Mixing Conditions for Multivariate Infinitely Divisible Processes with an Application to Mixed Moving Averages and the supOU Stochastic Volatility Model

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We consider strictly stationary infinitely divisible processes and first extend the mixing conditions given in Maruyama [18] and Rosiński and Żak [23] from the univariate to the d-dimensional case.

Thereafter, we show that multivariate Lévy-driven mixed moving average processes satisfy these conditions and hence a wide range of well-known processes such as superpositions of Ornstein-Uhlenbeck (supOU) processes or (fractionally integrated) continuous time autoregressive moving average (CARMA) processes are always mixing. Finally, mixing of the log-returns and the integrated volatility process of a multivariate supOU type stochastic volatility model, recently introduced in Barndorff-Nielsen and Stelzer [5], is established.

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1 Introduction

Lévy-driven continuous-time moving averages, i.e. processes $(X_t)_{t\in\mathbb{R}}$ of the form $X_t = \int_{\mathbb{R}} f(t-s)\,dL_s$ with f a deterministic function and L a Lévy process, are frequently used in applications. Particularly popular examples include, for instance, multivariate CARMA processes (see [7, 17]) or the increments of fractionally integrated Lévy processes (see [6, 16]). By allowing f to depend on an additional parameter and replacing the Lévy process by a Lévy basis one arrives at so-called mixed

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moving averages. One example are the supOU processes of [2, 4, 11]. They are particularly interesting, as they constitute a continuous-time time series model capable of exhibiting both jumps and long memory. For applications it is of high importance to understand the dependence structure of these processes. In this paper we consider conditions implying mixing of multivariate infinitely divisible processes and show that Lévy-driven mixed moving average processes are always mixing when they exist. This has important implications for the statistical inference on the parameters of such processes. For moment based estimation methods, like the GMM approach of [13], it implies that the estimators are consistent. This is used for supOU processes in [27], but also seems to be important for non-causal CARMA processes, since the proof of the strong mixing property of causal CARMA processes (see [17, Proposition 3.34]) requires Markovianity which is not given in the general case.

Furthermore, financial data typically call for models allowing for jumps and rather slowly decaying autocovariance functions of the squared log-returns, sometimes even for long memory. These properties are both part of the so-called "stylized facts" of financial data (see, e.g., [9, 12]). One possible model with these features is the supOU stochastic volatility model of [5], where a latent stochastic volatility (covariance matrix) process is modelled as a supOU process. We establish that the log-returns over an equidistant time grid, the only quantity typically observed, are mixing. Thus our results allow e.g. establishing consistency of estimators for this model as well.

Let $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued strictly stationary process defined on the canonical probability space $\big((\mathbb{R}^d)^{\mathbb{R}},\mathscr{F},\mathbb{P}\big)$ with $\mathscr{F}=\mathscr{B}\big((\mathbb{R}^d)^{\mathbb{R}}\big)$. Then $(X_t)_{t\in\mathbb{R}}$ is said to be **ergodic** if $\frac{1}{T}\int_0^T \mathbb{P}(A\cap S^tB)\,dt \overset{T\to\infty}{\to} \mathbb{P}(A)\mathbb{P}(B)$, **weakly mixing** if $\frac{1}{T}\int_0^T |\mathbb{P}(A\cap S^tB)-\mathbb{P}(A)\mathbb{P}(B)|\,dt \overset{T\to\infty}{\to} 0$ and **mixing** if

$$\mathbb{P}(A\cap S^tB) \overset{t\to\infty}{\to} \mathbb{P}(A)\mathbb{P}(B) \tag{1.1}$$

where $(S^t)_{t\in\mathbb{R}}$ is the induced group of shift transformations on $(\mathbb{R}^d)^{\mathbb{R}}$ (i.e. $S^t(x_s)_{s\in\mathbb{R}}=(x_{s-t})_{s\in\mathbb{R}}$ for any $(x_s)_{s\in\mathbb{R}}\in(\mathbb{R}^d)^{\mathbb{R}}$ and $t\in\mathbb{R}$) and $A,B\in\mathscr{F}$. It is obvious that mixing implies weakly mixing and weakly mixing implies ergodic, respectively. However, the inverse implications are not true in general, since in ergodic theory there are examples of flows $(S^t)_{t\in\mathbb{R}}$ that are weakly mixing, but not mixing and ergodic, but not weakly mixing, respectively (cf. [10], [20]). Usually, the weak mixing property is much closer to mixing than to ergodicity (cf. [24, Proposition 1]). However, Rosiński and Żak [24, Theorem 1] have proven the equivalence of weak mixing and ergodicity for real-valued stationary infinitely divisible processes.

Recall that a stochastic process is said to be infinitely divisible (i.d.) if all its finite dimensional margins are i.d. The description of the mixing property for univariate real-valued strictly stationary i.d. processes can be characterized in terms of their Lévy characteristics. More precisely, in the fundamental paper [18], Maruyama showed that such a process $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if

(M1) the covariance function $r(t)$ of its Gaussian part tends to 0 as $t\to\infty$,

(M2) $\lim\limits_{t\to\infty}\nu_{0t}(|xy|>\delta)=0$ for every $\delta>0$ and

(M3) $\lim\limits_{t\to\infty}\int_{\{0<x^2+y^2\le 1\}} xy\,\nu_{0t}(dx,dy)=0$

where ν_{0t} is the Lévy measure of the distribution of (X_0,X_t). This result has been essentially improved by [15], where the implication (M2)$\Rightarrow$(M3) has been established. However, condition (M2) is not very easy to verify even for symmetric stable processes as mentioned in [23]. Therefore, [23, Theorem 1] provides another useful criterion for mixing of i.d. processes: if ν_0, the Lévy measure of $\mathscr{L}(X_0)$, has no atoms in $2\pi\mathbb{Z}$, then $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if $\lim\limits_{t\to\infty}\mathbb{E}[e^{i(X_t-X_0)}]=\big|\mathbb{E}[e^{iX_0}]\big|^2$.

The outline of this paper is as follows. Below we briefly summarize some notation before we then generalize Maruyama's mixing condition and the condition of Rosiński and Żak to the multivariate case in Section 2. Interestingly, a multivariate i.d. process is mixing if and only if all bivariate marginal processes are mixing. An alternative formulation of the mixing condition in terms of the codifference for i.d. processes concludes that part. In the third section we briefly recall the definition of Lévy bases and Lévy-driven mixed moving average processes, then show that these processes are always mixing and finish with supOU processes as an example. The last section considers the multivariate supOU type stochastic volatility model, recently introduced in [5], and mixing of the log-returns and the integrated volatility process over an equally spaced time grid of that model, is established.

Notation

Given the real numbers $\mathbb{R}$ we use the convention $\mathbb{R}_+ := (0,\infty)$. For the minimum of two real numbers $a,b \in \mathbb{R}$ we write shortly $a \wedge b$ and for the maximum $a \vee b$. The real and imaginary part of a complex number $z \in \mathbb{C}$ is written as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. The collection of $n \times d$ matrices over the field $\mathbb{R}$ is denoted by $M_{n\times d}(\mathbb{R})$. We set $M_d(\mathbb{R}) := M_{d\times d}(\mathbb{R})$ and define $\mathbb{S}_d(\mathbb{R})$ as the linear subspace of symmetric matrices. The positive semidefinite cone is denoted by $\mathbb{S}_d^+(\mathbb{R})$, the transpose of $A \in M_{n\times d}(\mathbb{R})$ is written as A' and the identity matrix in $M_d(\mathbb{R})$ shall be denoted by I_d.

On $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ the Euclidean norm is denoted by $|\cdot|$ whereas on $\mathbb{K}^d$ it will be written as $\|\cdot\|$. Recall the fact that two norms on a finite dimensional linear space are always equivalent which is why our results remain true if we replace the Euclidean norm by any other norm. A scalar product on linear spaces is written as $\langle\cdot,\cdot\rangle$; in $\mathbb{R}^d$ and $\mathbb{C}^d$, we again take the Euclidean one. If X and Y are normed linear spaces, let $B(X,Y)$ be the set of bounded linear operators from X into Y and in particular we equip $M_{n\times d}(\mathbb{R}) = B(\mathbb{R}^d,\mathbb{R}^n)$ with the corresponding operator norm if not stated otherwise. If X is a topological space, we denote by $\mathscr{B}(X)$ the Borel σ-algebra on X.

For a random variable X defined on some probability space $(\Omega,\mathscr{F},\mathbb{P})$ the image measure $X(\mathbb{P})$ (distribution of X) is written as $\mathscr{L}(X)$. For two random variables X and Y the notation $X \stackrel{\mathscr{D}}{=} Y$ means equality in distribution. If we consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$, we shall denote convergence in distribution (weak convergence) of the sequence to some random variable X by $X_n \xrightarrow{w} X$.

2 Mixing of Multivariate Infinitely Divisible Processes

We denote the j-th component of an $\mathbb{R}^d$-valued stochastic process $(X_t)_{t\in\mathbb{R}}$ by $\left(X_t^{(j)}\right)_{t\in\mathbb{R}}$. Generalizing [23, Theorem 1] we show the following:

Theorem 2.1. *Let $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued strictly stationary i.d. process such that ν_0, the Lévy measure of $\mathscr{L}(X_0)$, satisfies $\nu_0\left(\left\{x=(x_1,\ldots,x_d)' \in \mathbb{R}^d : \exists j \in \{1,\ldots,d\}, x_j \in 2\pi\mathbb{Z}\right\}\right)=0$. Then $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if*

$$\lim_{t\to\infty} \mathbb{E}\left[e^{i\left(X_t^{(j)}-X_0^{(k)}\right)}\right] = \mathbb{E}\left[e^{iX_0^{(j)}}\right]\cdot\mathbb{E}\left[e^{-iX_0^{(k)}}\right] \tag{2.1}$$

for any $j,k=1,\ldots,d$.

