Tail Behavior of Multivariate Lévy-Driven Mixed Moving Average Processes and supOU Stochastic Volatility Models

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Multivariate Lévy-driven mixed moving average (MMA) processes of the type $X_t = \int\int f(A, t-s)\Lambda(dA, ds)$ cover a wide range of well known and extensively used processes such as Ornstein-Uhlenbeck processes, superpositions of Ornstein-Uhlenbeck (supOU) processes, (fractionally integrated) CARMA processes and increments of fractional Lévy processes. In this paper, we introduce multivariate MMA processes and give conditions for their existence and regular variation of the stationary distributions. Furthermore, we study the tail behavior of multivariate supOU processes and of a stochastic volatility model, where a positive semidefinite supOU process models the stochastic volatility.

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1 Introduction

In many areas of application Lévy-driven processes are used for modeling time series. One elementary example of the processes used is the Lévy-driven Ornstein-Uhlenbeck (OU) type process

$$X_t = \int_{-\infty}^{t} e^{-a(t-s)} dL_s,$$

where L is a Lévy process (see Sato (2002) for a detailed introduction). These processes are used, for instance, to model the variance (i.e., the volatility in the terminology of mathematical finance) in the OU type stochastic volatility model of Barndorff-Nielsen

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and Shephard (2001), which has been extended to the multivariate setting by Pigorsch and Stelzer (2009). Even though this model has many nice properties (e.g. stochastic volatility with jumps and clustering, heavy tails etc.) it does not account for the long memory effects that can often be found in real data. This problem can be bypassed by the superposition of OU type processes which leads to supOU processes of the type

$$X_t = \int_{\mathbb{R}} \int_{-\infty}^{t} e^{a(t-s)} \Lambda(da, ds),$$

where Λ is a so-called Lévy basis. These processes have been introduced by Barndorff-Nielsen (2001), extended to a multivariate setting by Barndorff-Nielsen and Stelzer (2011a) and they are used in the multivariate supOU type stochastic volatility model of Barndorff-Nielsen and Stelzer (2011b). Bayesian estimation of univariate supOU stochastic volatility models is e.g. carried out in Griffin and Steel (2010).
The aim of this paper is to analyze the tail behavior of the multivariate mixed moving average (MMA) processes

$$X_t = \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s) \Lambda(dA, ds)$$

that allow for a general kernel function $f : M_d^- \times \mathbb{R} \mapsto M_{n,d}$ (Λ is an $\mathbb{R}^d$-valued Lévy basis in this setting). They reach back to Surgailis et al. (1993) and they cover both, OU and supOU processes, as well as CARMA processes, fractionally integrated CARMA processes (cf. Brockwell (2004), Marquardt (2007)) and increments of fractional Lévy processes (cf. Marquardt (2006), Bender et al. (2011) and references therein). The tail behavior of univariate MMA processes has already been studied by Fasen (2005) and Jacobsen et al. (2009) and we extend the results to a multivariate setting and analyze also the special case of supOU processes and the related stochastic volatility model given by

$$\begin{aligned} dX_t &= a_t dt + \Sigma_t^{1/2} dW_t + \Psi(dL_t) \\ X_0 &= 0, \end{aligned}$$

where a is an $\mathbb{R}^d$-valued predictable process, W is the standard d-dimensional Brownian motion, L is the Lévy process associated with Λ, $\Psi : \mathbb{S}_d \mapsto \mathbb{R}^d$ is a linear operator and the stochastic volatility process $(\Sigma_t)_{t\in\mathbb{R}}$ is a matrix-valued positive semidefinite supOU process. The multivariate extension is non-trivial, since the definition of regular variation is considerably more involved in the multivariate setting and we have to take the peculiarities created by the use of matrices into account.
In finance understanding the tail behavior is of great importance for risk assessment and risk management. Moreover, our results allow one to understand how one can model the so-called "correlation breakdown" effect (viz. in times of extreme crisis basically all correlations get close to one) which is regarded by econometrics to be typically present in observed financial data.
The paper is structured as follows. We start by giving some general notation in Section 2.1. In Section 2.2 we will give a short excursion to multivariate regular variation that we need when we analyze the tail behavior of the processes given. An introduction to Lévy bases and conditions for the existence of integrals with respect to Lévy bases will be given in Section 2.3. Based on these preliminaries, we can then define and analyze multivariate mixed moving average processes in Section 3. We give sufficient conditions

for the mixed moving average processes to be regularly varying given that the driving Lévy basis is regularly varying. Furthermore, we examine the restrictiveness of the conditions by establishing closely related necessary conditions. In Section 4 we apply these results to multivariate supOU processes and give some more accessible conditions for this special case. Finally, we consider a stochastic volatility model that is based on positive semidefinite supOU processes and analyze its tail behavior in Section 5, which is very important for risk assessment.

