{2}\left((2h+1)^m - 1\right)$ such that

$$= \sum_{h=0}^{\infty} f\left(\frac{1}{2}\Big((2h+1)^m - 1\Big)\right)\mathbb{1}_{\{\theta_{h+1}\le u<\theta_h\}}.$$

Let us assume that f is monotonically increasing, sub-multiplicative and $f(0) = 0$ such that $f\left(\frac{1}{2}\Big((2h+1)^m - 1\Big)\right) \le f(2^{m-1})f((h+1)^m) = \sum_{k=0}^{h} f(2^{m-1})\left(f\left((k+1)^m\right) - f\left(k^m\right)\right)$. Finally we can deduce

$$\le \sum_{h=0}^{\infty} f(2^{m-1})f\left((h+1)^m\right)\mathbb{1}_{\{\theta_{h+1}\le u<\theta_h\}} = f(2^{m-1})\sum_{h=0}^{\infty}\left(f\left((h+1)^m\right) - f\left(h^m\right)\right)\mathbb{1}_{\{u<\theta_h\}}.$$

Applying the above result for $f(x) = x^v$ with $v = (p-1)\frac{(r-1)}{(r-p)}$ and noting that $(h+1)^{vm} - h^{vm} \le vm(h+1)^{vm-1}$ for $vm \ge 1$ and $(h+1)^{vm} - h^{vm} \le vm\ h^{vm-1}$ for $vm < 1$, $h > 0$ by the mean value theorem, we get that for a constant $C = 2^{v(m-1)}\theta_1 > 0$

$$\left(\int_0^{\|X\|_1} (\tilde{\theta}(u))^{(p-1)\frac{(r-1)}{(r-p)}}du\right)^{(r-p)}$$

$$\leq \begin{cases} \left(C + \int_0^{\|X\|_1} 2^{v(m-1)} vm \sum_{h=0}^{\infty} (h+1)^{vm-1} \mathbb{1}_{\{u<\theta_h\}} du\right)^{(r-p)}, & \text{if } vm \geq 1 \\ \left(C + \int_0^{\|X\|_1} 2^{v(m-1)} vm \sum_{h=1}^{\infty} h^{vm-1} \mathbb{1}_{\{u<\theta_h\}} du\right)^{(r-p)}, & \text{if } vm < 1 \end{cases}$$

$$= \begin{cases} \left(C + 2^{v(m-1)} vm \sum_{h=0}^{\infty} (h+1)^{vm-1} \theta_h\right)^{(r-p)}, & \text{if } vm \geq 1 \\ \left(C + 2^{v(m-1)} vm \sum_{h=1}^{\infty} h^{vm-1} \theta_h\right)^{(r-p)}, & \text{if } vm < 1 \end{cases}$$

$$\leq \max\left(1, 2^{r-p-1}\right) \left(C^{r-p} + \left((vm) 2^{v(m-1)} \left(\sum_{h=0}^{\infty} (h+1)^{vm-1} \theta_h\right)\right)^{(r-p)}\right),$$

which concludes the proof. ∎

Proof of Theorem 2.5. In order to use [27, Theorem 1] we need to show that

$$\sum_{k \in V_0} |E[X_k E_{|k|}[X_k]]| < \infty.$$

By Lemma 5.2 and Lemma 5.3 the result is proven if $\theta_h \in \mathcal{O}(h^{-\alpha})$ with $\alpha > m(1 + \frac{1}{\delta})$. Finally, since X is stationary an application of Lemma 5.1 concludes.

5.2 Proofs of Section 3.3

Proof of Proposition 3.11.

(i) Let $t \in \mathbb{R}^m$, $\psi > 0$. We restrict the MMAF X to a finite support and define the truncated sequence

$$X_t^{(\psi)} = \int_S \int_{A_t \setminus V_t^{\psi}} f(A, t-s) \Lambda(dA, ds). \tag{5.6}$$

Note that the kernel function f is square integrable such that (3.6), (3.7) and (3.8) hold. Therefore, f is Λ-integrable. Since $E[X_t X_t'] < \infty$ for all $t \in \mathbb{R}^m$ by Proposition 3.6 we can derive an upper bound of the expectation

$$\begin{aligned} E\Big[\|X_t - X_t^{(\psi)}\|\Big] &= E\left[\left\| \int_S \int_{A_t \cap V_t^{\psi}} f(A, t-s) \Lambda(dA, ds) \right\|\right] \\ &\leq E\left[\left\| \int_S \int_{A_t \cap V_t^{\psi}} f(A, t-s) \Lambda(dA, ds) \right\|^2\right]^{\frac{1}{2}} \\ &= \left(\sum_{\kappa=1}^{n} E\left[\left(\left(\int_S \int_{A_t \cap V_t^{\psi}} f(A, t-s) \Lambda(dA, ds)\right)^{(\kappa)}\right)^2\right]\right)^{\frac{1}{2}}. \end{aligned}$$

Using Proposition 3.7 and the translation invariance of A_t and V_t^{ψ} this is equal to

$$\left(\int_S \int_{A_0 \cap V_0^{\psi}} \mathrm{tr}(f(A, -s) \Sigma_{\Lambda} f(A, -s)') ds \pi(dA)\right)^{\frac{1}{2}}.$$

Now let $G \in \mathcal{F}$ and $F \in \mathcal{F}^*$, i.e. F, G are bounded with $\|F\|_\infty, \|G\|_\infty \leq 1$ and G is additionally Lipschitz-continuous, $u \in \mathbb{N}^*, h \in \mathbb{R}^+, \Gamma = \{i_1, \ldots, i_u\} \in (\mathbb{R}^m)^u$ and $j \in \mathbb{R}^m$ as in Definition 2.2 such that $i_1, \ldots, i_u \in V_j^h$. For $a \in \{1, \ldots, u\}$ define

$$\begin{aligned} X_{i_a} &= \int_S \int_{A_{i_a}} f(A, i_a - s)\Lambda(dA, ds) \text{ and} \\ X_j^{(\psi)} &= \int_S \int_{A_j \setminus V_j^\psi} f(A, j - s)\Lambda(dA, ds). \end{aligned}$$