Proof. We extend the proof of [23, Theorem 1] to the multivariate set-up.
"$\Rightarrow$": Let $(X_t)_{t\in\mathbb{R}}$ be mixing which implies

$$\mathbb{E}\left[e^{i\langle\theta_1,X_0\rangle+i\langle\theta_2,X_t\rangle}\right] \stackrel{t\to\infty}{\to} \mathbb{E}\left[e^{i\langle\theta_1,X_0\rangle}\right]\cdot\mathbb{E}\left[e^{i\langle\theta_2,X_0\rangle}\right]$$

for any $\theta_1, \theta_2 \in \mathbb{R}^d$ (see e.g. [10], [14] or [20]) and in particular, setting $(\theta_1, \theta_2) = (-e_k, e_j),\ j,k = 1,\dots,d$, with e_j the j-th unit vector in $\mathbb{R}^d$, (2.1) holds.
"$\Leftarrow$": Assume that (2.1) holds for every $j,k = 1,\dots,d$. Note first that then

$$\mathbb{E}\left[e^{i\left(X_t^{(j)}+X_0^{(k)}\right)}\right] \overset{t\to\infty}{\to} \mathbb{E}\left[e^{iX_0^{(j)}}\right] \cdot \mathbb{E}\left[e^{iX_0^{(k)}}\right] \tag{2.2}$$

holds for every $j,k = 1,\dots,d$ as well (cf. [23, Theorem 1, Step 1]).

We now prove that (2.1) and (2.2) imply:

(M1) the covariance matrix function $\Sigma(t)$ of the Gaussian part of $(X_t)_{t\in\mathbb{R}}$ tends to 0 as $t \to \infty$ and

(M2) $\lim_{t\to\infty} \nu_{0t}(\|x\| \cdot \|y\| > \delta) = 0$ for every $\delta > 0$

where ν_{0t} is the Lévy measure of $\mathscr{L}(X_0, X_t)$ on $(\mathbb{R}^{2d}, \mathscr{B}(\mathbb{R}^{2d}))$. Having established (M1) and (M2), we will conclude with the upcoming Theorem 2.3 which shows that these two conditions imply mixing.

As to (M1), since (X_0, X_t) has a $2d$-dimensional i.d. distribution, its characteristic function can be written, due to the Lévy-Khintchine formula, for every $(\theta_1, \theta_2) \in \mathbb{R}^d \times \mathbb{R}^d$, as

$$\begin{aligned}
&\mathbb{E}\left[e^{i\langle\theta_1,X_0\rangle+i\langle\theta_2,X_t\rangle}\right] \\
&\quad= \exp\Bigg\{ i\left\langle \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right\rangle - \frac{1}{2}\left\langle \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \Sigma \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\right\rangle \\
&\qquad + \int_{\mathbb{R}^{2d}} e^{i\langle\theta_1,x\rangle+i\langle\theta_2,y\rangle} - 1 - (i\langle\theta_1,x\rangle + i\langle\theta_2,y\rangle)\, \mathbb{1}_{[0,1]}\left(\left\| (x',y')' \right\|\right) \nu_{0t}(d(x,y)) \Bigg\}
\end{aligned} \tag{2.3}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}^d$, $\Sigma \in \mathbb{S}^+_{2d}(\mathbb{R})$ and ν_{0t} is the Lévy measure of $\mathscr{L}(X_0, X_t)$ on $(\mathbb{R}^{2d}, \mathscr{B}(\mathbb{R}^{2d}))$. Since $\mathscr{L}(X_0) = \mathscr{L}(X_t)$, we observe that

$$\Sigma = \begin{pmatrix} \Sigma(0) & \Sigma(t) \\ \Sigma(t)' & \Sigma(0) \end{pmatrix} \tag{2.4}$$

with $\Sigma(t)$ being the covariance matrix function of the Gaussian part of $(X_t)_{t\in\mathbb{R}}$. If we denote the generating triplet of $\mathscr{L}(X_0)$ by $(\gamma, \Sigma(0), \nu_0)$, we can use [25, Proposition 11.10] in order to deduce

$$\gamma_1 = \gamma - \int_{\mathbb{R}^{2d}} x \left(\mathbb{1}_{[0,1]}(\|x\|) - \mathbb{1}_{[0,1]}\left(\left\| (x',y')' \right\|\right)\right) \nu_{0t}(d(x,y)) \qquad \text{and} \tag{2.5}$$

$$\gamma_2 = \gamma - \int_{\mathbb{R}^{2d}} y \left(\mathbb{1}_{[0,1]}(\|y\|) - \mathbb{1}_{[0,1]}\left(\left\| (x',y')' \right\|\right)\right) \nu_{0t}(d(x,y)). \tag{2.6}$$

Putting (2.3)-(2.6) together, the characteristic function of (X_0, X_t) at the point $(\theta_1, \theta_2) \in \mathbb{R}^d \times \mathbb{R}^d$ can be written as

$$\begin{aligned}
&\mathbb{E}\left[e^{i\langle\theta_1,X_0\rangle+i\langle\theta_2,X_t\rangle}\right] \\
&\quad= \exp\Bigg\{ i\langle\theta_1+\theta_2,\gamma\rangle - \frac{1}{2}\Big(\langle\theta_1,\Sigma(0)\theta_1\rangle + 2\langle\theta_1,\Sigma(t)\theta_2\rangle + \langle\theta_2,\Sigma(0)\theta_2\rangle\Big) \\
&\qquad + \int_{\mathbb{R}^{2d}} e^{i\langle\theta_1,x\rangle+i\langle\theta_2,y\rangle} - 1 - i\langle\theta_1,x\rangle\mathbb{1}_{[0,1]}(\|x\|) - i\langle\theta_2,y\rangle\mathbb{1}_{[0,1]}(\|y\|)\, \nu_{0t}(d(x,y)) \Bigg\}.
\end{aligned} \tag{2.7}$$

By substituting $(-e_k, e_j)$, $(0, e_j)$ and $(-e_k, 0)$, $j,k = 1,\dots,d$, for (θ_1, θ_2) in (2.7) we get the description of (2.1) in terms of the covariance matrix function of the Gaussian part and the Lévy measure ν_{0t}, namely

$$\lim_{t\to\infty}\mathbb{E}\left[e^{i\left(X_t^{(j)}-X_0^{(k)}\right)}\right]\cdot\left(\mathbb{E}\left[e^{iX_t^{(j)}}\right]\cdot\mathbb{E}\left[e^{-iX_0^{(k)}}\right]\right)^{-1}$$
$$=\lim_{t\to\infty}\exp\left\{\sigma_{jk}(t)+\int_{\mathbb{R}^{2d}}\left(e^{i(y^{(j)}-x^{(k)})}-e^{iy^{(j)}}-e^{-ix^{(k)}}+1\right)\nu_{0t}(d(x,y))\right\}=1$$

for arbitrary $j,k=1,\dots,d$, where $\sigma_{jk}(t)$ is the (k,j)-th element of $\Sigma(t)$. Next, taking logarithms on both sides and using the identity $\operatorname{Re}\left(e^{i(y-x)}-e^{iy}-e^{-ix}+1\right)=(\cos x-1)(\cos y-1)+\sin x\sin y$ we obtain

$$\lim_{t\to\infty}\sigma_{jk}(t)+\int_{\mathbb{R}^{2d}}\left((\cos x^{(k)}-1)(\cos y^{(j)}-1)+\sin x^{(k)}\sin y^{(j)}\right)\nu_{0t}(d(x,y))=0 \tag{2.8}$$

for any $j,k=1,\dots,d$.

Likewise, we get

$$\lim_{t\to\infty}-\sigma_{jk}(t)+\int_{\mathbb{R}^{2d}}\left((\cos x^{(k)}-1)(\cos y^{(j)}-1)-\sin x^{(k)}\sin y^{(j)}\right)\nu_{0t}(d(x,y))=0 \tag{2.9}$$

for every $j,k=1,\dots,d$.

Adding (2.8) and (2.9) yields, due to the consistency of Lévy measures (see again [25, Proposition 11.10]),

$$\lim_{t\to\infty}\int_{\mathbb{R}^{2d}}(\cos x^{(k)}-1)(\cos y^{(j)}-1)\,\nu_{0t}(d(x,y))=\lim_{t\to\infty}\int_{\mathbb{R}^2}(\cos x-1)(\cos y-1)\,\nu_{0t}^{(jk)}(dx,dy)=0 \tag{2.10}$$

for every $j,k=1,\dots,d$, where $\nu_{0t}^{(jk)}$ denotes the Lévy measure of $\mathscr{L}\left(X_0^{(k)},X_t^{(j)}\right)$ on $(\mathbb{R}^2,\mathscr{B}(\mathbb{R}^2))$.