2 Preliminaries

2.1 Notation

Given the real numbers $\mathbb{R}$ we use the notation $\mathbb{R}^+$ for the positive real numbers and $\mathbb{R}^-$ for the negative real numbers, both without 0. The Borel sets are denoted by $\mathcal{B}$, where $\mathcal{B}_b$ are the bounded Borel sets and $\mathcal{B}_\mu := \{B \in \mathcal{B} : \mu(\partial B) = 0\}$ describes all Borel sets with no μ-mass at the boundary ∂B. The closure of a set B is given by $\overline{B}$. S is the unit sphere, λ is the Lebesgue measure on $\mathbb{R}$ and $N(0, I_d)$ is the standard normal distribution in $\mathbb{R}^d$.
For matrices, $M_{n,d}$ is the set of all $n \times d$ matrices and M_d the set of all $d \times d$ matrices. M_d^- is the set of all $d \times d$ matrices with eigenvalues having strictly negative real part. I_d is the $d \times d$ identity matrix, $\mathbb{S}_d$ denotes the symmetric $d \times d$ matrices and $\mathbb{S}_d^+$ the positive semidefinite $d \times d$ matrices. We write A^T for the transposed of a matrix A and $\|A\|$ for its matrix norm. Since all norms are equivalent, the type of norm is not important for our results, but if we make no further specifications, we use the operator norm induced by the Euclidean norm. $j(A) := \min_{\|x\|=1} \|Ax\|$ is the modulus of injectivity of A. vec(A) is the well known operation that creates a vector by stacking the columns of an $n \times n$ matrix A below each other to obtain an $\mathbb{R}^{n^2}$-valued vector and $\otimes$ is the tensor product of two matrices.
Vague convergence is denoted by $\xrightarrow{v}$. It is defined on the one-point uncompactification $\overline{\mathbb{R}}^d \backslash \{0\}$, which assures that the sets $B \subseteq V_r := \{x : \|x\| > r\}$, $r > 0$, that are bounded away from the origin can be referred to as the relatively compact sets in the vague topology. In this topology, the compact sets shall be denoted by $\mathcal{K}$ and the open sets by $\mathcal{G}$.

2.2 Multivariate Regular Variation

For the analysis of the tail behavior of multivariate stochastic processes, we use the well established concept of regular variation. However, there is not only one single definition of multivariate regular variation, but many different equivalent ones. For detailed and very good introductions into the different approaches to multivariate regular variation, we refer the reader to Resnick (2007) and Lindskog (2004). We start with a well-known definition of multivariate regular variation (cf. Resnick (1986) and Hult and Lindskog (2006)).

Definition 2.1 (Multivariate Regular Variation). *A random vector $X \in \mathbb{R}^d$ is called* regularly varying *with index $\alpha > 0$, if there exists a slowly varying function $l : \mathbb{R} \mapsto \mathbb{R}$ and a nonzero Radon measure μ defined on $\mathcal{B}(\overline{\mathbb{R}}^d \backslash \{0\})$ with $\mu(\overline{\mathbb{R}}^d \backslash \mathbb{R}^d) = 0$ such that, as $u \to \infty$,*

$$u^\alpha l(u) P(u^{-1} X \in \cdot) \xrightarrow{v} \mu(\cdot)$$

on $\mathcal{B}(\overline{\mathbb{R}}^d \backslash \{0\})$. We write $X \in RV(\alpha, l, \mu)$.

Similarly, we call a Radon measure ν regularly varying, if α, l and μ exist as above with

$$u^{\alpha} l(u)\nu(u\cdot) \xrightarrow{v} \mu(\cdot)$$

for $u \to \infty$ and we write $\nu \in RV(\alpha, l, \mu)$.
A stochastic process $(X_t)_{t\in\mathbb{R}} \in \mathbb{R}^d$ is called regularly varying with index α, if all its finite dimensional distributions are regularly varying with index α.

The measure μ is homogeneous, i.e. it necessarily satisfies the condition

$$\mu(tB) = t^{-\alpha}\mu(B)$$

for all $B \in \mathcal{B}(\overline{\mathbb{R}}^d \backslash \{0\})$ and $t > 0$. We make use of this property throughout this paper. In this paper, we will deal with infinitely divisible random variables and processes. For those, the following very useful connection between regular variation of the random variable and its Lévy measure exists.