W.l.o.g. we assume that $i_a \leq_{lex} i_u$ for all $a \in \{1, \ldots, u\}$. If there exists a ψ such that $A_{i_u} \cap A_j \setminus V_j^\psi = \emptyset$, then $A_{i_a} \cap A_j \setminus V_j^\psi = \emptyset$.
Now, A is translation invariant with initial sphere of influence A_0. Furthermore, A_0 satisfies (3.13). Then, for $\psi(h)$ as defined in (3.16) it holds $A_{i_u} \cap A_j \setminus V_j^\psi = \emptyset$.
From now on we set $\psi = \psi(h)$. We then get that $I_a = S \times A_{i_a}$ and $J = S \times A_j \setminus V_j^\psi$ are disjoint or have intersection on a set $S \times O$, where $O \subset \mathbb{R}^m$ and $\dim(O) < m$. Since $(\pi \times \lambda)(S \times O) = 0$, by the definition of a Lévy basis X_{i_a} and $X_j^{(\psi)}$ are independent for all $a \in \{1, \ldots, u\}$. Finally, we get that X_Γ and $X_j^{(\psi)}$ are independent and therefore also $F(X_\Gamma)$ and $G(X_j^{(\psi)})$. Now

$$\begin{aligned} &|Cov(F(X_\Gamma), G(X_j))| \\ &\leq |Cov(F(X_\Gamma), G(X_j^{(\psi)}))| + |Cov(F(X_\Gamma), G(X_j) - G(X_j^{(\psi)}))| \\ &= |E[(G(X_j) - G(X_j^{(\psi)}))F(X_\Gamma)] - E[G(X_j) - G(X_j^{(\psi)})]E[F(X_\Gamma)]| \\ &\leq 2\|F\|_\infty E\Big[|G(X_j) - G(X_j^{(\psi)})|\Big] \leq 2Lip(G)\|F\|_\infty E[\|X_j - X_j^{(\psi)}\|], \end{aligned}$$

and using the above inequality for $E[\|X_t - X_t^{(\psi)}\|]$ with ψ as described above we conclude

$$\leq 2Lip(G)\|F\|_\infty \Big(\int_S \int_{A_0 \cap V_0^\psi} \mathrm{tr}(f(A, -s)\Sigma_\Lambda f(A, -s)')ds\pi(dA) \Big)^{\frac{1}{2}}.$$

Therefore X is θ-lex weakly dependent with θ-lex-coefficients

$$\theta_X(h) \leq 2\Big(\int_S \int_{A_0 \cap V_0^\psi} \mathrm{tr}(f(A, -s)\Sigma_\Lambda f(A, -s)')ds\pi(dA) \Big)^{\frac{1}{2}},$$

which converge to zero as h goes to infinity by applying the dominated convergence theorem.

(ii) Let $t \in \mathbb{R}^m$, $\psi > 0$. As in Proposition 3.11 we define $X_t^{(\psi)}$. For the upper bound of the expectation we can derive with the help of Proposition 3.7

$$\begin{aligned} E\Big[\|X_t - X_t^{(\psi)}\|\Big] \leq \Bigg(&\int_S \int_{A_0 \cap V_t^\psi} \mathrm{tr}(f(A, t - s)\Sigma_\Lambda f(A, t - s)')ds\pi(dA) \\ &+ \Big\| \int_S \int_{A_0 \cap V_t^\psi} f(A, t - s)\mu_\Lambda ds\pi(dA) \Big\|^2 \Bigg)^{\frac{1}{2}}. \end{aligned}$$

Finally, we can proceed as in the proof of Proposition 3.11 and we obtain a bound for the θ-lex-coefficients.

(iii) Since the kernel function f is in L^1 the Equations (3.9) and (3.10) hold and f is Λ-integrable and $E[X_t] < \infty$ by Proposition 3.6. In the following we use the notation of Proposition 3.11. Let $t \in \mathbb{R}^m$ and $\psi > 0$. Then, we can derive with the help of Proposition 3.7

$$\begin{aligned} &E\Big[\|X_t - X_t^{(\psi)}\|\Big] \\ &\leq \Big(\int_S \int_{A_0 \cap V_0^{\psi}} \Big\|f(A,-s)\gamma_0\Big\| ds\pi(dA) + \int_S \int_{A_0 \cap V_0^{\psi}} \int_{\mathbb{R}^d} \Big\|f(A,-s)y\Big\| \nu(dy) ds\pi(dA)\Big), \end{aligned}$$

where we used that $E[\int_E f(t)d\mu(t)] = \int_E f(t)d\nu(t)$ for a Poisson random measure μ with corresponding intensity measure ν and arbitrary set E.
Now for F, G, X_Γ, X_j and $\psi = \psi(h)$ and as described in the proof of Proposition 3.11 we get

$$\begin{aligned} |Cov(F(X_\Gamma), G(X_j))| \leq 2Lip(G)\|F\|_\infty \Big(&\int_S \int_{A_0 \cap V_0^{\psi(h)}} \Big\|f(A,-s)\gamma_0\Big\| ds\pi(dA) \\ &+ \int_S \int_{A_0 \cap V_0^{\psi(h)}} \int_{\mathbb{R}^d} \Big\|f(A,-s)y\Big\| \nu(dy) ds\pi(dA)\Big) \end{aligned}$$

Therefore X is θ-lex weakly dependent with θ-lex-coefficients

$$\begin{aligned} \theta_X(h) \leq 2\Big(&\int_S \int_{A_0 \cap V_0^{\psi(h)}} \Big\|f(A,-s)\gamma_0\Big\| ds\pi(dA) \\ &+ \int_S \int_{A_0 \cap V_0^{\psi(h)}} \int_{\mathbb{R}^d} \Big\|f(A,-s)y\Big\| \nu(dy) ds\pi(dA)\Big), \end{aligned}$$

which converge to zero as h goes to infinity by applying the dominated convergence theorem.

(iv) We use the notation of Proposition 3.11 and realize Λ in distribution as the sum of two $\mathbb{R}^d$-valued independent Lévy bases Λ_1 and Λ_2 with characteristic quadruplets $(\gamma, \Sigma, \nu\big|_{\|x\|\leq 1}, \pi)$ and $(0, 0, \nu\big|_{\|x\|>1}, \pi)$. Since $f \in L^1 \cap L^2$ we know that both integrals $X_t^{(\Lambda_1)} = \int_S \int_{\mathbb{R}^m} f(A, t-s)\Lambda_1(dA, ds)$ and $X_t^{(\Lambda_2)} = \int_S \int_{\mathbb{R}^m} f(A, t-s)\Lambda_2(dA, ds)$ exist and additionally it holds $\int_S \int_{\mathbb{R}^m} f(A, t-s)\Lambda(dA, ds) = X_t^{(\Lambda_1)} + X_t^{(\Lambda_2)}$. Let us note that

$$\begin{aligned} E\Big[\|X_t - X_t^{(\psi)}\|\Big] &\leq E\Big[\|X_t^{(\Lambda_1)} - (X_t^{(\Lambda_1)})^{(\psi)}\|\Big] + E\Big[\|X_t^{(\Lambda_2)} - (X_t^{(\Lambda_2)})^{(\psi)}\|\Big] \\ &\leq E\Bigg[\Big\|X_t^{(\Lambda_1)} - (X_t^{(\Lambda_1)})^{(\psi)}\Big\|^2\Bigg]^{\frac{1}{2}} + E\Bigg[\Big\|X_t^{(\Lambda_2)} - (X_t^{(\Lambda_2)})^{(\psi)}\Big\|\Bigg]. \end{aligned}$$

Following the proof of (ii) (for the first summand) and (iii) (for the second summand) we obtain the stated bound for the θ-lex-coefficients.