Now fix $j,k\in\{1,\dots,d\}$ and observe that the family $\left\{\mathscr{L}\left(X_0^{(k)},X_t^{(j)}\right)\right\}_{t\in\mathbb{R}}$ is tight. Indeed, letting $B_r:=\left\{(x,y)\in\mathbb{R}^2: x^2+y^2\le r^2\right\}$, we have by stationarity $\mathbb{P}\left(\left(X_0^{(k)},X_t^{(j)}\right)\notin B_r\right)\le\mathbb{P}\left(\left|X_0^{(k)}\right|^2>\frac{r^2}{2}\right)+\mathbb{P}\left(\left|X_0^{(j)}\right|^2>\frac{r^2}{2}\right)$ and hence $\lim_{r\to\infty}\sup_{t\in\mathbb{R}}\mathbb{P}\left(\left(X_0^{(k)},X_t^{(j)}\right)\notin B_r\right)=0$. Thus, due to Prohorov's Theorem, the family is relatively compact (in the topology of weak convergence). Choose any sequence $\tau_n\to\infty$, $\tau_n\in\mathbb{R}$, and let F_{jk} be an accumulation point of $\left\{\mathscr{L}\left(X_0^{(k)},X_{\tau_n}^{(j)}\right)\right\}_{n\in\mathbb{N}}$. Then F_{jk} is an i.d. distribution on $\mathbb{R}^2$ with some Lévy measure ν_{jk} (cf. [25, Lemma 7.8]). Now let $(t_n)_{n\in\mathbb{N}}$ be a subsequence of $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\mathscr{L}\left(X_0^{(k)},X_{t_n}^{(j)}\right)\xrightarrow{w}F_{jk}\quad\text{as } n\to\infty. \tag{2.11}$$

Then, for every $\delta>0$ with $\nu_{jk}(\partial B_\delta)=0$,

$$\nu_{0t_n}^{(jk)}\Big|_{B_\delta^c}\xrightarrow{w}\nu_{jk}\big|_{B_\delta^c}\quad\text{as } n\to\infty \tag{2.12}$$

which is an immediate consequence of [25, Theorem 8.7]. Since $(\cos x-1)(\cos y-1)\ge 0$, we deduce

$$0\le\int_{B_\delta^c}(\cos x-1)(\cos y-1)\,\nu_{jk}(dx,dy)\overset{(2.12)}{=}\lim_{n\to\infty}\int_{B_\delta^c}(\cos x-1)(\cos y-1)\,\nu_{0t_n}^{(jk)}(dx,dy)$$
$$\le\lim_{n\to\infty}\int_{\mathbb{R}^2}(\cos x-1)(\cos y-1)\,\nu_{0t_n}^{(jk)}(dx,dy)\overset{(2.10)}{=}0.$$

Since δ can be taken arbitrarily small we infer that every Lévy measure ν_{jk} is concentrated on $\{(x,y)\in\mathbb{R}^2 : x\in 2\pi\mathbb{Z} \text{ or } y\in 2\pi\mathbb{Z}\}$.

By the stationarity of the process and (2.11), the projection of ν_{jk} onto the first and second axis coincides with $\nu_0^{(k)}$ and $\nu_0^{(j)}$, respectively, on the complement of every neighborhood of zero. Hence, by our assumption on ν_0, we have for every $m\in\mathbb{Z}$, $m\neq 0$,

$$\begin{aligned}\nu_{jk}(\{2\pi m\}\times\mathbb{R}) &= \nu_0^{(k)}(\{2\pi m\}) = \nu_0(\underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{k-1}\times\{2\pi m\}\times\underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{d-k})\\ &\le \nu_0\left(\left\{x\in\mathbb{R}^d : \exists l\in\{1,\ldots,d\},\, x_l\in 2\pi\mathbb{Z}\right\}\right)=0\end{aligned}$$

and similarly $\nu_{jk}(\mathbb{R}\times\{2\pi m\})=0$. This shows that every ν_{jk}, $j,k=1,\ldots,d$, is actually concentrated on the axes of $\mathbb{R}^2$ and on each of them coincides with $\nu_0^{(k)}$ and $\nu_0^{(j)}$, respectively.

Now, observe that, for every $t\in\mathbb{R}$,

$$\int_{B_\delta}|xy|\,\nu_{0t}^{(jk)}(dx,dy)\le\frac{1}{2}\int_{\{|x|\le\delta\}}x^2\,\nu_0^{(k)}(dx)+\frac{1}{2}\int_{\{|y|\le\delta\}}y^2\,\nu_0^{(j)}(dy)<\varepsilon \tag{2.13}$$

for any positive ε and any $j,k=1,\ldots,d$, if only δ is small enough. Then (2.13) yields, for every $j,k=1,\ldots,d$ and any n, $\int_{B_\delta}|\sin x\sin y|\,\nu_{0t_n}^{(jk)}(dx,dy)\le\int_{B_\delta}|xy|\,\nu_{0t_n}^{(jk)}(dx,dy)<\varepsilon$ for sufficiently small $\delta>0$. Since every ν_{jk} is concentrated on the axes of $\mathbb{R}^2$, (2.12) implies $\lim_{n\to\infty}\int_{B_\delta^c}\sin x\sin y\,\nu_{0t_n}^{(jk)}(dx,dy)=0$. Thus

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\sin x\sin y\,\nu_{0t_n}^{(jk)}(dx,dy)=0 \tag{2.14}$$

for every $j,k=1,\ldots,d$.

From (2.8), (2.10) and (2.14) we infer that $\sigma_{jk}(t_n)\to 0$ as $n\to\infty$ for all $j,k=1,\ldots,d$. Since (t_n) is a subsequence of an arbitrary sequence $\tau_n\to\infty$, it follows that $\sigma_{jk}(t)\to 0$ as $t\to\infty$ and thus $\Sigma(t)\to 0$ as $t\to\infty$, i.e. (M1) holds.

To prove (M2), observe that, for any $n\in\mathbb{N}$,

$$\nu_{0t_n}\big(\underbrace{\|x\|^2\cdot\|y\|^2}_{=\sum_{j,k=1}^d\left(x^{(k)}y^{(j)}\right)^2}>\delta^2\big)\le\sum_{j,k=1}^d\nu_{0t_n}^{(jk)}\left(\left|x^{(k)}y^{(j)}\right|\ge\frac{\delta}{d}\right).$$

By virtue of (2.12), we have $\limsup_{n\to\infty}\nu_{0t_n}^{(jk)}\left(\left|x^{(k)}y^{(j)}\right|\ge\delta/d\right)\le\nu_{jk}\left(\left|x^{(k)}y^{(j)}\right|\ge\delta/d\right)=0$ for every $j,k=1,\ldots,d$ and thus $\lim_{n\to\infty}\nu_{0t_n}(\|x\|\cdot\|y\|>\delta)=0$ for every $\delta>0$. Again, since (t_n) is a subsequence of any arbitrary sequence $\tau_n\to\infty$, it follows that $\lim_{t\to\infty}\nu_{0t}(\|x\|\cdot\|y\|>\delta)=0$ for any $\delta>0$, i.e. (M2) is shown.

We can now conclude with the upcoming Theorem 2.3. □

To establish Theorem 2.3 we need the following multivariate generalization of [15, Lemma 1].

Lemma 2.2.
Assume that $\lim_{t\to\infty}\nu_{0t}(\|x\|\cdot\|y\|>\delta)=0$ *for every* $\delta>0$. *Then one has*

(M3) $\lim_{t\to\infty}\int_{\{0<\|x\|^2+\|y\|^2\le 1\}}\|x\|\cdot\|y\|\,\nu_{0t}(d(x,y))=0.$

Proof. Fix $\varepsilon > 0$ and define for any $\delta \in (0,1)$ the sets $B_\delta := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \|x\|^2 + \|y\|^2 \le \delta^2\}$ and $R_\delta := B_1 \backslash B_\delta$. Then, for every $\delta \in (0,1)$,

$$\int_{\{0<\|x\|^2+\|y\|^2\le 1\}} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) = \int_{B_\delta} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) + \int_{R_\delta} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y))$$
$$=: I_1 + I_2.$$

Taking advantage of stationarity of $(X_t)_{t\in\mathbb{R}}$, we obtain

$$|I_1| \le \frac{1}{2} \int_{B_\delta} \|x\|^2 + \|y\|^2 \, \nu_{0t}(d(x,y)) = \int_{\{\|x\|\le\delta\}} \|x\|^2 \, \nu_0(dx) \le \frac{\varepsilon}{2}$$

for every δ sufficiently small.

We fix such a δ and set $l := \min\{\delta/2, \varepsilon/8q\}$ with $q := \nu_0(\{\|x\|^2 > \delta^2/2\}) < \infty$ and $C := R_\delta \cap \{\|x\| \cdot \|y\| > l\}$. Then

$$|I_2| = \int_C \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) + \int_{R_\delta \backslash C} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) \le \frac{1}{2}\nu_{0t}(C) + \frac{\varepsilon}{8q}\nu_{0t}(R_\delta \backslash C)$$
$$\le \frac{1}{2}\nu_{0t}(C) + \frac{\varepsilon}{8q}\left[\nu_{0t}\left(\|x\|^2 > \frac{\delta^2}{2}\right) + \nu_{0t}\left(\|y\|^2 > \frac{\delta^2}{2}\right)\right] \le \frac{1}{2}\nu_{0t}(\|x\| \cdot \|y\| > l) + \frac{\varepsilon}{4}.$$

Since $\nu_{0t}(\|x\| \cdot \|y\| > l) \le \varepsilon/2$ if only t is large enough, we obtain

$$\int_{\{0<\|x\|^2+\|y\|^2\le 1\}} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) \le \varepsilon$$

for sufficiently large t. Letting $\varepsilon \searrow 0$ yields the desired result. □

The next theorem, which is a multivariate generalization of Maruyama's mixing condition [18, Theorem 6], shows that conditions (M1) and (M2) together imply mixing and thus concludes the proof of Theorem 2.1.

Theorem 2.3.
Let $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued strictly stationary i.d. process. Then $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if

(M1) the covariance matrix function $\Sigma(t)$ of its Gaussian part tends to 0 as $t \to \infty$ and

(M2) $\lim_{t\to\infty} \nu_{0t}(\|x\| \cdot \|y\| > \delta) = 0$ for every $\delta > 0$.

Proof. We have already shown in the proof of Theorem 2.1 that mixing implies (M1) and (M2). Conversely, assume that (M1) and (M2) hold and note that, due to Lemma 2.2, also condition (M3) must hold. We now generalize the proof of [18, Theorem 6] to a multivariate setting. We shall denote $X_\tau = (X'_{s_1}, \ldots, X'_{s_m})'$ for any $\tau = (s_1, \ldots, s_m)' \in \mathbb{R}^m$. Then (cf. [18, (5.13)]) it is sufficient for $(X_t)_{t\in\mathbb{R}}$ to be mixing that for all $\tau = (s_1, \ldots, s_m)'$, $\mu = (u_1, \ldots, u_m)' \in \mathbb{R}^m$ and $z_1, z_2 \in \mathbb{R}^{md}$,

$$\lim_{t\to\infty} \mathbb{E}\left[e^{i\langle z_1, X_\tau\rangle + i\langle z_2, X_{\mu+t}\rangle}\right] = \mathbb{E}\left[e^{i\langle z_1, X_\tau\rangle}\right] \cdot \mathbb{E}\left[e^{i\langle z_2, X_\mu\rangle}\right] \tag{2.15}$$

where $\mu + t := (u_1 + t, \ldots, u_m + t)'$.