Theorem 2.2 (Hult and Lindskog (2006), Proposition 3.1). *Let $X \in \mathbb{R}^d$ be an infinitely divisible random vector with Lévy measure ν. Then $X \in RV(\alpha, l, \mu)$ if and only if $\nu \in RV(\alpha, l, \mu)$.*

Furthermore, we will also need regular variation of matrix-valued random variables and processes. If we take into account the well known vec operation that creates a vector by stacking the columns of a matrix below each other, we can simply apply the above definition. This allows us to use all known results for the $\mathbb{R}^d$-valued case also in the matrix-valued case.

2.3 Lévy Bases and Integration

In this chapter we recall $\mathbb{R}^d$-valued Lévy bases, which are generalizations of Lévy processes, and the related integration theory. For a general introduction to Lévy processes and infinitely divisible distributions see Sato (2002). Lévy bases are also called infinitely divisible independently scattered random measures (i.d.i.s.r.m.) in the literature. For more details on Lévy bases see Rajput and Rosiński (1989) and Pedersen (2003).

Definition 2.3 (Lévy basis). *An $\mathbb{R}^d$-valued random measure $\Lambda = (\Lambda(B))$ with $B \in \mathcal{B}_b(M_d^- \times \mathbb{R})$ is called a* Lévy basis, *if:*

- *The distribution of $\Lambda(B)$ is infinitely divisible for all $B \in \mathcal{B}_b(M_d^- \times \mathbb{R})$.*
- *For any n the random variables $\Lambda(B_1), \dots, \Lambda(B_n)$ are independent for pairwise disjoint sets $B_1, \dots, B_n \in \mathcal{B}_b(M_d^- \times \mathbb{R})$.*
- *For any pairwise disjoint sets $(B_i)_{i\in N} \in \mathcal{B}_b(M_d^- \times \mathbb{R})$ with $\bigcup_{n\in N} B_n \in \mathcal{B}_b(M_d^- \times \mathbb{R})$ we have $\Lambda(\bigcup_{n\in N} B_n) = \sum_{n\in N} \Lambda(B_n)$ almost surely.*

In this paper we restrict ourselves to time-homogeneous and factorisable Lévy bases, i.e. Lévy bases with characteristic function

$$\mathbb{E}\left(e^{iu^T\Lambda(B)}\right) = e^{\varphi(u)\Pi(B)} \tag{2.1}$$

for all $u \in \mathbb{R}^d$ and $B \in \mathcal{B}_b(M_d^- \times \mathbb{R})$, where $\Pi = \lambda \times \pi$ is the product of a probability measure π on $M_d^-(\mathbb{R})$ and the Lebesgue measure λ on $\mathbb{R}$ and

$$\varphi(u) = iu^T\gamma - \frac{1}{2}u^T\Sigma u + \int_{\mathbb{R}^d} \left(e^{iu^Tx} - 1 - iu^Tx\mathbb{1}_{[-1,1]}(\|x\|)\right)\nu(dx)$$

is the cumulant transform of an infinitely divisible distribution with characteristic triplet (γ, Σ, ν). By L we denote the underlying Lévy process associated with (γ, Σ, ν) and given by $L_t = \Lambda(M_d^- \times (0,t])$ and $L_{-t} = \Lambda(M_d^- \times [-t,0))$ for $t \in \mathbb{R}^+$. The quadruple $(\gamma, \Sigma, \nu, \pi)$ determines the distribution of the Lévy basis completely and therefore it is called the *generating quadruple*. A definition of $\mathbb{S}_d$-valued Lévy bases follows along the same lines.

Remark 2.4. Considering only time homogeneous and factorisable Lévy bases is motivated by possible applications, where models with too many parameters are of no real help, and the so far developed theory of special cases, particularly the supOU process, where this assumption is also made. However, it should be noted that this assumption is not overly restrictive, because stationarity of a Lévy-driven MMA requires obviously in general a time-homogeneous Lévy basis, i.e. the Lebesgue measure has to be used on the time axis. In our work it appears very natural only to consider stationary cases. Hence, the only possible generalization would be to allow the infinitely divisible distribution to depend on $A \in M_d^-$. We could have $\varphi(A,u)$ instead of $\varphi(u)$. Then we would also have a characteristic triplet $(\gamma(A), \Sigma(A), \nu(A,dx))$ with ν being a "Lévy kernel" and all our results would have immediate extensions to this case noting that as far as regular variation is concerned one would have to demand that $\Lambda(B)$ has to be regularly varying for all sets B with same index α and slowly varying function l (or "degenerately α-regularly varying", i.e. $u^\alpha l(u) P(u^{-1}\Lambda(B \in \cdot)) \to 0$) and the measure of regular variation would have to be given via a nontrivial kernel $\mu_\nu(A, \cdot)$. Like the univariate literature (see Fasen (2005), Fasen (2009) and Fasen and Klüppelberg (2007)) we refrain from stating our results on this level of generality, since it would not add real insight, but lead to overly technical statements not relevant for applications.