Proof of Proposition 3.12. In order to show that the MMAF Z is well defined we need to show that $g(A,s)$ is Λ-integrable as described in Theorem 3.3, i.e. $g(A,s)$ satisfies the conditions (3.6), (3.7) and (3.8). Let us consider an induction over k. For the sake of brevity we will consider the norm $\|(x_1,\dots,x_m)\| = \|x_1\| + \dots + \|x_m\|$ for $x_i \in \mathbb{R}$ for $i = 1,\dots,m$. Then, for $k=1$ we consider

$$g(A,s) = \Big(f\Big(A,s\Big), f\Big(A,s-s_1,\dots,f\Big(A,s-s_{|S_k|}\Big)', \text{ where } |S_k| = 2\cdot 3^{m-1}.$$

Note that for $x \in \mathbb{R}^d$

$$\begin{aligned}
\mathbb{1}_{[0,1]}(\|g(A,s)x\|) &\le \mathbb{1}_{[0,1]}(\|f(A,s)x\|),\\
\mathbb{1}_{[0,1]}(\|g(A,s)x\|) &\le \mathbb{1}_{[0,1]}(\|f(A,s-s_1)x\|),\\
&\dots\\
\mathbb{1}_{[0,1]}(\|g(A,s)x\|) &\le \mathbb{1}_{[0,1]}(\|f(A,s-s_{|S_k|})x\|),
\end{aligned}$$

such that

$$\begin{aligned}
&\int_S\int_{\mathbb{R}^m}\Big\|g(A,s)\gamma+\int_{\mathbb{R}^d} g(A,s)x\left(\mathbb{1}_{[0,1]}(\|g(A,s)x\|)-\mathbb{1}_{[0,1]}(\|x\|)\right)\nu(dx)\Big\|ds\pi(dA)\\
&\le \int_S\int_{\mathbb{R}^m}\Big\|f(A,s)\gamma+\int_{\mathbb{R}^d} f(A,s)x\left(\mathbb{1}_{[0,1]}(\|f(A,s)x\|)-\mathbb{1}_{[0,1]}(\|x\|)\right)\nu(dx)\Big\|ds\pi(dA)\\
&+\int_S\int_{\mathbb{R}^m}\Big\|f(A,s-s_1)\gamma+\int_{\mathbb{R}^d} f(A,s-s_1)x\left(\mathbb{1}_{[0,1]}(\|f(A,s-s_1)x\|)-\mathbb{1}_{[0,1]}(\|x\|)\right)\nu(dx)\Big\|ds\pi(dA)\\
&+\dots+\int_S\int_{\mathbb{R}^m}\Big\|f(A,s-s_{|S_k|})\gamma+\int_{\mathbb{R}^d} f(A,s-s_{|S_k|})x\big(\mathbb{1}_{[0,1]}(\|f(A,s-s_{|S_k|})\|-\mathbb{1}_{[0,1]}(\|x\|)\big)\nu(dx)\Big\|ds\pi(dA).
\end{aligned}$$

Since f is Λ-integrable we can conclude that the above expression is finite and (3.6) holds. Now

$$\begin{aligned}
&\int_S\int_{\mathbb{R}^m}\|g(A,s)\Sigma g(A,s)'\|ds\pi(dA)\\
&=\int_S\int_{\mathbb{R}^m}\|f(A,s)\Sigma f(A,s)'\|ds\pi(dA)+\int_S\int_{\mathbb{R}^m}\|f(A,s-s_1)\Sigma f(A,s-s_1)'\|ds\pi(dA)\\
&\quad+\dots+\int_S\int_{\mathbb{R}^m}\|f(A,s-s_{|S_k|})\Sigma f(A,s-s_{|S_k|})'\|ds\pi(dA),
\end{aligned}$$

is finite since f is Λ-integrable and (3.7) holds. Since $\left(\sum_{i=1}^n a_i\right)^2 \le n\sum_{i=1}^n a_i^2$ we have

$$\|g(A,s)\|^2 \le |S_k|\Big(\|f(A,s)\|^2 + \|f(A,s-s_1)\|^2 + \dots + \|f(A,s-s_{|S_k|})\|^2\Big),$$

and finally

$$\begin{aligned}
&\int_S\int_{\mathbb{R}^m}\int_{\mathbb{R}^d}\Big(1\wedge\|g(A,s)x\|^2\Big)\nu(dx)ds\pi(dA)\\
&\le |S_k|\Bigg(\int_S\int_{\mathbb{R}^m}\int_{\mathbb{R}^d}\Big(1\wedge\|f(A,s)\|^2\Big)\nu(dx)ds\pi(dA)+\int_S\int_{\mathbb{R}^m}\int_{\mathbb{R}^d}\Big(1\wedge\|f(A,s-s_1)\|^2\Big)\nu(dx)ds\pi(dA)\\
&\quad+\dots+\int_S\int_{\mathbb{R}^m}\int_{\mathbb{R}^d}\Big(1\wedge\|f(A,s-s_{|S_k|})\|^2\Big)\nu(dx)ds\pi(dA)\Bigg),
\end{aligned}$$

which is finite since f satisfies (3.8). Thus, g is Λ-integrable and Z is an (A,Λ)-influenced MMAF.
Assume X satisfies the assumptions of Proposition (i) 3.11 and consider $\psi(h)$ as defined in (3.16). Then

$$\begin{aligned}
\theta_Z^{(i)}(h) \leq& 2\Bigg(\int_S\int_{A_0\cap V_0^{\psi(h)}} \mathrm{tr}\Big(g(A,-s)\Sigma_\Lambda g(A,-s)'\Big)ds\pi(dA)\Bigg)^{\frac{1}{2}}\\
=&2\Bigg(\int_S\int_{A_0\cap V_0^{\psi(h)}} \mathrm{tr}\Big(f(A,-s)\Sigma_\Lambda f(A,-s)'\Big)ds\pi(dA)\\
&+\int_S\int_{A_0\cap V_0^{\psi(h)}} \mathrm{tr}\Big(f(A,s_1-s)\Sigma_\Lambda f(A,s_1-s)'\Big)ds\pi(dA)\\
&+\ldots+\int_S\int_{A_0\cap V_0^{\psi(h)}} \mathrm{tr}\Big(f(A,s_{|S_k|}-s)\Sigma_\Lambda f(A,s_{|S_k|}-s)'\Big)ds\pi(dA)\Bigg)^{\frac{1}{2}}\\
\leq& 2|S_k|^{\frac{m}{2}}\Bigg(\int_S\int_{A_0\cap V_0^{\psi(h)-k}} \mathrm{tr}\Big(f\Big(A,-s\Big)\Sigma_\Lambda f\Big(A,-s\Big)'\Big)ds\pi(dA)\Bigg)^{\frac{1}{2}}\\
=&|S_k|^{\frac{m}{2}}\hat{\theta}_X^{(i)}(h-\psi^{-1}(k)).
\end{aligned}$$

where ψ^{-1} denotes the inverse of ψ, for all $\psi(h)>k$. Thus, Z is a $(k+1)(2k+1)^{m-1}$-dimensional θ-lex-weakly dependent MMAF. Similar calculations lead to the other statements in (3.22).