The family of $\mathbb{R}^{2md}$-valued i.d. random vectors $\{(X_\tau, X_{\mu+t})\}_{t\in\mathbb{R}}$ is tight since

$$\mathbb{P}\left((X_\tau, X_{\mu+t}) \notin B_{\sqrt{2m}r}\right) \le \sum_{j=1}^{m} \mathbb{P}\left(\|X_{s_j}\| > r\right) + \mathbb{P}\left(\|X_{u_j+t}\| > r\right) = 2m\cdot\mathbb{P}(\|X_0\| > r) \to 0$$

as $r\to\infty$, where $B_r := \{x\in\mathbb{R}^{2md} : \|x\|^2 \le r^2\}$. Hence the family is relatively compact with respect to the weak topology (i.e. the topology generated by weak convergence).

Let $(\gamma^1, \Sigma^1, \nu^1)$ and $(\gamma^2, \Sigma^2, \nu^2)$ be the characteristic triplets of $\mathscr{L}(X_\tau)$ and $\mathscr{L}(X_\mu)$, respectively. Consider an arbitrary sequence $\eta_n \in \mathbb{R}$, $\eta_n \to \infty$ and an accumulation point F of the associated sequence $\{\mathscr{L}(X_\tau, X_{\mu+\eta_n})\}_{n\in\mathbb{N}}$ as $n\to\infty$, i.e. there is a subsequence $(t_n)_{n\in\mathbb{N}}$ of $(\eta_n)_{n\in\mathbb{N}}$ such that $\mathscr{L}(X_\tau, X_{\mu+t_n}) \xrightarrow{w} F$ as $n\to\infty$ where the accumulation point F is obviously (see [25, Lemma 7.8]) an i.d. distribution on $\mathbb{R}^{2md}$ with some generating triplet (γ, Σ, ν). We denote by $(\gamma_n, \Sigma_n, \nu_n)$ the characteristic triplet of $\mathscr{L}(X_\tau, X_{\mu+t_n})$ for any $n\in\mathbb{N}$ and by $\Phi_n(z_1,z_2)$ its characteristic function at the point $(z_1,z_2)\in\mathbb{R}^{md}\times\mathbb{R}^{md}$. The logarithm of Φ_n can be written (cf. proof of Theorem 2.1) as

$$\begin{aligned}
\log\Phi_n(z_1,z_2) = {} & i\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} \gamma^1 \\ \gamma^2 \end{pmatrix}\rangle - \frac{1}{2}\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \Sigma_n \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\rangle \\
& + \int_{\{\|x\|<\delta,\, \|y\|<\delta\}} e^{i\langle z_1,x\rangle + i\langle z_2,y\rangle} - 1 - i\langle z_1,x\rangle \mathbb{1}_{[0,1]}(\|x\|) - i\langle z_2,y\rangle \mathbb{1}_{[0,1]}(\|y\|)\,\nu_n(d(x,y)) \\
& + \int_{\{\|x\|\ge\delta \text{ or } \|y\|\ge\delta\}} e^{i\langle z_1,x\rangle + i\langle z_2,y\rangle} - 1 - i\langle z_1,x\rangle \mathbb{1}_{[0,1]}(\|x\|) - i\langle z_2,y\rangle \mathbb{1}_{[0,1]}(\|y\|)\,\nu_n(d(x,y)) \\
& =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We shall prove that $\log\Phi_n(z_1,z_2) \to \log\Phi_1(z_1) + \log\Phi_2(z_2)$ as $n\to\infty$ for all $z_1,z_2\in\mathbb{R}^{md}$ where Φ_1 and Φ_2 are the characteristic functions of X_τ and X_μ, respectively.

Obviously $I_1 = i\langle z_1,\gamma^1\rangle + i\langle z_2,\gamma^2\rangle$ and due to the assumption (M1) the second term I_2 converges to $-1/2\langle z_1, \Sigma^1 z_1\rangle - 1/2\langle z_2, \Sigma^2 z_2\rangle$ as $n\to\infty$.

As to I_4, we have (cf. (2.12))

$$\begin{aligned}
I_4 \overset{n\to\infty}{\to} & \int_{\{\|x\|\ge\delta \text{ or } \|y\|\ge\delta\}} e^{i\langle z_1,x\rangle + i\langle z_2,y\rangle} - 1 - i\langle z_1,x\rangle \mathbb{1}_{[0,1]}(\|x\|) - i\langle z_2,y\rangle \mathbb{1}_{[0,1]}(\|y\|)\,\nu(d(x,y)) \\
= & \int_{\{\|x\|\ge\delta\}} e^{i\langle z_1,x\rangle} - 1 - i\langle z_1,x\rangle \mathbb{1}_{[0,1]}(\|x\|)\,\nu^1(dx) + \int_{\{\|y\|\ge\delta\}} e^{i\langle z_2,y\rangle} - 1 - i\langle z_2,y\rangle \mathbb{1}_{[0,1]}(\|y\|)\,\nu^2(dy)
\end{aligned}$$

since, letting $x = \left(x^{(1)\prime}, \ldots, x^{(m)\prime}\right)' \in \left(\mathbb{R}^d\right)^m$ and $y = \left(y^{(1)\prime}, \ldots, y^{(m)\prime}\right)' \in \left(\mathbb{R}^d\right)^m$,

$$\nu\left(\|x\|\cdot\|y\| > \delta\right) \le \liminf_{n\to\infty} \nu_n\left(\|x\|\cdot\|y\| > \delta\right) \le \liminf_{n\to\infty} \sum_{j,k=1}^{m} \nu_{0,u_k-s_j+t_n}\left(\left\|x^{(j)}\right\|\cdot\left\|y^{(k)}\right\| > \frac{\delta}{m}\right) \overset{(M2)}{=} 0$$

for any $\delta > 0$ which shows in particular that $\nu(\|x\|\cdot\|y\| > 0) = 0$.

Analogously to x and y we denote the j-th $\mathbb{R}^d$-component of z_1 and z_2 by $z_1^{(j)}$ and $z_2^{(j)}$, respectively.

Concerning I_3, a simple Taylor expansion yields for any $\delta > 0$ small enough

$$I_3 = -\frac{1}{2}\Bigg[\int_{\{\|x\|<\delta,\,\|y\|<\delta\}} \left(\sum_{j=1}^{m} \langle z_1^{(j)}, x^{(j)}\rangle\right)^2 + \left(\sum_{j=1}^{m} \langle z_2^{(j)}, y^{(j)}\rangle\right)^2 \nu_n(d(x,y))$$
$$+2\int_{\{\|x\|<\delta,\,\|y\|<\delta\}} \left(\sum_{j,k=1}^{m} \langle z_1^{(j)}, x^{(j)}\rangle\langle z_2^{(k)}, y^{(k)}\rangle\right) \nu_n(d(x,y))\Bigg] + R$$

with

$$\begin{aligned}
6\,|R| &\le \int_{\{\|x\|<\delta,\,\|y\|<\delta\}} |\langle z_1, x\rangle + \langle z_2, y\rangle|^3 + o\left(\left(\|x\|^2 + \|y\|^2\right)^{3/2}\right) \nu_n(d(x,y)) \\
&\le 2\left\|\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right\|^3 \cdot \int_{\{\|x\|<\delta,\,\|y\|<\delta\}} \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^3 \nu_n(d(x,y)) \\
&\le 2\left\|\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right\|^3 \cdot \sqrt{2}\delta \cdot \left(\int_{\{0<\|x\|<\delta\}} \|x\|^2\, \nu^1(dx) + \int_{\{0<\|y\|<\delta\}} \|y\|^2\, \nu^2(dy)\right)
\end{aligned}$$

and thus $6\,|R| < \varepsilon$ for any positive ε if only δ is sufficiently small. Moreover, we obtain for every $j,k = 1,\dots,m$ and any $\delta \in (0, \frac{1}{2}\sqrt{2})$

$$\begin{aligned}
&\int_{\{\|x\|<\delta,\,\|y\|<\delta\}} \left|\langle z_1^{(j)}, x^{(j)}\rangle\langle z_2^{(k)}, y^{(k)}\rangle\right| \nu_n(d(x,y)) \\
&\qquad \le \|z_1^{(j)}\| \cdot \|z_2^{(k)}\| \cdot \int_{\{\|x\|<\delta,\,\|y\|<\delta\}} \|x^{(j)}\| \cdot \|y^{(k)}\|\, \nu_n(d(x,y)) \\
&\qquad \le \|z_1^{(j)}\| \cdot \|z_2^{(k)}\| \cdot \int_{\left\{0<\|x^{(j)}\|^2+\|y^{(k)}\|^2\le 1\right\}} \|x^{(j)}\| \cdot \|y^{(k)}\|\, \nu_{0,u_k-s_j+t_n}\left(d\left(x^{(j)}, y^{(k)}\right)\right) \overset{n\to\infty}{\to} 0
\end{aligned}$$

by virtue of (M3). Finally

$$\left|\frac{1}{2}\int_{\{\|x\|<\delta,\,\|y\|<\delta\}} \langle z_1, x\rangle^2\, \nu_n(d(x,y)) + \int_{\{0<\|x\|<\delta\}} e^{i\langle z_1, x\rangle} - 1 - i\langle z_1, x\rangle \mathbb{1}_{[0,1]}(\|x\|)\, \nu^1(dx)\right| \le J_1 + J_2$$

with

$$\begin{aligned}
J_1 &= \left|\frac{1}{2}\int_{\{0<\|x\|<\delta,\,\|y\|<\delta\}} \langle z_1, x\rangle^2\, \nu_n(d(x,y)) - \frac{1}{2}\int_{\{0<\|x\|<\delta\}} \langle z_1, x\rangle^2\, \nu_n(d(x,y))\right| \\
&\le \int_{\{0<\|x\|<\delta\}} \langle z_1, x\rangle^2\, \nu_n(d(x,y)) \le \|z_1\|^2 \cdot \int_{\{0<\|x\|<\delta\}} \|x\|^2\, \nu^1(dx)
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \left|\int_{\{0<\|x\|<\delta\}} \frac{1}{2}\langle z_1, x\rangle^2 + e^{i\langle z_1, x\rangle} - 1 - i\langle z_1, x\rangle \mathbb{1}_{[0,1]}(\|x\|)\, \nu^1(dx)\right| \\
&\le \|z_1\|^3\, \delta \cdot \int_{\{0<\|x\|<\delta\}} \|x\|^2\, \nu^1(dx).
\end{aligned}$$

An analogous result is obviously true for the second addend of the first term of I_3.