The main focus of this paper, the mixed moving average processes, are defined by integrating over a function f with respect to a Lévy basis. Regarding the existence of these integrals we recall the following multivariate extension of Theorem 2.7 in Rajput and Rosiński (1989).

Theorem 2.5. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic function of the form (2.1) and let $f : M_d^- \times \mathbb{R} \mapsto M_{n,d}$ be a measurable function. Then f is Λ-integrable as a limit in probability in the sense of Rajput and Rosiński (1989), if and only if*

$$\int_{M_d^-} \int_{\mathbb{R}} \left\| f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x \left(\mathbb{1}_{[0,1]}(\|f(A,s)x\|) - \mathbb{1}_{[0,1]}(\|x\|) \right) \nu(dx) \right\| ds\,\pi(dA) < \infty, \tag{2.2}$$

$$\int_{M_d^-} \int_{\mathbb{R}} \|f(A,s)\Sigma f(A,s)^T\| ds\,\pi(dA) < \infty \quad \textit{and} \tag{2.3}$$

$$\int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(A,s)x\|^2\right) \nu(dx) ds\,\pi(dA) < \infty. \tag{2.4}$$

If f is Λ-integrable, the distribution of $X_0 = \int_{M_d^-} \int_{\mathbb{R}^+} f(A,s)\Lambda(dA,ds)$ is infinitely divisible with characteristic triplet $(\gamma_{int}, \Sigma_{int}, \nu_{int})$ given by

$$\gamma_{int} = \int_{M_d^-} \int_{\mathbb{R}} \left(f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x \left(\mathbb{1}_{[0,1]}(\|f(A,s)x\|) - \mathbb{1}_{[0,1]}(\|x\|) \right) \nu(dx) \right) ds\,\pi(dA),$$

$$\Sigma_{int} = \int_{M_d^-} \int_{\mathbb{R}} f(A,s)\Sigma f(A,s)^T ds\,\pi(dA) \quad \textit{and}$$

$$\nu_{int}(B) = \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_B(f(A,s)x)\nu(dx)ds\pi(dA) \text{ for all Borel sets } B \subseteq \mathbb{R}^d.$$

We give now some more accessible sufficient conditions for the special case of a regular varying driving Lévy measure ν. Therefore, we define the set

$$\mathbb{L}^\delta(\lambda \times \pi) := \left\{ f : M_d^- \times \mathbb{R} \mapsto M_{n,d} \text{ measurable, } \int_{M_d^-} \int_{\mathbb{R}} \|f(A,s)\|^\delta ds\pi(dA) < \infty \right\}.$$

The following theorem is a multivariate analogue of Proposition 3.1 in Fasen (2005), which is non-trivial due to the peculiarities arising from the used matrices.

Theorem 2.6. *Let Λ be a Lévy basis with values in $\mathbb{R}^d$ and characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$, let ν be regularly varying with index α and let $f : M_d^- \times \mathbb{R} \mapsto M_{n,d}$. Then f is Λ-integrable in the sense of Rajput and Rosiński (1989) and X_0 is well defined and infinitely divisible with the characteristic triplet given in Theorem 2.5, if one of the following conditions hold:*

(i) L_1 is α-stable with $\alpha \in (0,2)\backslash\{1\}$ and $f \in \mathbb{L}^\alpha \cap \mathbb{L}^1$.

(ii) f is bounded and $f \in \mathbb{L}^\delta$ for some $\delta < \alpha$, $\delta \le 1$.

(iii) f is bounded, $\mathbb{E}\, L_1 = 0$, $\alpha > 1$ and $f \in \mathbb{L}^\delta$ for some $\delta < \alpha$, $\delta \le 2$.