5.3 Proof of Section 3.4

Proof of Theorem 3.14. Let us first consider X to be univariate. In order to use [27, Theorem 1] we need to show

$$\sum_{k\in V_0} |X_k E_{|k|}[X_0]| \in L^1. \tag{5.7}$$

The Hölder inequality implies

$$\|X_k E_{|k|}(X_0)\|_1 \leq \|X_k\|_2 \|E[X_0|\mathcal{F}_{V_0^{|k|}}]\|_2,$$

where $\|X_k\|_2 < C$ for all k and a constant C. Furthermore, we note that $\|E[X|\mathcal{F}]\|_2 = \|E[E[X|\mathcal{G}]|\mathcal{F}]\|_2 \leq \|E[X|\mathcal{G}]\|_2$ holds for a σ-algebra $\mathcal{G}$, a sub σ-algebra $\mathcal{F}$ and an L^2 random variable X, as the conditional expectation is the orthogonal projection in L^2. Now, using (3.12)

$$\begin{aligned}
\|E[X_0|\mathcal{F}_{V_0^{|k|}}]\|_2 =& \left\|E\left[\int_S\int_{V_0} \mathbb{1}_{A_0}(s)f(A,-s)\Lambda(dA,ds)\middle|\sigma(X_l : l\in V_0^{|k|})\right]\right\|_2\\
\leq& \left\|E\left[\int_S\int_{V_0} \mathbb{1}_{A_0}(s)f(A,-s)\Lambda(dA,ds)\middle|\sigma(\Lambda(B) : B\in\mathcal{B}(V_0^{|k|}))\right]\right\|_2
\end{aligned}$$

$$
\begin{aligned}
=&\Bigg\| E\Bigg[\int_S \int_{V_0 \cap V_0^{|k|}} \mathbb{1}_{A_0}(s) f(A,-s)\Lambda(dA,ds) \\
&+ \int_S \int_{V_0 \setminus V_0^{|k|}} \mathbb{1}_{A_0}(s) f(A,-s)\Lambda(dA,ds) \Big| \sigma(\Lambda(B) : B \in \mathcal{B}(V_0^{|k|})) \Bigg] \Bigg\|_2 \\
=&\Bigg\| E\left[\int_S \int_{V_0^{|k|}} \mathbb{1}_{A_0}(s) f(A,-s)\Lambda(dA,ds) \Big| \sigma(\Lambda(B) : B \in \mathcal{B}(V_0^{|k|})) \right] \\
&+ E\left[\int_S \int_{V_0 \setminus V_0^{|k|}} \mathbb{1}_{A_0}(s) f(A,-s)\Lambda(dA,ds) \Big| \sigma(\Lambda(B) : B \in \mathcal{B}(V_0^{|k|})) \right] \Bigg\|_2 .
\end{aligned}
$$

We note that $\int_S \int_{V_0^{|k|}} \mathbb{1}_{A_0}(s) f(A,-s)\Lambda(dA,ds)$ is measurable with respect to $\sigma(\Lambda(B) : B \in \mathcal{B}(V_0^{|k|}))$. Since Λ is a Lévy basis (in particular independent for disjoint sets) we get that $\int_S \int_{V_0 \setminus V_0^{|k|}} \mathbb{1}_{A_0}(s) f(A,-s)\ \Lambda(dA,ds)$ is independent of $\sigma(\Lambda(B) : B \in \mathcal{B}(V_0^{|k|}))$, such that the above equation is equal to

$$
= \left\| \int_S \int_{V_0^{|k|}} \mathbb{1}_{A_0}(s) f(A,-s)\Lambda(dA,ds) + E\left[\int_S \int_{V_0 \setminus V_0^{|k|}} \mathbb{1}_{A_0}(s) f(A,-s)\Lambda(dA,ds) \right] \right\|_2 .
$$

Since $\gamma + \int_{\|x\|>1} x\nu(dx) = 0$ the second summand is equal to zero and we arrive at

$$
\begin{aligned}
&= \left\| \int_S \int_{V_0^{|k|}} \mathbb{1}_{A_0}(s) f(A,-s) f(A,-s)\Lambda(dA,ds) \right\|_2 \\
&= \Big(\int_S \int_{A_0 \cap V_0^{|k|}} \mathrm{tr}(f(A,-s)\Sigma_\Lambda f(A,-s)')ds\pi(dA) \Big)^{\frac{1}{2}} = \theta_X(|k|),
\end{aligned}
$$

using Proposition 3.7.
The stated result then follows from [27, Theorem 1] using the dominated convergence theorem. The Cramér-Wold device establishes the multivariate case straightforwardly.

5.4 Proofs of Section 3.5

Proof of Proposition 3.19.

(i) Let $t \in \mathbb{R}^m$ and $\psi > 0$. We truncate the MMAF to a finite support, i.e.

$$
X_t^{(\psi)} = \int_S \int_{\mathbb{R}^m} f(A,t-s)\mathbb{1}_{(-\psi,\psi)^m}(t-s)\Lambda(dA,ds) = \int_S \int_{(t-\psi,t+\psi)^m} f(A,t-s)\Lambda(dA,ds). \tag{5.8}
$$

Note that the kernel function f is square integrable such that (3.6), (3.7) and (3.8) hold. Therefore, f is Λ-integrable. Since $E[X_t X_t'] < \infty$ for all $t \in \mathbb{R}^m$ by Proposition 3.6 we can derive an upper bound of the expectation

$$
E\Big[\|X_t - X_t^{(\psi)}\| \Big]
$$

$$= E\Big[\Big\| \int_S \int_{\big((t-\psi,t+\psi)^m\big)^c} f(A,t-s)\Lambda(dA,ds)\Big\|\Big] \leq E\Big[\Big\| \int_S \int_{\big((t-\psi,t+\psi)^m\big)^c} f(A,t-s)\Lambda(dA,ds)\Big\|^2\Big]^{\frac{1}{2}}$$

$$= \Bigg(\sum_{\kappa=1}^{n} E\Bigg[\Bigg(\Bigg(\int_S \int_{\big((t-\psi,t+\psi)^m\big)^c} f(A,t-s)\Lambda(dA,ds)\Bigg)^{(\kappa)}\Bigg)^2\Bigg]\Bigg)^{\frac{1}{2}},$$

where $x^{(\kappa)}$ denotes the κth coordinate of $x \in \mathbb{R}^n$. Using Proposition 3.7 and the stationarity of X this is equal to

$$\Bigg(\int_S \int_{\big((\psi,\psi)^m\big)^c} \operatorname{tr}(f(A,-s)\Sigma_\Lambda f(A,-s)')ds\pi(dA)\Bigg)^{\frac{1}{2}}.$$