Putting all this together we obtain

$$\lim_{n\to\infty} \log \Phi_n(z_1,z_2) = \log \Phi_1(z_1) + \log \Phi_2(z_2) \quad \text{for all } z_1, z_2 \in \mathbb{R}^{md}$$

and thus the desired result in (2.15) which completes the proof. □

From Theorem 2.1 we can immediately derive the following corollary:

Corollary 2.4. *A d-dimensional strictly stationary i.d. process $X = (X_t)_{t\in\mathbb{R}}$ with*

$$\nu_0\big(\{x = (x_1,\dots,x_d)' \in \mathbb{R}^d : \exists j \in \{1,\dots,d\}, x_j \in 2\pi\mathbb{Z}\}\big) = 0 \tag{2.16}$$

is mixing if and only if the bivariate processes $(X^{(j)}, X^{(k)})$, $j,k \in \{1,\dots,d\}$, $j<k$, are all mixing.

If we use the asymptotic independence of X_0 and X_t as a natural interpretation of the mixing property, Corollary 2.4 means that pairwise asymptotic independence yields asymptotic independence for strictly stationary i.d. processes which satisfy the technical assumption (2.16). This is in a way consistent to the well-known fact that pairwise independence and independence are equivalent for random vectors with i.d. distribution (cf. [25, Exercises 12.9 and 12.10]).

The next corollary can be seen as the multivariate generalization of [23, Corollary 3].

Corollary 2.5. *Let $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued strictly stationary i.d. process. Then, with the previous notation, $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if*

$$\lim_{t\to\infty} \left\{ \|\Sigma(t)\| + \int_{\mathbb{R}^{2d}} (1 \wedge \|x\|\cdot\|y\|)\, \nu_{0t}(d(x,y)) \right\} = 0. \tag{2.17}$$

Proof. Obviously (2.17) implies (M1) and (M2) and thus, due to Theorem 2.3, also mixing.

Conversely if $(X_t)_{t\in\mathbb{R}}$ is mixing, then (M1) holds (cf. Theorem 2.3). Moreover (cf. (2.12))

$$\nu_{0t}^{(jk)}\Big|_{B_\delta^c} \xrightarrow{w} \nu_{jk}\Big|_{B_\delta^c} \quad \text{as } t\to\infty \tag{2.18}$$

for every $\delta > 0$ s.t. $\nu_{jk}(\partial B_\delta) = 0$ and any $j,k = 1,\dots,d$. From the proof of Theorem 2.1 we further know that the Lévy measures ν_{jk} are concentrated on the axes of $\mathbb{R}^2$. Now choose $\delta > 0$ s.t. (2.13) and (2.18) hold, then we have

$$\limsup_{t\to\infty} \int_{\mathbb{R}^2} (1\wedge|xy|)\, \nu_{0t}^{(jk)}(dx,dy) \le \varepsilon + \limsup_{t\to\infty} \int_{B_\delta^c} (1\wedge|xy|)\, \nu_{0t}^{(jk)}(dx,dy) = \varepsilon.$$

Letting $\varepsilon \searrow 0$ we deduce $\lim_{t\to\infty} \int_{\mathbb{R}^2} (1\wedge|xy|)\, \nu_{0t}^{(jk)}(dx,dy) = 0$ for any $j,k = 1,\dots,d$. Finally

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \left(1 \wedge \sum_{k=1}^d |x_k| \cdot \sum_{j=1}^d |y_j|\right) \nu_{0t}(d(x,y)) &\le \sum_{j,k=1}^d \int_{\mathbb{R}^{2d}} \left(1\wedge|x_k y_j|\right) \nu_{0t}(d(x,y)) \\ &= \sum_{j,k=1}^d \int_{\mathbb{R}^2} (1\wedge|xy|)\, \nu_{0t}^{(jk)}(dx,dy) \overset{t\to\infty}{\to} 0.\end{aligned}$$

This clearly implies $\lim_{t\to\infty} \int_{\mathbb{R}^{2d}} (1\wedge\|x\|\cdot\|y\|)\, \nu_{0t}(d(x,y)) = 0$ as well and hence (2.17) is shown. □

Let us conclude this section with an alternative formulation of Theorem 2.1. Therefore recall that the **codifference** $\tau(X_1,X_2)$ of an i.d. real bivariate random vector (X_1,X_2) is defined as follows

$$\tau(X_1,X_2) := \log \mathbb{E}\left[e^{i(X_1-X_2)}\right] - \log \mathbb{E}\left[e^{iX_1}\right] - \log \mathbb{E}\left[e^{-iX_2}\right].$$

If we now consider a univariate strictly stationary i.d. process $(X_t)_{t\in\mathbb{R}}$, then $\tau(X_s,X_t) = \tau(X_0,X_{t-s})$ for any $s,t \in \mathbb{R}$ and hence we can define the function

$$\tau(t) := \tau(X_0,X_t), \quad t \in \mathbb{R},$$

which is positive semidefinite (see [24, Section 2]). We call τ **autocodifference** function of $(X_t)_{t\in\mathbb{R}}$. Analogously to the univariate case we define the autocodifference function for an $\mathbb{R}^d$-valued strictly stationary i.d. process $(X_t)_{t\in\mathbb{R}}$ by $\tau(t) = \left(\tau^{(jk)}(t)\right)_{j,k=1,\dots,d}$ with $\tau^{(jk)}(t) := \tau\left(X_0^{(k)}, X_t^{(j)}\right)$. Hence we have the following mixing condition in terms of the autocodifference function:

Corollary 2.6. *Let $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued strictly stationary i.d. process such that ν_0, the Lévy measure of $\mathscr{L}(X_0)$, satisfies $\nu_0\left(\left\{x=(x_1,\dots,x_d)' \in \mathbb{R}^d : \exists j \in \{1,\dots,d\}, x_j \in 2\pi\mathbb{Z}\right\}\right) = 0$. Then $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if $\tau(t) \to 0$ as $t \to \infty$.*

3 Mixed Moving Average Processes

The central result of this section shows that mixed moving average processes are always mixing.

Let us first recall the definition of $\mathbb{R}^d$-valued Lévy bases, which are generalizations of Lévy processes, and the related integration theory. For a general introduction to Lévy processes and i.d. distributions see [25]. Lévy bases are also called infinitely divisible independently scattered random measures (i.d.i.s.r.m.) in the literature. For more details on Lévy bases see [22] and [19]. In the following, S denotes a non-empty topological space, $\mathscr{B}(S)$ is the Borel σ-field on S and π is some probability measure on $(S,\mathscr{B}(S))$. The collection of all Borel sets in $S\times\mathbb{R}$ with finite $\pi\otimes\lambda^1$-measure, where λ^1 denotes the one-dimensional Lebesgue measure, is written as $\mathscr{B}_0(S\times\mathbb{R})$.

Definition 3.1 (Lévy Basis). *A d-dimensional* Lévy basis *on $S\times\mathbb{R}$ is an $\mathbb{R}^d$-valued random measure $\Lambda = \{\Lambda(B) : B \in \mathscr{B}_0(S\times\mathbb{R})\}$ satisfying:*

(i) the distribution of $\Lambda(B)$ is infinitely divisible for all $B \in \mathscr{B}_0(S\times\mathbb{R})$,

(ii) for arbitrary $n\in\mathbb{N}$ and pairwise disjoint sets $B_1,\dots,B_n \in \mathscr{B}_0(S\times\mathbb{R})$ the random variables $\Lambda(B_1),\dots,\Lambda(B_n)$ are independent and

(iii) for any pairwise disjoint sets $B_1,B_2,\dots \in \mathscr{B}_0(S\times\mathbb{R})$ with $\bigcup_{n\in\mathbb{N}} B_n \in \mathscr{B}_0(S\times\mathbb{R})$ we have, almost surely, $\Lambda(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \Lambda(B_n)$.

In this paper we restrict ourselves to time-homogeneous and factorisable Lévy bases, i.e. Lévy bases with characteristic function

$$\mathbb{E}\left[e^{i\langle z,\Lambda(B)\rangle}\right] = e^{\psi(z)\Pi(B)} \tag{3.1}$$

for all $z\in\mathbb{R}^d$ and $B \in \mathscr{B}_0(S\times\mathbb{R})$, where $\Pi = \pi\otimes\lambda^1$ is the product of the probability measure π on S and the Lebesgue measure λ^1 on $\mathbb{R}$ and

$$\psi(z) = i\langle\gamma,z\rangle - \frac{1}{2}\langle z,\Sigma z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle \mathbb{1}_{[0,1]}(\|x\|)\right)\nu(dx)$$

is the cumulant transform of an i.d. distribution with characteristic triplet (γ,Σ,ν). By L we denote the underlying Lévy process associated with (γ,Σ,ν) and given by $L_t=\Lambda\big(S\times(0,t]\big)$ and $L_{-t}=-\Lambda\big(S\times(-t,0)\big)$ for $t\in\mathbb{R}_+$. The quadruple (γ,Σ,ν,π) determines the distribution of the Lévy basis completely and therefore it is called the **generating quadruple**. A definition of $\mathbb{S}_d(\mathbb{R})$-valued Lévy bases on $S\times\mathbb{R}$ follows along the same lines.