Proof. We will prove the result by validating the conditions (2.2), (2.3) and (2.4) given in Theorem 2.5 in each of the three settings.
(i). From Theorem 14.3 of Sato (2002) we know that in the α-stable case $\Sigma = 0$, which makes condition (2.3) trivial. Furthermore, there is a finite measure θ on the unit sphere S such that

$$\nu(B) = \int_S \int_0^\infty \frac{\mathbb{1}_B(r\xi)}{r^{1+\alpha}} dr\theta(d\xi) \quad \text{ for } B \in \mathcal{B}^d.$$

For condition (2.2), this yields

$$\begin{aligned}
&\int_{M_d^-} \int_{\mathbb{R}} \left\| f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x \left(\mathbb{1}_{[0,1]}\left(\|f(A,s)x\|\right) - \mathbb{1}_{[0,1]}\left(\|x\|\right)\right) \nu(dx) \right\| ds\pi(dA) \\
&= \int_{M_d^-} \int_{\mathbb{R}} \left\| f(A,s)\gamma + \int_S \int_0^\infty f(A,s)\xi \left(\mathbb{1}_{[0,1]}\left(\|f(A,s)r\xi\|\right) - \mathbb{1}_{[0,1]}\left(\|r\xi\|\right)\right) \frac{dr}{r^\alpha}\theta(d\xi)\right\| ds\pi(dA) \\
&= \int_{M_d^-} \int_{\mathbb{R}} \left\| f(A,s)\gamma + f(A,s) \int_S \xi \int_1^{\|f(A,s)\xi\|^{-1}} r^{-\alpha} dr\theta(d\xi)\right\| ds\pi(dA) \\
&= \int_{M_d^-} \int_{\mathbb{R}} \left\| f(A,s)\gamma + f(A,s) \int_S \xi \frac{1}{1-\alpha} \left(\|f(A,s)\xi\|^{\alpha-1} - 1\right)\theta(d\xi)\right\| ds\pi(dA) \\
&\le \int_{M_d^-} \int_{\mathbb{R}} \left(\|f(A,s)\|\,\gamma + \frac{\|f(A,s)\|^\alpha}{1-\alpha}\theta(S) + \frac{\|f(A,s)\|}{1-\alpha}\theta(S)\right) ds\pi(dA) \\
&< \infty,
\end{aligned}$$

where we used $f \in \mathbb{L}^\alpha \cap \mathbb{L}^1$. For condition (2.4) we get

$$\int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \left(1 \wedge \|f(A,s)x\|^2\right) \nu(dx)ds\pi(dA) =$$

$$= \int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathbb{1}_{\{\|f(A,s)x\|\geq 1\}}\nu(dx)ds\pi(dA) + \tag{2.5}$$

$$+ \int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \|f(A,s)x\|^2 \mathbb{1}_{\{\|f(A,s)x\|\leq 1\}}\nu(dx)ds\pi(dA).$$

The first term on the right hand side can be bounded by

$$\int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathbb{1}_{\{\|f(A,s)x\|\geq 1\}}\nu(dx)ds\pi(dA) =$$

$$= \frac{1}{\alpha}\int_{M_d^-}\int_{\mathbb{R}}\int_{S} \|f(A,s)\xi\|^\alpha \theta(d\xi)ds\pi(dA)$$

$$\leq \frac{\theta(S)}{\alpha}\int_{M_d^-}\int_{\mathbb{R}} \|f(A,s)\|^\alpha ds\pi(dA) < \infty$$

and for the second term on the right hand side we get

$$\int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \|f(A,s)x\|^2 \mathbb{1}_{\{\|f(A,s)x\|\leq 1\}}\nu(dx)ds\pi(dA) =$$

$$= \frac{1}{2-\alpha}\int_{M_d^-}\int_{\mathbb{R}}\int_{S} \|f(A,s)\xi\|^\alpha \theta(d\xi)ds\pi(dA)$$

$$\leq \frac{\theta(S)}{2-\alpha}\int_{M_d^-}\int_{\mathbb{R}} \|f(A,s)\|^\alpha ds\pi(dA) < \infty.$$