Now let $F, G \in \mathcal{F}$, i.e. bounded and additionally Lipschitz-continuous, $(u,v) \in \mathbb{N}^* \times \mathbb{N}^*, h \in \mathbb{R}^+, \Gamma_i = \{i_1,\ldots,i_u\} \in (\mathbb{R}^m)^u$ and $\Gamma_j = \{j_1,\ldots,j_v\} \in (\mathbb{R}^m)^v$ as in Definition 2.1 such that $dist(\Gamma_i,\Gamma_j) \geq h$. For $a \in \{1,\ldots,u\}$ and $b \in \{1,\ldots,v\}$ define

$$X_{i_a}^{(\psi)} = \int_S \int_{(i_a-\psi,i_a+\psi)^m} f(A,i_a-s)\Lambda(dA,ds) \text{ and}$$
$$X_{j_b}^{(\psi)} = \int_S \int_{(j_b-\psi,j_b+\psi)^m} f(A,j_b-s)\Lambda(dA,ds).$$

Now consider $a \in \{1,\ldots,u\}$ and $b \in \{1,\ldots,v\}$ such that $\inf_{1\leq x\leq u,1\leq y\leq v} \|i_x - j_y\|_\infty = \|i_a - j_b\|_\infty$. Define the two sets $I_a = S \times (i_a-\psi, i_a+\psi)^m$ and $J_b = S \times (j_b-\psi, j_b+\psi)^m$. Furthermore, consider $\psi = \frac{h}{2}$ and since $\|i_a - j_b\|_\infty \geq h$ it holds that I_a and J_b are disjoint as well as $I_{\tilde{a}}$ and $J_{\tilde{b}}$ for all $\tilde{a} = 1,\ldots,u$ and $\tilde{b} = 1,\ldots,v$. By the definition of a Lévy basis $X_{i_a}^{(\psi)}$ and $X_{j_b}^{(\psi)}$ are independent for all $a \in \{1,\ldots,u\}$ and $b \in \{1,\ldots,v\}$. Finally, we get that $X_{\Gamma_i}^{(\psi)}$ and $X_{\Gamma_j}^{(\psi)}$ are independent and therefore also $F(X_{\Gamma_i}^{(\psi)})$ and $G(X_{\Gamma_j}^{(\psi)})$.
Now

$$\begin{aligned}
&|Cov(F(X_{\Gamma_i}), G(X_{\Gamma_j}))| \\
&\leq |Cov(F(X_{\Gamma_i}) - F(X_{\Gamma_i}^{(\psi)}), G(X_{\Gamma_j}))| + |Cov(F(X_{\Gamma_i}^{(\psi)}), G(X_{\Gamma_j}) - G(X_{\Gamma_i}^{(\psi)}))| \\
&= |E[(F(X_{\Gamma_i}) - F(X_{\Gamma_i}^{(\psi)}))G(X_{\Gamma_j})] - E[F(X_{\Gamma_i}) - F(X_{\Gamma_i}^{(\psi)})]E[G(X_{\Gamma_j})]| \\
&\ + |E[(G(X_{\Gamma_j}) - G(X_{\Gamma_j}^{(\psi)}))F(X_{\Gamma_i}^{(\psi)})] - E[G(X_{\Gamma_j}) - G(X_{\Gamma_j}^{(\psi)})]E[F(X_{\Gamma_i}^{(\psi)})]| \\
&\leq 2\Big(\|G\|_\infty E\Big[|F(X_{\Gamma_i}) - F(X_{\Gamma_i}^{(\psi)})|\Big] + \|F\|_\infty E\Big[|G(X_{\Gamma_j}) - G(X_{\Gamma_i}^{(\psi)})|\Big]\Big) \\
&\leq 2\Big(\|G\|_\infty Lip(F) \sum_{l=1}^{u} E[\|X_{i_l} - X_{i_l}^{(\psi)}\|] + \|F\|_\infty Lip(G) \sum_{k=1}^{v} E[\|X_{j_k} - X_{j_k}^{(\psi)}\|]\Big),
\end{aligned}$$

and using the above inequality for $E[\|X_t - X_t^{(\psi)}\|]$ with ψ chosen as described above we conclude

$$\leq 2(u\|G\|_\infty Lip(F) + v\|F\|_\infty Lip(G))\Bigg(\int_S \int_{\big(\big(-\frac{h}{2},\frac{h}{2}\big)^m\big)^c} \operatorname{tr}(f(A,-s)\Sigma_\Lambda f(A,-s)')ds\pi(dA)\Bigg)^{\frac{1}{2}}.$$

Therefore X is η-weakly dependent with η-coefficients

$$\eta_X(h) \leq 2\Big(\int_S \int_{\big(\big(-\frac{h}{2},\frac{h}{2}\big)^m\big)^c} \operatorname{tr}(f(A,-s)\Sigma_\Lambda f(A,-s)')ds\pi(dA)\Big)^{\frac{1}{2}},$$

which converge to zero as h goes to infinity by applying the dominated convergence theorem.

(iii) Since the kernel function f is in L^1 the Equations (3.9) and (3.10) hold and f is Λ-integrable. Let $t \in \mathbb{R}^m$, $\psi > 0$ and $X_t^{(\psi)}$ as in Proposition 3.19. Moreover, Proposition 3.6 implies that $E[X_t] < \infty$. Then, using Proposition 3.7 we arrive at

$$E\Big[\|X_t - X_t^{(\psi)}\|\Big]$$
$$\leq \Big(\int_S \int_{\big((t-\psi,t+\psi)^m\big)^c} \Big\|f(A,-s)\gamma_0\Big\|ds\pi(dA) + \int_S \int_{\big((t-\psi,t+\psi)^m\big)^c} \int_{\mathbb{R}^d} \Big\|f(A,-s)x\Big\|\nu(dx)ds\pi(dA)\Big).$$

Now for F, G, X_{Γ_i} and X_{Γ_j} and ψ as described in the proof of Proposition 3.11 we get

$$|Cov(F(X_{\Gamma_i}), G(X_{\Gamma_j}))| \leq 2(u\|G\|_\infty Lip(F) + v\|F\|_\infty Lip(G))$$
$$\Bigg(\int_S \int_{\big(\big(-\frac{h}{2},\frac{h}{2}\big)^m\big)^c} \Big\|f(A,-s)\gamma_0\Big\|ds\pi(dA) + \int_S \int_{\big(\big(-\frac{h}{2},\frac{h}{2}\big)^m\big)^c} \int_{\mathbb{R}^d} \Big\|f(A,-s)x\Big\|\nu(dx)ds\pi(dA)\Bigg).$$

Therefore, X is η weakly dependent with η-coefficients

$$\eta_X(h) \leq 2\Bigg(\int_S \int_{\big(\big(-\frac{h}{2},\frac{h}{2}\big)^m\big)^c} \Big\|f(A,-s)\gamma_0\Big\|ds\pi(dA) + \int_S \int_{\big(\big(-\frac{h}{2},\frac{h}{2}\big)^m\big)^c} \int_{\mathbb{R}^d} \Big\|f(A,-s)x\Big\|\nu(dx)ds\pi(dA)\Bigg),$$

which converge to zero as h goes to infinity by applying the dominated convergence theorem.