Regarding the existence of integrals with respect to a Lévy basis we recall the following multivariate extension of [22, Theorem 2.7].

Theorem 3.2. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic function of the form (3.1) and let $f:S\times\mathbb{R}\to M_{n\times d}(\mathbb{R})$ be a measurable function. Then f is Λ-integrable as a limit in probability in the sense of Rajput and Rosiński [22], if and only if*

$$\int_S\int_{\mathbb{R}}\Big\|f(A,s)\gamma+\int_{\mathbb{R}^d}f(A,s)x\left(\mathbb{1}_{[0,1]}(\|f(A,s)x\|)-\mathbb{1}_{[0,1]}(\|x\|)\right)\nu(dx)\Big\|ds\,\pi(dA)<\infty,$$

$$\int_S\int_{\mathbb{R}}\|f(A,s)\Sigma f(A,s)'\|\,ds\,\pi(dA)<\infty\qquad\textit{and}$$

$$\int_S\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left(1\wedge\|f(A,s)x\|^2\right)\nu(dx)\,ds\,\pi(dA)<\infty.$$

If f is Λ-integrable, the distribution of $\int_S\int_{\mathbb{R}}f(A,s)\Lambda(dA,ds)$ is infinitely divisible with characteristic triplet $(\gamma_{int},\Sigma_{int},\nu_{int})$ given by

$$\gamma_{int}=\int_S\int_{\mathbb{R}}\Big(f(A,s)\gamma+\int_{\mathbb{R}^d}f(A,s)x\left(\mathbb{1}_{[0,1]}(\|f(A,s)x\|)-\mathbb{1}_{[0,1]}(\|x\|)\right)\nu(dx)\Big)ds\,\pi(dA),$$

$$\Sigma_{int}=\int_S\int_{\mathbb{R}}f(A,s)\Sigma f(A,s)'\,ds\,\pi(dA)\qquad\textit{and}$$

$$\nu_{int}(B)=\int_S\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbb{1}_B(f(A,s)x)\,\nu(dx)\,ds\,\pi(dA)\quad\textit{for all Borel sets } B\subseteq\mathbb{R}^n\backslash\{0\}.$$

Before we prove that mixed moving average processes are always mixing, let us briefly recall the definition of these processes.

Definition 3.3 (Mixed Moving Average Process). *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $S\times\mathbb{R}$ and let $f:S\times\mathbb{R}\to M_{n\times d}(\mathbb{R})$ be a measurable function. If the process*

$$X_t:=\int_S\int_{\mathbb{R}}f(A,t-s)\,\Lambda(dA,ds)$$

exists in the sense of Theorem 3.2 for all $t\in\mathbb{R}$, it is called an n-dimensional mixed moving average process *(*MMA process *for short). The function f is said to be its* kernel function*.*

MMA processes have been first introduced in [26] in the univariate stable case. Note that an MMA process is an i.d. process and obviously always strictly stationary.

Now Corollary 2.5 immediately yields the following mixing condition for MMA processes.

Lemma 3.4. *Let $(X_t)_{t\in\mathbb{R}}\overset{\mathscr{D}}{=}\left(\int_S\int_{\mathbb{R}}f(A,t-s)\,\Lambda(dA,ds)\right)_{t\in\mathbb{R}}$ be an MMA process where Λ is an $\mathbb{R}^d$-valued Lévy basis on $S\times\mathbb{R}$ with generating quadruple (γ,Σ,ν,π) and $f:S\times\mathbb{R}\to M_{n\times d}(\mathbb{R})$ is mea-*

surable. Then $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if

$$\lim_{t\to\infty}\Big\{\Big\|\int_S\int_{\mathbb{R}} f(A,-s)\Sigma f(A,t-s)'\,ds\,\pi(dA)\Big\| \\ +\int_S\int_{\mathbb{R}}\int_{\mathbb{R}^d}(1\wedge\|f(A,-s)x\|\cdot\|f(A,t-s)x\|)\,\nu(dx)\,ds\,\pi(dA)\Big\}=0. \quad (3.2)$$

Proof. Since we can write

$$\begin{pmatrix}X_0\\X_t\end{pmatrix}=\int_S\int_{\mathbb{R}}\begin{pmatrix}f(A,-s)\\f(A,t-s)\end{pmatrix}\Lambda(dA,ds),\quad t\in\mathbb{R},$$

we immediately obtain the covariance matrix function of the Gaussian part of $(X_t)_{t\in\mathbb{R}}$ (cf. Theorem 3.2) by

$$\Sigma(t)=\int_S\int_{\mathbb{R}} f(A,-s)\Sigma f(A,t-s)'\,ds\,\pi(dA),\quad t\in\mathbb{R}.$$

The Lévy measure ν_{0t} of $\mathscr{L}(X_0,X_t)$ is given (see again Theorem 3.2) by

$$\nu_{0t}(B)=\int_S\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbb{1}_B(f(A,-s)x,f(A,t-s)x)\,\nu(dx)\,ds\,\pi(dA)$$

for all Borel sets $B\subseteq\mathbb{R}^{2n}\backslash\{0\}$. Thus

$$\int_{\mathbb{R}^{2n}}(1\wedge\|x\|\cdot\|y\|)\,\nu_{0t}(d(x,y))=\int_S\int_{\mathbb{R}}\int_{\mathbb{R}^d}(1\wedge\|f(A,-s)x\|\cdot\|f(A,t-s)x\|)\,\nu(dx)\,ds\,\pi(dA)$$

and Corollary 2.5 completes the proof. □

The following theorem shows that the mixing condition of Lemma 3.4 is indeed always satisfied for MMA processes. Note that in the univariate moving average and stable mixed moving average case this result is already known from [8, Section 7, Example 1] and [26, Theorem 3], respectively. The multivariate case is, however, considerably more involved, because norms are only submultiplicative whereas $|f(a,-s)x|=|f(a,-s)|\cdot|x|$ could be used for a proof of the general univariate case.

Theorem 3.5. *Let $(X_t)_{t\in\mathbb{R}}\overset{\mathscr{D}}{=}(\int_S\int_{\mathbb{R}} f(A,t-s)\,\Lambda(dA,ds))_{t\in\mathbb{R}}$ be an MMA process where Λ is an $\mathbb{R}^d$-valued Lévy basis on $S\times\mathbb{R}$ with generating quadruple (γ,Σ,ν,π) and $f:S\times\mathbb{R}\to M_{n\times d}(\mathbb{R})$ is measurable. Then $(X_t)_{t\in\mathbb{R}}$ is mixing.*

Proof. By virtue of Lemma 3.4 we have to show that

$$\|\Sigma(t)\|=\Big\|\int_S\int_{\mathbb{R}} f(A,-s)\Sigma f(A,t-s)'\,ds\,\pi(dA)\Big\|\overset{t\to\infty}{\to}0$$

and

$$\int_{\mathbb{R}^{2n}}(1\wedge\|x\|\cdot\|y\|)\,\nu_{0t}(d(x,y))=\int_S\int_{\mathbb{R}}\int_{\mathbb{R}^d}(1\wedge\|f(A,-s)x\|\cdot\|f(A,t-s)x\|)\,\nu(dx)\,ds\,\pi(dA)\overset{t\to\infty}{\to}0.$$

We first prove that $\|\Sigma(t)\|\to0$ as $t\to\infty$. Therefore, note that the existence of the MMA process implies (cf. Theorem 3.2)

$$\int_S\int_{\mathbb{R}}\Big\|f(A,t-s)\Sigma^{\frac12}\Big\|^2\,ds\,\pi(dA)<\infty$$

for any $t \in \mathbb{R}$, where $\Sigma^{\frac{1}{2}}$ denotes the unique square root of Σ. Thus, for any t, the function $g_t : S \times \mathbb{R} \to \mathbb{R}$, $(A,s) \mapsto \left\| f(A,t-s)\Sigma^{\frac{1}{2}} \right\|$ is an element of $L^2\big(S \times \mathbb{R}, \mathscr{B}(S \times \mathbb{R}), \pi \otimes \lambda^1; \mathbb{R}\big)$. Since the measure $\pi \otimes \lambda^1$ is σ-finite, every such L^2-function can be approximated (in the L^2-norm) by an elementary function in

$$\mathscr{E} := \bigg\{ f \in L^2\big(S \times \mathbb{R}, \mathscr{B}(S \times \mathbb{R}), \pi \otimes \lambda^1; \mathbb{R}\big) : f = \sum_{i=1}^{n} c_i \mathbb{1}_{D_i \times [a_i, b_i)},\, n \in \mathbb{N},$$
$$c_i \in \mathbb{R},\, D_i \in \mathscr{B}(S),\, -\infty < a_i < b_i < \infty,\, i = 1, \ldots, n \bigg\}.$$

Now fix an arbitrary $\varepsilon > 0$ and choose $\widetilde{g} \in \mathscr{E}$ such that

$$\|g_0 - \widetilde{g}\|_{L^2} = \left(\int_S \int_{\mathbb{R}} |g_0(A,s) - \widetilde{g}(A,s)|^2 \, ds \, \pi(dA) \right)^{\frac{1}{2}} < \varepsilon.$$

Then, due to the Cauchy-Schwarz Inequality,

$$\begin{aligned}
\|\Sigma(t)\| &= \left\| \int_S \int_{\mathbb{R}} f(A,-s)\Sigma^{\frac{1}{2}} \left(f(A,t-s)\Sigma^{\frac{1}{2}} \right)' ds \, \pi(dA) \right\| \leq \int_S \int_{\mathbb{R}} g_0(A,s) \cdot g_t(A,s) \, ds \, \pi(dA) \\
&\leq \varepsilon \cdot \|g_t\|_{L^2} + \|\widetilde{g}\|_{L^2} \cdot \left(\int_S \int_{\mathbb{R}} |g_t(A,s) - \widetilde{g}(A,s-t)|^2 \, ds \, \pi(dA) \right)^{\frac{1}{2}} \\
&\qquad\qquad + \int_S \int_{\mathbb{R}} |\widetilde{g}(A,s)\widetilde{g}(A,s-t)| \, ds \, \pi(dA) \\
&\leq \varepsilon \cdot \|g_0\|_{L^2} + (\varepsilon + \|g_0\|_{L^2}) \cdot \varepsilon
\end{aligned}$$

for sufficiently large t. This yields $\|\Sigma(t)\| \to 0$ as $t \to \infty$.