(ii) and (iii). Condition (2.3) can be bounded by

$$\int_{M_d^-}\int_{\mathbb{R}} \|f(A,s)\Sigma\, f(A,s)^T\| ds\pi(dA) \leq \|\Sigma\| \int_{M_d^-}\int_{\mathbb{R}} \|f(A,s)\|^2 ds\pi(dA) < \infty,$$

which follows from the boundedness of f together with $f \in \mathbb{L}^\delta$ for some $\delta \leq 2$. For condition (2.4) we use (2.5) again. For the first term on the right hand side of (2.5) we use the inequality

$$\|f(A,s)\|\,\|x\| \geq \|f(A,s)x\| \geq 1$$

which implies

$$\|x\| \geq \frac{1}{\|f(A,s)\|}.$$

This yields

$$\int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathbb{1}_{\{\|f(A,s)x\|\geq 1\}}\nu(dx)ds\pi(dA) \leq \int_{M_d^-}\int_{\mathbb{R}} \nu\left(\left\{\|x\| \geq \frac{1}{\|f(A,s)\|}\right\}\right) ds\pi(dA).$$

Now we can apply the Potter bounds (Resnick (2007), Proposition 2.6. (ii)), giving the existence of some t_0 such that for all $t \geq t_0$ a regular varying function (in our case ν) can be bounded. Therefore, we distinguish the cases $1/\|f(A,s)\| > t_0$ and $1/\|f(A,s)\| < t_0$. For the first case we set $\widetilde{C} := \sup\{\|f(A,s)\| : \|f(A,s)\| < 1/t_0\} \leq 1/t_0$. Then we can apply the Potter bounds for $t = 1/\widetilde{C} \geq t_0$ to get

$$\int_{M_d^-} \int_{\mathbb{R}} \mathbb{1}_{\{1/\|f(A,s)\|>t_0\}} \, \nu\left(\left\{\|x\| \geq \frac{1}{\|f(A,s)\|}\right\}\right) ds\pi(dA) \leq$$

$$\leq (1+\alpha-\delta) \int_{M_d^-} \int_{\mathbb{R}} \mathbb{1}_{\{1/\|f(A,s)\|>t_0\}} \, \nu\left(\left\{\|x\| \geq \frac{1}{\widetilde{C}}\right\}\right) \left(\frac{\|f(A,s)\|}{\widetilde{C}}\right)^{\delta} ds\pi(dA)$$

$$< \infty.$$

In the other case we set $C := \sup \|f(A,s)\| < \infty$ and obtain

$$\int_{M_d^-} \int_{\mathbb{R}} \mathbb{1}_{\{1/\|f(A,s)\|\leq t_0\}} \, \nu\left(\left\{\|x\| \geq \frac{1}{\|f(A,s)\|}\right\}\right) ds\pi(dA) \leq$$

$$\leq \int_{M_d^-} \int_{\mathbb{R}} \mathbb{1}_{\{1/\|f(A,s)\|\leq t_0\}} \, \nu\left(\left\{\|x\| \geq \frac{1}{C}\right\}\right) ds\pi(dA)$$

$$= \nu\left(\left\{\|x\| \geq \frac{1}{C}\right\}\right) \quad \pi \times \lambda\left(\left\{(A,s) : \|f(A,s)\| \geq \frac{1}{t_0}\right\}\right)$$

$$< \infty,$$

since $f \in \mathbb{L}^{\delta}$. The second term on the right hand side of (2.5) can be bounded by

$$\int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|f(A,s)x\|^2 \mathbb{1}_{\{\|f(A,s)x\|\leq 1\}} \nu(dx) ds\pi(dA) =$$

$$= \int_{M_d^-} \int_{\mathbb{R}} \int_{\|x\|<1} \|f(A,s)x\|^2 \mathbb{1}_{\{\|f(A,s)x\|\leq 1\}} \nu(dx)$$

$$+ \int_{\|x\|\geq 1} \|f(A,s)x\|^2 \mathbb{1}_{\{\|f(A,s)x\|\leq 1\}} \nu(dx) ds\pi(dA)$$

$$\leq \int_{M_d^-} \int_{\mathbb{R}} \|f(A,s)\|^2 ds\pi(dA) \int_{\|x\|<1} \|x\|^2 \nu(dx)$$

$$+ \int_{M_d^-} \int_{\mathbb{R}} \|f(A,s)\|^{\delta} ds\pi(dA) \int_{\|x\|\geq 1} \|x\|^{\delta} \nu(dx)$$

$$< \infty,$$

where we used the fact that for bounded functions f the assumption $f \in \mathbb{L}^{\delta}$, $\delta < 2$, implies $f \in \mathbb{L}^2$. Moreover, note that $\int_{\|x\|\geq 1} \|x\|^{\delta} \nu(dx) < \infty$ by Sato (2002), Corollary 25.8, since $0 < \delta < \alpha$ and hence the underlying Lévy process has a finite δth moment. Condition (2.2) in Theorem 2.5 can be reformulated as