5.5 Proofs of Section 3.8

Proof of Theorem 3.36. Let $\|A\|_F = \sqrt{tr(AA')}$ for $A \in M_{n\times d}(\mathbb{R})$ denote the Frobenius norm and $\|x\|_1 = \sum_{\nu=1}^m |x^{(\nu)}|$ for $x \in \mathbb{R}^m$. From Proposition 3.19 it follows that X is η-weakly dependent with η-coefficients

$$\eta_X(h) = \left(\int_{\big(\big(-\frac{h}{2},\frac{h}{2}\big)^m\big)^c} \operatorname{tr}(g(-s)\Sigma_L g(-s)')ds\right)^{\frac{1}{2}} = \left(\int_{\big(\big(-\frac{h}{2},\frac{h}{2}\big)^m\big)^c} \|g(-s)\Sigma^{\frac{1}{2}}\|_F^2 ds\right)^{\frac{1}{2}}$$
$$\leq \left(\|\Sigma^{\frac{1}{2}}\|_F^2 \int_{\big(\big(-\frac{h}{2},\frac{h}{2}\big)^m\big)^c} \|g(-s)\|_F^2 ds\right)^{\frac{1}{2}} \leq \left(\|\Sigma^{\frac{1}{2}}\|_F^2 M \int_{\big(\big(-\frac{h}{2},\frac{h}{2}\big)^m\big)^c} e^{-K\|s\|_1} ds\right)^{\frac{1}{2}}$$

$$= \|\Sigma^{\frac{1}{2}}\|_F M^{\frac{1}{2}} \left(\int_{\mathbb{R}^m} e^{-K\|s\|_1} ds - \int_{\left(-\frac{h}{2},\frac{h}{2}\right)^m} e^{-K\|s\|_1} ds \right)^{\frac{1}{2}}$$

$$= \|\Sigma^{\frac{1}{2}}\|_F M^{\frac{1}{2}} \left(\left(\frac{1}{2K}\right)^m - \left(\frac{1}{2K} - \frac{e^{-\frac{K}{2}h}}{2K}\right)^m \right)^{\frac{1}{2}} = \frac{\|\Sigma^{\frac{1}{2}}\|_F M^{\frac{1}{2}}}{(2K)^{\frac{m}{2}}} \left(1 - \left(1 - e^{-\frac{K}{2}h}\right)^m\right)^{\frac{1}{2}}$$

$$\leq \frac{m\|\Sigma^{\frac{1}{2}}\|_F M^{\frac{1}{2}}}{(2K)^{\frac{m}{2}}} e^{-\frac{K}{4}h},$$

where the last inequality follows from Bernoulli's inequality.

5.6 Proofs of Section 4.2

Proof of Proposition 4.4.

(i) Let $(t,x) \in \mathbb{R} \times \mathbb{R}^m$, $\psi > 0$. We define the two truncated sequences

$$\begin{aligned} \tilde{Y}_t^{(\psi)}(x) &= \int_{A_t(x)\setminus V^{\psi}_{(t,x)}} l(x-\xi, t-s)\sigma_s(\xi)\Lambda(d\xi, ds) \text{ and} \\ Y_t^{(\psi)}(x) &= \int_{A_t(x)\setminus V^{\psi}_{(t,x)}} l(x-\xi, t-s)\sigma_s^{(\psi)}(\xi)\Lambda(d\xi, ds), \end{aligned} \tag{5.9}$$

where

$$\sigma_t^{(\psi)}(x) = \int_S \int_{A^{\sigma}_{(t,x)}(x)\setminus V^{\psi}_{(t,x)}} j(x-\xi, t-s)\Lambda^{\sigma}(dA, d\xi, ds).$$

Since the kernel function j is square integrable we have that (3.6), (3.7) and (3.8) hold. Therefore, j is Λ^{σ}-integrable and σ is well-defined and stationary. Now, by Proposition 3.6 it holds that $\sigma_t(x) \in L^2(\Omega)$. Since additionally $l \in L^2(\mathbb{R}^m \times \mathbb{R})$ and σ is stationary it holds that $l\sigma \in L^2(\Omega \times \mathbb{R}^m \times \mathbb{R})$. This implies $l\sigma \in L^2(\mathbb{R}^m \times \mathbb{R})$ almost surely. Then, $l\sigma$ satisfies (3.6), (3.7) and (3.8) almost surely and the ambit field Y is well-defined. Analogous to Proposition 3.11 we derive an upper bound of the expectation using Proposition 4.3

$$\begin{aligned} E\Big[|Y_t(x) - Y_t^{(\psi)}(x)|\Big] &\leq E\Big[|Y_t(x) - \tilde{Y}_t^{(\psi)}(x)|\Big] + E\Big[|\tilde{Y}_t^{(\psi)}(x) - Y_t^{(\psi)}(x)|\Big] = \\ &= E\Bigg[\Big|\int_{A_t(x)\cap V^{\psi}_{(t,x)}} l(x-\xi, t-s)\sigma_s(\xi)\Lambda(d\xi, ds)\Big|\Bigg] \\ &+E\Bigg[\Big|\int_{A_t(x)\setminus V^{\psi}_{(t,x)}} l(x-\xi, t-s)(\sigma_s(\xi) - \sigma_s^{(\psi)}(\xi))\Lambda(d\xi, ds)\Big|\Bigg] \\ &\leq E\left[\left(\int_{A_t(x)\cap V^{\psi}_{(t,x)}} l(x-\xi, t-s)\sigma_s(\xi)\Lambda(d\xi, ds)\right)^2\right]^{\frac{1}{2}} \\ &+E\left[\left(\int_{A_t(x)\setminus V^{\psi}_{(t,x)}} l(x-\xi, t-s)(\sigma_s(\xi) - \sigma_s^{(\psi)}(\xi))\Lambda(d\xi, ds))\right)^2\right]^{\frac{1}{2}}. \end{aligned}$$