It remains to show that $\int_{\mathbb{R}^{2n}} (1 \wedge \|x\| \cdot \|y\|) \, \nu_{0t}(d(x,y)) \to 0$ as $t \to \infty$. We fix again an arbitrary $\varepsilon > 0$ and set $B_r := \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \|x\|^2 + \|y\|^2 \leq r^2\big\}$. Note that in order to establish (2.12) we did not use the assumption that the process is mixing. Hence there is some $R > 1$ and some $t_0 > 0$ such that

$$\sup_{t \geq t_0} \nu_{0t}\big(\mathbb{R}^{2n} \backslash B_R\big) \leq \varepsilon.$$

Thus, for all $t \geq t_0$, we deduce $\int_{\mathbb{R}^{2n}} (1 \wedge \|x\| \cdot \|y\|) \, \nu_{0t}(d(x,y)) \leq \int_{B_R} (1 \wedge \|x\| \cdot \|y\|) \, \nu_{0t}(d(x,y)) + \varepsilon$. Since

$$\min\{\|u\| \cdot \|v\|, 1\} \leq R \cdot \min\{\|u\|, 1\} \cdot \min\{\|v\|, 1\},$$

provided that $\max\{\|u\|, \|v\|\} \leq R$, we obtain

$$\begin{aligned}
\int_{B_R} (1 \wedge \|x\| \cdot \|y\|) \, \nu_{0t}(d(x,y)) &\leq R \cdot \int_{B_R} (1 \wedge \|x\|) \cdot (1 \wedge \|y\|) \, \nu_{0t}(d(x,y)) \\
&\leq R \cdot \int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} (1 \wedge \|f(A,-s)x\|) \cdot (1 \wedge \|f(A,t-s)x\|) \, \nu(dx) \, ds \, \pi(dA).
\end{aligned}$$

Analogously to above, the existence of the MMA process shows that, for any $t \in \mathbb{R}$, the function $h_t : S \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $h_t(A,s,x) := 1 \wedge \|f(A,t-s)x\|$ is an element of $L^2\big(S \times \mathbb{R} \times \mathbb{R}^d, \mathscr{B}(S \times \mathbb{R} \times \mathbb{R}^d), \pi \otimes \lambda^1 \otimes \nu; \mathbb{R}\big)$. Since every Lévy measure is σ-finite, the product measure $\pi \otimes \lambda^1 \otimes \nu$ is σ-finite as well and hence we can use the same approximation argument as above in order to show that

$$\int_S \int_{\mathbb{R}} \int_{\mathbb{R}^d} (1 \wedge \|f(A,-s)x\|) \cdot (1 \wedge \|f(A,t-s)x\|)\, \nu(dx)\, ds\, \pi(dA) \overset{t\to\infty}{\to} 0.$$

Consequently $\lim_{t\to\infty} \int_{\mathbb{R}^{2n}} (1 \wedge \|x\| \cdot \|y\|)\, \nu_{0t}(d(x,y)) = 0$ and the MMA process $(X_t)_{t\in\mathbb{R}}$ is mixing. □

Example 3.6 (supOU Processes). Superpositions of Ornstein-Uhlenbeck-type (supOU) processes provide a class of continuous time processes capable of exhibiting long memory behavior. The theory of univariate supOU processes as introduced in [2] has been extended to a multivariate setting in [4]. Intuitively supOU processes are obtained by "adding up" (i.e., integrating) independent OU-type processes with different mean reversion coefficients.

Given an $\mathbb{R}^d$-valued Lévy basis Λ on $M_d^-(\mathbb{R}) \times \mathbb{R}$, where $M_d^-(\mathbb{R})$ denotes the collection of all real $d \times d$ matrices with eigenvalues having strictly negative real part, and given that the process

$$X_t := \int_{M_d^-(\mathbb{R})} \int_{-\infty}^{t} e^{(t-s)A}\, \Lambda(dA, ds)$$

exists for all $t \in \mathbb{R}$, it is said to be a d-dimensional supOU process.

One easily verifies that supOU processes are MMA processes with special kernel function

$$f(A,s) = e^{sA} \mathbb{1}_{[0,\infty)}(s).$$

Consequently, (multivariate) supOU processes are always mixing by virtue of Theorem 3.5.

4 The supOU Stochastic Volatility Model

The well-known Ornstein-Uhlenbeck type stochastic volatility (OU type SV) model introduced in [3] has recently been extended to a multivariate set-up in [21]. Whereas the OU type SV model is capable of reproducing most of the so-called stylized facts (stochastic volatility exhibiting jumps, volatility clustering, heavy tails, ...) which are usually present in observed financial return series, it is not capable of producing long memory in the volatility or log-returns. Therefore one could use a (multivariate) supOU type SV model (see [5]) where the volatility or instantaneous covariance matrix is modelled via a positive semidefinite supOU process.

Let us briefly recall the definition of positive semidefinite supOU processes. Suppose we have given an $\mathbb{S}_d(\mathbb{R})$-valued Lévy basis Λ on $M_d^-(\mathbb{R}) \times \mathbb{R}$ with generating quadruple $(\gamma, 0, \nu, \pi)$ with

$$\gamma_0 := \gamma - \int_{\{\|x\|\le 1\}} \|x\|\, \nu(dx) \in \mathbb{S}_d^+(\mathbb{R}) \tag{4.1}$$

and ν being a Lévy measure on $\mathbb{S}_d(\mathbb{R})$ satisfying

$$\nu\big(\mathbb{S}_d(\mathbb{R}) \backslash \mathbb{S}_d^+(\mathbb{R})\big) = 0, \quad \int_{\{\|x\|>1\}} \log(\|x\|)\, \nu(dx) < \infty \quad \text{and} \quad \int_{\{\|x\|\le 1\}} \|x\|\, \nu(dx) < \infty. \tag{4.2}$$

Moreover, assume there exist measurable functions $\rho : M_d^-(\mathbb{R}) \to \mathbb{R}_+$ and $\kappa : M_d^-(\mathbb{R}) \to [1,\infty)$ such that

$$\left\|e^{sA}\right\| \le \kappa(A) e^{-\rho(A)s} \quad \forall s \in [0,\infty),\ \pi-\text{almost surely}, \quad \text{and} \tag{4.3}$$

$$\int_{M_d^-(\mathbb{R})} \frac{\kappa(A)^2}{\rho(A)}\, \pi(dA) < \infty. \tag{4.4}$$

Then the process $(\Sigma_t)_{t\in\mathbb{R}}$ given by

$$\Sigma_t := \int_{M_d^-(\mathbb{R})} \int_{-\infty}^{t} e^{(t-s)A}\, \Lambda(dA,ds)\, e^{(t-s)A'}$$

is well defined for all $t \in \mathbb{R}$ and strictly stationary. Moreover, with $\otimes$ being the tensor (Kronecker) product of two matrices and vec the well-known vectorization operator that maps the $d\times d$ matrices to $\mathbb{R}^{d^2}$ by stacking the columns of a matrix below one another beginning with the left one, we have

$$\operatorname{vec}(\Sigma_t) = \int_{M_d^-(\mathbb{R})} \int_{-\infty}^{t} e^{(t-s)(A\otimes \mathrm{I}_d + \mathrm{I}_d \otimes A)} \operatorname{vec}(\Lambda)(dA,ds) \tag{4.5}$$

and Σ_t is positive semidefinite for all $t \in \mathbb{R}$ (see [4, Theorem 4.1]). The process $(\Sigma_t)_{t\in\mathbb{R}}$ is said to be a **positive semidefinite supOU process**.

We also recall the definition of a supOU type stochastic volatility model (cf. [5, Definition 3.1]):

Definition 4.1 (supOU Stochastic Volatility Model). *Let W be a d-dimensional standard Brownian motion and Λ be an $\mathbb{S}_d(\mathbb{R})$-valued Lévy basis on $M_d^-(\mathbb{R})\times\mathbb{R}$, independent of W, with generating quadruple $(\gamma,0,\nu,\pi)$ satisfying conditions (4.1)-(4.4). Let, moreover L be the underlying Lévy process of Λ and finally $\beta,\psi \in B\big(\mathbb{S}_d(\mathbb{R}),\mathbb{R}^d\big)$. Assume that $X=(X_t)_{t\geq 0}$ is given by*

$$dX_t = (\mu + \beta\Sigma_{t-})\,dt + r(\Sigma_{t-})\,dW_t + \psi(dL_t), \qquad X_0 = 0, \tag{4.6}$$

for some $\mu \in \mathbb{R}^d$, a continuous function $r : \mathbb{S}_d^+(\mathbb{R}) \to M_d(\mathbb{R})$ such that $x = r(x)r(x)'$ and where

$$\Sigma_t = \int_{M_d^-(\mathbb{R})} \int_{-\infty}^{t} e^{(t-s)A}\, \Lambda(dA,ds)\, e^{(t-s)A'} \quad \forall t \in [0,\infty).$$

Then we say that X follows a multivariate supOU type stochastic volatility (SV) model with leverage.