$$\int_{M_d^-} \int_{\mathbb{R}} \left\| f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x \left(\mathbb{1}_{[0,1]}\left(\|f(A,s)x\|\right) - \mathbb{1}_{[0,1]}\left(\|x\|\right)\right) \nu(dx) \right\| ds\pi(dA) =$$

$$= \int_{M_d^-} \int_{\mathbb{R}} \Bigg\| f(A,s)\gamma + \int_{\|x\|>1} f(A,s)x\mathbb{1}_{\{\|f(A,s)x\|\leq 1\}}\nu(dx) - \int_{\|x\|\leq 1} f(A,s)x\mathbb{1}_{\{\|f(A,s)x\|>1\}}\nu(dx)\Bigg\| ds\pi(dA) =: T.$$

In case (ii) we use $\|f(A,s)\| \leq C$ and thus T can be bounded by

$$T \leq \int_{M_d^-} \int_{\mathbb{R}} \|f(A,s)\|^\delta \left(C^{1-\delta}|\gamma| + \int_{\|x\|>1} \|x\|^\delta \nu(dx) + C^{1-\delta} \int_{\|x\|\in(\frac{1}{C},1]} \|x\|^\delta \nu(dx) \right) ds\pi(dA).$$

In case (iii), we know $\gamma = -\int_{\|x\|>1} x\nu(dx)$. Since $\alpha > 1$ and $\delta < \alpha$, we can arbitrarily choose a $\xi \in (\delta, \alpha)$ with $\xi > 1$. This yields

$$\begin{aligned}
T &= \int_{M_d^-} \int_{\mathbb{R}} \Bigg\| -\int_{\|x\|>1} f(A,s)x\nu(dx) + \int_{\|x\|>1} f(A,s)x\mathbb{1}_{\{\|f(A,s)x\|\leq 1\}}\nu(dx) \\
&\qquad - \int_{\|x\|\leq 1} f(A,s)x\mathbb{1}_{\{\|f(A,s)x\|>1\}}\nu(dx)\Bigg\| ds\pi(dA) \\
&\leq \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|f(A,s)x\|\mathbb{1}_{\{\|f(A,s)x\|>1\}}\nu(dx)ds\pi(dA) \\
&\leq \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|f(A,s)x\|^\xi \mathbb{1}_{\{\|x\|>\frac{1}{C}\}}\nu(dx)ds\pi(dA) \\
&\leq C^{\xi-\delta} \int_{M_d^-} \int_{\mathbb{R}} \|f(A,s)\|^\delta ds\pi(dA) \int_{\|x\|>\frac{1}{C}} \|x\|^\xi \nu(dx) < \infty.
\end{aligned}$$

□

3 Mixed Moving Average Processes

Mixed Moving Average (short MMA) processes have been first introduced by Surgailis et al. (1993) in the univariate stable case. As we have already mentioned in the previous chapters, they are integrals over a given kernel function with respect to a Lévy basis.

Definition 3.1 (Mixed Moving Average Process). *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ and let $f : M_d^- \times \mathbb{R} \mapsto M_{n,d}$ be a measurable function (*kernel function*). If the process*

$$X_t := \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s)\Lambda(dA, ds)$$

*exists in the sense of Theorem 2.5 for all $t \in \mathbb{R}$, it is called an n-*dimensional mixed moving average process *(short* MMA process*).*

Note that we could also define "generalized MMA" processes by integrating over a slightly more general function $g : M_d^- \times \mathbb{R} \times \mathbb{R} \mapsto M_{n,d}$, which gives us

$$X_t = \int_{M_d^-} \int_{\mathbb{R}} g(A, t, s)\Lambda(dA, ds).$$

However, the extension of all upcoming results is trivial, so we stated the results for the notationally easier case of Definition 3.1. Moreover, an MMA process is obviously always stationary and this needs not to be true for generalized MMA processes. Note also that M_d^- can obviously be replaced by M_d or basically any other Borel set. Again we state everything for M_d^-, because this eases notation and is the canonical choice in the supOU case.
Existence of the MMA processes follows directly from Theorem 2.5 and Theorem 2.6. Especially Theorem 2.6 turns out to be very useful in this setting, since it is based on similar conditions compared to the key conditions of the following theorem: Regular variation of the driving Lévy measure ν and $f \in \mathbb{L}^\alpha(\lambda \times \pi)$.
The theorem is the multivariate analog of (3.1) in Proposition 3.2 of Fasen (2005), where the same conditions, simplified to the univariate set-up, are used. A similar result also exists for the special case of a univariate filtered Lévy process, where the kernel function f is continuous and of compact support, see Hult and Lindskog (2005), Theorem 22.