Using Proposition 4.3 and the translation invariance of $A_t(x)$ and $V^{\psi}_{(t,x)}$ this is equal to

$$\Big(\Sigma_{\Lambda} E[\sigma_0(0)^2] \int_{A_0(0)\cap V^{\psi}_{(0,0)}} l(-\xi,-s)^2 d\xi ds\Big)^{\frac{1}{2}}$$

$$+\Bigg(E\Bigg[\Big(\int_S \int_{A_0^{\sigma}(0)\cap V^{\psi}_{(0,0)}} j(A,-\xi,-s)\Lambda^{\sigma}(dA,d\xi,ds)\Big)^2\Bigg] \Sigma_{\Lambda} \int_{A_0(0)\setminus V^{\psi}_{(0,0)}} l(-\xi,-s)^2 d\xi ds\Bigg)^{\frac{1}{2}}$$

$$= \Big(\Sigma_{\Lambda} E[\sigma_0(0)^2] \int_{A_0(0)\cap V^{\psi}_{(0,0)}} l(-\xi,-s)^2 d\xi ds\Big)^{\frac{1}{2}}$$

$$+\Bigg(\Bigg(\Sigma_{\Lambda^{\sigma}} \int_S \int_{A_0^{\sigma}(0)\cap V^{\psi}_{(0,0)}} j(A,-\xi,-s)^2 d\xi ds \pi(dA)$$

$$+\mu^2_{\Lambda^{\sigma}} \Bigg(\int_S \int_{A_0^{\sigma}(0)\cap V^{\psi}_{(0,0)}} j(A,-\xi,-s) d\xi ds \pi(dA)\Bigg)^2\Bigg) \times \Sigma_{\Lambda} \int_{A_0(0)\setminus V^{\psi}_{(0,0)}} l(-\xi,-s)^2 d\xi ds\Bigg)^{\frac{1}{2}}.$$

Now let $G \in \mathcal{F}$ and $F \in \mathcal{F}^*$, i.e. F, G are bounded with $\|F\|_\infty, \|G\|_\infty \leq 1$ and G additionally Lipschitz-continuous, $u \in \mathbb{N}^*, h \in \mathbb{R}^+, \Gamma = \{(t_{i_1}, x_{i_1}), \dots, (t_{i_u}, x_{i_u})\} \in (\mathbb{R}\times\mathbb{R}^m)^u$ and $(t_j, x_j) \in \mathbb{R}\times\mathbb{R}^m$ as in Definition 2.2 such that $(t_{i_1}, x_{i_1}), \dots, (t_{i_u}, x_{i_u}) \in V^h_{(t_j,x_j)}$. For $a \in \{1,\dots,u\}$ define

$$Y_{t_{i_a}}(x_{i_a}) = \int_{A_{t_{i_a}}(x_{i_a})} l(x_{i_a}-\xi, t_{i_a}-s)\sigma_s(\xi)\Lambda(d\xi,ds) \text{ and}$$
$$Y^{(\psi)}_{t_j}(x_j) = \int_{A_{t_j}(x_j)\setminus V^{\psi(h)}_{(t_j,x_j)}} l(x_j-\xi, t_j-s)\sigma^{(\psi)}_s(\xi)\Lambda(d\xi,ds).$$

W.l.o.g. we assume that $(t_{i_a}, x_{i_a}) \leq_{lex} (t_{i_u}, x_{i_u})$ for all $a \in \{1,\dots,u\}$. Since $A_0(0)\cup A_0^{\sigma}(0)$ satisfy (3.13) we find analogous to (3.16) a function $\psi(h) = \frac{-hb}{2\sqrt{m+1}}$, such that $A^{\sigma}_{s_1}(\xi_1)$ and $A^{\sigma}_{s_2}(\xi_2)\setminus V^{\psi(h)}_{(s_2,\xi_2)}$ are disjoint for all $(s_1,\xi_1) \in A_{(t_{i_u},x_{i_u})}$ and $(s_2,\xi_2) \in A_{(t_j,x_j)}\setminus V^{\psi(h)}_{(t_j,x_j)}$ or have intersection with zero Lebesgue measure. Then, by the definition of a Lévy basis we get that $\sigma_{s_1}(\xi_1)$ and $\sigma^{\psi(h)}_{s_2}(\xi_2)$ are independent. Furthermore, it holds that $A_{t_{i_u}}(x_{i_u})$ and $A_{t_j}(x_j)\setminus V^{\psi(h)}_{(t_j,x_j)}$ are disjoint. We set $\psi = \psi(h)$. Finally, we get that $Y_{t_{i_a}}(x_{i_a})$ and $Y^{(\psi(h))}_{t_j}(x_j)$ are independent for all $a \in \{1,\dots,u\}$ and therefore also $F(Y_\Gamma)$ and $G(Y^{(\psi(h))}_{t_j}(x_j))$. Now

$$|Cov(F(Y_\Gamma), G(Y_{t_j}(x_j)))|$$
$$\leq |Cov(F(X_\Gamma), G(Y^{(\psi(h))}_{t_j}(x_j)))| + |Cov(F(Y_\Gamma), G(Y_{t_j}(x_j)) - G(Y^{(\psi(h))}_{t_j}(x_j)))|$$
$$= |E[(G(Y_{t_j}(x_j)) - G(Y^{(\psi(h))}_{t_j}(x_j)))F(X_\Gamma)] - E[G(Y_{t_j}(x_j)) - G(Y^{(\psi(h))}_{t_j}(x_j))]E[F(X_\Gamma)]|$$
$$\leq 2\|F\|_\infty E\Big[|(G(Y_{t_j}(x_j)) - G(Y^{(\psi(h))}_{t_j}(x_j)))|\Big] \leq 2Lip(G)\|F\|_\infty E[\|Y_{t_j}(x_j) - Y^{(\psi(h))}_{t_j}(x_j)\|],$$

and using the above inequality for $E\Big[|Y_t(x) - Y_t^{(\psi(h))}(x)|\Big]$ we conclude

$$\leq 2Lip(G)\|F\|_\infty\Bigg(\bigg(\Sigma_\Lambda E[\sigma_0(0)^2]\int_{A_0(0)\cap V_{(0,0)}^{\psi(h)}} l(-\xi,-s)^2 d\xi ds\bigg)^{\frac{1}{2}}$$

$$+\Bigg(\bigg(\Sigma_{\Lambda^\sigma}\int_S\int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi(h)}} j(A,-\xi,-s)^2 d\xi ds\pi(dA)$$

$$+\mu_{\Lambda^\sigma}^2\bigg(\int_S\int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi(h)}} j(A,-\xi,-s) d\xi ds\pi(dA)\bigg)^2\bigg)\Sigma_\Lambda\int_{A_0(0)\setminus V_{(0,0)}^{\psi(h)}} l(-\xi,-s)^2 d\xi ds\Bigg)^{\frac{1}{2}}\Bigg).$$

Therefore Y is θ-lex weakly dependent with θ-lex-coefficients

$$\theta_Y(h)\leq 2\Bigg(\bigg(\Sigma_\Lambda E[\sigma_0(0)^2]\int_{A_0(0)\cap V_{(0,0)}^{\psi(h)}} l(-\xi,-s)^2 d\xi ds\bigg)^{\frac{1}{2}}$$

$$+\Bigg(\bigg(\Sigma_{\Lambda^\sigma}\int_S\int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi(h)}} j(A,-\xi,-s)^2 d\xi ds\pi(dA)$$

$$+\mu_{\Lambda^\sigma}^2\bigg(\int_S\int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi(h)}} j(A,-\xi,-s) d\xi ds\pi(dA)\bigg)^2\bigg)\times\Sigma_\Lambda\int_{A_0(0)\setminus V_{(0,0)}^{\psi(h)}} l(-\xi,-s)^2 d\xi ds\Bigg)^{\frac{1}{2}}\Bigg),$$

which converges to zero as h goes to infinity by applying the dominated convergence theorem.