When thinking about X as the log-price processes of d financial assets, it is clear that one typically will observe neither X continuously nor the volatility process Σ, but only X at a discrete set of times. In the following we assume that we observe X at an equally spaced time grid with given grid size $\Delta > 0$. Then one is typically interested in the log-returns $Y=(Y_n)_{n\in\mathbb{N}}$ over the grid intervals as well as the integrated volatility $V=(V_n)_{n\in\mathbb{N}}$ over them (for more background see [21]). They are defined by

$$Y_n := X_{n\Delta} - X_{(n-1)\Delta} \overset{(4.6)}{=} \mu\Delta + \beta\Big(\int_{(n-1)\Delta}^{n\Delta} \Sigma_{t-}\,dt\Big) + \int_{(n-1)\Delta}^{n\Delta} r(\Sigma_{t-})\,dW_t + \psi(L_{n\Delta} - L_{(n-1)\Delta}) \quad \text{and}$$

$$V_n := \int_{(n-1)\Delta}^{n\Delta} \Sigma_{t-}\,dt, \qquad n \in \mathbb{N}.$$

Of course, we have to ensure that the stochastic integrals involving Σ as integrand do indeed exist. To this end we suppose throughout the whole section that the conditions of [4, Theorem 4.3 (ii) and (iii)], namely

$$\int_{M_d^-(\mathbb{R})} \kappa(A)^2\,\pi(dA) < \infty, \tag{4.7}$$

$$\int_{M_d^-(\mathbb{R})} \frac{(\|A\|\vee 1)\kappa(A)^2}{\rho(A)}\,\pi(dA) < \infty \qquad \text{and} \qquad \int_{M_d^-(\mathbb{R})} \|A\|\kappa(A)^2\,\pi(dA) < \infty, \tag{4.8}$$

are satisfied.

Our central result of this section is the following theorem:

Theorem 4.2. *Both processes, the log-returns $Y = (Y_n)_{n\in\mathbb{N}}$ over the grid intervals as well as the integrated volatility $V = (V_n)_{n\in\mathbb{N}}$ over them are mixing.*

Proof. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of $N(0,\mathrm{I}_d)$-distributed random vectors, independent of Σ and L. Then using the independence between Λ and W we obtain

$$(Y_n)_{n\in\mathbb{N}} \stackrel{\mathscr{D}}{=} \big(\mu\Delta+\beta V_n + r(V_n)\varepsilon_n + \psi(L_{n\Delta}-L_{(n-1)\Delta})\big)_{n\in\mathbb{N}}. \tag{4.9}$$

Now observe that the process $G = (G_t)_{t\geq 0}$ given by

$$G_t := \begin{pmatrix} \mathrm{vec}(\Sigma_t) \\ \mathrm{vec}(L_{t+\Delta}-L_t) \end{pmatrix} \stackrel{(4.5)}{=} \int_{M_d^-(\mathbb{R})}\int_{\mathbb{R}} \begin{pmatrix} e^{(t-s)(A\otimes \mathrm{I}_d + \mathrm{I}_d\otimes A)}\mathbb{1}_{[0,\infty)}(t-s) \\ \mathrm{I}_{d^2}\mathbb{1}_{[-\Delta,0)}(t-s)\end{pmatrix} \mathrm{vec}(\Lambda)(dA,ds), \quad t\in[0,\infty),$$

is a $2d^2$-dimensional MMA process with respect to the $\mathbb{R}^{d^2}$-valued Lévy basis $\mathrm{vec}(\Lambda) = \{\mathrm{vec}(\Lambda(B)) : B \in \mathscr{B}_0(M_d^-(\mathbb{R})\times\mathbb{R})\}$ and hence it is mixing by virtue of Theorem 3.5. Using [1, Theorem 3.2.7] we easily deduce that the process

$$(H_n)_{n\in\mathbb{N}} := \begin{pmatrix} (\Sigma_s)_{s\in[(n-1)\Delta,n\Delta]} \\ (L_{s+\Delta}-L_s)_{s\in[(n-1)\Delta,n\Delta]} \end{pmatrix}_{n\in\mathbb{N}}$$

is mixing as well. Since $(\varepsilon_n)_{n\in\mathbb{N}}$ is mixing and independent of $(H_n)_{n\in\mathbb{N}}$ we easily deduce from the definition (1.1) that

$$(R_n)_{n\in\mathbb{N}} := \begin{pmatrix} H_n \\ \varepsilon_n \end{pmatrix}_{n\in\mathbb{N}}$$

is also mixing. Finally, condition (4.7) ensures that the integrated volatility V_n can be seen ω-wise as a Lebesgue integral (cf. [4, Theorem 4.3 (ii)]), i.e. V_n is a measurable transformation of H_n for any $n\in\mathbb{N}$ and hence mixing. In the same way the right-hand side of (4.9) is a measurable transformation of R_n and thus the log-returns $(Y_n)_{n\in\mathbb{N}}$ are mixing as well. $\square$

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References

[1] ASH, R. B., AND GARDNER, M. F. *Topics in Stochastic Processes*, vol. 27 of *Probability and Mathematical Statistics*. Academic Press, New York, 1975.

[2] BARNDORFF-NIELSEN, O. E. Superposition of Ornstein-Uhlenbeck type processes. *Teor. Veroyatnost. i Primenen. 45* (2000), 289–311.

[3] BARNDORFF-NIELSEN, O. E., AND SHEPHARD, N. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol. 63* (2001), 167–241.

[4] BARNDORFF-NIELSEN, O. E., AND STELZER, R. Multivariate supOU processes. *Ann. Appl. Probab. 21* (2011), 140–182.

[5] BARNDORFF-NIELSEN, O. E., AND STELZER, R. The multivariate supOU stochastic volatility model. *Math. Finance* (2011). Accepted for publication, available from http://www-m4.ma.tum.de/Papers.

[6] BENDER, C., LINDNER, A., AND SCHICKS, M. Finite variation of fractional Lévy processes. *J. Theoret. Probab.* (2011).

[7] BROCKWELL, P. J. Lévy-driven continuous-time ARMA processes. In *Handbook of Financial Time Series* (Berlin, 2009), T. G. Andersen, R. Davis, J.-P. Kreiß, and T. Mikosch, Eds., Springer, pp. 457 – 480.

[8] CAMBANIS, S., PODGÓRSKI, K., AND WERON, A. Chaotic behavior of infinitely divisible processes. *Studia Math. 115* (1995), 109–127.

[9] CONT, R., AND TANKOV, P. *Financial Modelling with Jump Processes*. CRC Financial Mathematics Series. Chapman & Hall, London, 2004.

[10] CORNFELD, I. P., FOMIN, S. V., AND SINAǏ, Y. G. *Ergodic Theory*, vol. 245 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1982.

[11] FASEN, V., AND KLÜPPELBERG, C. Extremes of supOU processes. In *Stochastic Analysis and Applications: The Abel Symposium 2005* (Berlin, 2007), F. E. Benth, G. Di Nunno, T. Lindstrom, B. Øksendal, and T. Zhang, Eds., vol. 2 of *Abel Symposia*, Springer, pp. 340–359.

[12] GUILLAUME, D. M., DACOROGNA, M. M., DAVÉ, R. R., MÜLLER, U. A., OLSEN, R. B., AND PICTET, O. V. From the bird's eye to the microscope: a survey of new stylized facts of the intra-daily foreign exchange markets. *Finance Stoch. 1* (1997), 95–129.

[13] HANSEN, L. P. Large sample properties of generalized method of moments estimators. *Econometrica 50* (1982), 1029–1054.

[14] KRENGEL, U. *Ergodic Theorems*, vol. 6 of *de Gruyter Studies in Mathematics*. Walter de Gruyter, Berlin, 1985.

[15] MAGDZIARZ, M. A note on Maruyama's mixing theorem. *Theory Probab. Appl. 54* (2010), 322–324.

[16] MARQUARDT, T. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli 12* (2006), 1099–1126.

[17] MARQUARDT, T., AND STELZER, R. Multivariate CARMA processes. *Stochastic Process. Appl. 117* (2007), 96–120.

[18] MARUYAMA, G. Infinitely divisible processes. *Theor. Probability Appl. 15* (1970), 1–22.

[19] PEDERSEN, J. The Lévy-Itô decomposition of an independently scattered random measure. MaPhySto research report 2, MaPhySto and University of Århus, 2003. Available from http://www.maphysto.dk.

[20] PETERSEN, K. *Ergodic Theory*, vol. 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, UK, 1983.

[21] PIGORSCH, C., AND STELZER, R. A Multivariate Ornstein-Uhlenbeck Type Stochastic Volatility Model. Available from http://www-m4.ma.tum.de/Papers (preprint), 2009.

[22] RAJPUT, B. S., AND ROSIŃSKI, J. Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields 82* (1989), 451–487.

[23] ROSIŃSKI, J., AND ŻAK, T. Simple conditions for mixing of infinitely divisible processes. *Stochastic Process. Appl. 61* (1996), 277–288.

[24] ROSIŃSKI, J., AND ŻAK, T. The equivalence of ergodicity and weak mixing for infinitely divisible processes. *J. Theoret. Probab. 10* (1997), 73–86.

[25] SATO, K. *Lévy Processes and Infinitely Divisible Distributions*, vol. 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, UK, 1999.

[26] SURGAILIS, D., ROSIŃSKI, J., MANDREKAR, V., AND CAMBANIS, S. Stable mixed moving averages. *Probab. Theory Related Fields 97* (1993), 543–558.

[27] TOSSTORFF, T., AND STELZER, R. Moment based estimation of supOU processes and a related stochastic volatility model. In preparation, 2011.