(ii) Let $(t,x)\in\mathbb{R}\times\mathbb{R}^m$, $\psi>0$. As in the proof of part (i) we define $Y_t^{(\psi)}(x)$ and $\tilde{Y}_t^{(\psi)}(x)$. For the upper bound of the expectation we can derive with the help of Proposition 4.3

$$E\Big[|Y_t(x)-\tilde{Y}_t^{(\psi)}(x)|\Big]+E\Big[|\tilde{Y}_t^{(\psi)}(x)-Y_t^{(\psi)}(x)|\Big]$$

$$\leq\Bigg(\bigg(\Sigma_\Lambda E[\sigma_0(0)^2]\int_{A_t(x)\cap V_{(t,x)}^{\psi}} l(x-\xi,t-s)^2 d\xi ds$$

$$+\mu_\Lambda E[\sigma_0(0)^2]\Big(\int_{A_t(x)\cap V_{(t,x)}^{\psi}} l(x-\xi,t-s) d\xi ds\Big)^2\bigg)^{\frac{1}{2}}$$

$$+\Bigg(\bigg(\Sigma_{\Lambda^\sigma}\int_S\int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi}} j(A,-\xi,-s)^2 d\xi ds\pi(dA)$$

$$+\mu_{\Lambda^\sigma}^2\bigg(\int_S\int_{A_0^\sigma(0)\cap V_{(0,0)}^{\psi}} j(A,-\xi,-s) d\xi ds\pi(dA)\bigg)^2\bigg)$$

$$\bigg(\Sigma_\Lambda\int_{A_t(x)\setminus V_{(t,x)}^{\psi}} l(x-\xi,t-s)^2 d\xi ds+\mu_\Lambda\Big(\int_{A_t(x)\setminus V_{(t,x)}^{\psi}} l(x-\xi,t-s) d\xi ds\Big)^2\bigg)\Bigg)^{\frac{1}{2}},$$

Finally, we can proceed as in the proof of part (i) to obtain the stated bound for the θ-lex-coefficients.

(iii) Note that the kernel function j is square integrable such that σ is well defined and stationary. Now, by Proposition 3.6 it holds that $\sigma \in L^1(\Omega)$. Since additionally $l \in L^1(\mathbb{R}^m \times \mathbb{R})$ and σ is stationary it holds that $l\sigma \in L^1(\Omega \times \mathbb{R}^m \times \mathbb{R})$. This implies $l\sigma \in L^1(\mathbb{R}^m \times \mathbb{R})$ almost surely. Then $l\sigma$ satisfies (3.9) and (3.10) almost surely and the ambit field Y is well defined. In the following we use the notation of part (i). Let $(t,x) \in \mathbb{R} \times \mathbb{R}^m$ and $\psi > 0$. Then, we can derive with the help of Proposition 4.3

$$\begin{aligned} &E\Big[|Y_t(x) - \tilde{Y}_t^{(\psi)}(x)|\Big] + E\Big[|\tilde{Y}_t^{(\psi)}(x) - Y_t^{(\psi)}(x)|\Big] \\ &\leq E[|\sigma_0(0)|]\Big(|\gamma_0| + \int_{\mathbb{R}^d} |y|\nu(dy)\Big)\Big(\int_{A_0(0)\cap V^{\psi}_{(0,0)}} |l(-\xi,-s)|d\xi ds\Big) \\ &+E[|\sigma_0(0) - \sigma_0^{(\psi)}(0)|]\Big(|\gamma_0| + \int_{\mathbb{R}^d} |y|\nu(dy)\Big)\Big(\int_{A_0(0)\setminus V^{\psi}_{(0,0)}} |l(-\xi,-s)|d\xi ds\Big). \end{aligned}$$

Finally, we can proceed as in the proof of Proposition 4.4 and we obtain a bound for the θ-lex-coefficients.

Proof of Proposition 4.5. Let $(t,x) \in \mathbb{R} \times \mathbb{R}^m$, $\psi > 0$. We define the truncated sequence

$$Y_t^{(\psi)}(x) = \int_{A_t(x)\setminus V^{\psi}_{(t,x)}} l(x-\xi, t-s)\sigma_s(\xi)\Lambda(d\xi, ds).$$

Since $l \in L^2(\mathbb{R}^m \times \mathbb{R})$, $\sigma \in L^2(\Omega)$ and σ is stationary it holds that $l\sigma \in L^2(\Omega \times \mathbb{R}^m \times \mathbb{R})$. This implies $l\sigma \in L^2(\mathbb{R}^m \times \mathbb{R})$ almost surely. Then $l\sigma$ satisfies (3.6), (3.7) and (3.8) almost surely and the ambit field Y is well defined. Finally, analogous to Proposition 3.11 we derive an upper bound of the expectation using Proposition 4.3

$$\begin{aligned} E\Big[|Y_t(x) - Y_t^{(\psi)}(x)|\Big] &= E\Big[\Big|\int_{A_t(x)\cap V^{\psi}_{(t,x)}} l(x-\xi,t-s)\sigma_s(\xi)\Lambda(d\xi,ds)\Big|\Big] \\ &\leq E\Big[\Big(\int_{A_t(x)\cap V^{\psi}_{(t,x)}} l(x-\xi,t-s)\sigma_s(\xi)\Lambda(d\xi,ds)\Big)^2\Big]^{\frac{1}{2}} \end{aligned}$$

Using Proposition 4.3 and the translation invariance of $A_t(x)$ and $V^{\psi}_{(t,x)}$ this is equal to

$$\Big(\Sigma_\Lambda E[\sigma_0(0)] \int_{A_0(0)\cap V^{\psi}_{(0,0)}} l(-\xi,-s)^2 d\xi ds\Big)^{\frac{1}{2}}.$$

Define $\Gamma \in (\mathbb{R} \times \mathbb{R}^m)^u$, $(t_j, x_j) \in \mathbb{R} \times \mathbb{R}^m$ and $\psi(h)$ as in the proof of Proposition 4.4. Since σ is p-dependent we get that Y_Γ and $Y_{t_j}^{\psi(h)}(x_j)$ are independent for a sufficiently big h. Then, for these sufficiently big h, Y is θ-lex weakly dependent with θ-lex-coefficients

$$\theta_Y(h) \leq 2\Big(\Sigma_\Lambda E[\sigma_0(0)^2] \int_{A_0(0)\cap V^{\psi(h)}_{(0,0)}} l(-\xi,-s)^2 d\xi ds\Big)^{\frac{1}{2}},$$

which converges to zero as h goes to infinity by applying the dominated convergence theorem.

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