Theorem 3.2. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ and let $\nu \in RV(\alpha, l, \mu_\nu)$. If $X_0 = \int_{M_d^-} \int_{\mathbb{R}^+} f(A,s)\Lambda(dA, ds)$ exists (in the sense of Theorem 2.5), $f \in \mathbb{L}^\alpha(\lambda \times \pi)$ and $\mu_\nu(f^{-1}(A,s)(\mathbb{R}^n \backslash \{0\})) = 0$ does not hold for $\pi \times \lambda$ almost-every (A, s), then $X_0 \in RV(\alpha, l, \mu_X)$ with*

$$\mu_X(B) := \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^d} \mathbb{1}_B \left(f(A,s)x\right) \mu_\nu(dx) ds \pi(dA).$$

Proof. From Theorem 2.5 we know that the distribution of X is infinitely divisible. Following Theorem 2.2 it is sufficient to prove that its Lévy measure ν_X is regularly varying. The concrete representation

$$\nu_X = \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^d} \mathbb{1}_B(f(A,s)x)\nu(dx) ds \pi(dA)$$

is also known from Theorem 2.5. Regular variation of ν then yields the existence of a constant $\alpha > 0$, a slowly varying function l and a Radon measure μ_ν on $\mathcal{B}(\overline{\mathbb{R}}^d \backslash \{0\})$ with $\mu_\nu(\overline{\mathbb{R}}^d \backslash \mathbb{R}^d) = 0$ such that, as $u \to \infty$,

$$u^\alpha l(u) \nu(u\,\cdot) \xrightarrow{v} \mu_\nu(\cdot).$$

Using Resnick (2007), Theorem 3.2, and Fatou's Lemma, we have that for all compact sets $B \in \mathcal{K}$

$$\begin{aligned}
\limsup_{u\to\infty} u^\alpha l(u) \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^d} \mathbb{1}_{uB} \left(f(A,s)x\right) \nu(dx) ds \pi(dA) \leq \\
\leq \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \limsup_{u\to\infty} u^\alpha l(u) \int\limits_{\mathbb{R}^d} \mathbb{1}_{uB} \left(f(A,s)x\right) \nu(dx) ds \pi(dA) \\
\leq \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^d} \mathbb{1}_B \left(f(A,s)x\right) \mu_\nu(dx) ds \pi(dA)
\end{aligned}$$

and conversely for all open sets $B \in \mathcal{G}$ that are relatively compact

$$\liminf_{u\to\infty} u^\alpha l(u) \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^d} \mathbb{1}_{uB} \left(f(A,s)x\right) \nu(dx) ds \pi(dA) \geq$$

$$\geq \int_{M_d^-} \int_{\mathbb{R}} \liminf_{u\to\infty} u^\alpha l(u) \int_{\mathbb{R}^d} \mathbb{1}_{uB}\left(f(A,s)x\right) \nu(dx) ds \pi(dA)$$

$$\geq \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_{B}\left(f(A,s)x\right) \mu_\nu(dx) ds \pi(dA).$$

Note here that for any set $B \in \mathcal{K}$ (resp. $\mathcal{G}$) also the preimage $f(A,s)^{-1}(B) \in \mathcal{K}$ (resp. $\mathcal{G}$) for all A, s, since $f(A,s)$ is for fixed A, s a linear mapping. This yields the vague convergence

$$u^\alpha l(u) \nu_X(u\,\cdot) = u^\alpha l(u) \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_{u\,\cdot}\left(f(A,s)x\right) \nu(dx) ds \pi(dA)$$

$$\xrightarrow{v} \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_{(\cdot)}\left(f(A,s)x\right) \mu_\nu(dx) ds \pi(dA) = \mu_X(\cdot).$$

It remains to prove that μ_X is again a Radon measure with $\mu_X(\overline{\mathbb{R}}^n \backslash \mathbb{R}^n) = 0$. The second property follows directly from the observation

$$\mathbb{1}_{\left(\overline{\mathbb{R}}^n \backslash \mathbb{R}^n\right)}(f(A,s)x) \leq \mathbb{1}_{\left(\overline{\mathbb{R}}^n \backslash \mathbb{R}^n\right)}(x).$$

For the local finiteness of μ_X, take some compact $B \in \mathcal{B}(\overline{\mathbb{R}}^d \backslash \{0\})$, i.e. there exists some finite $r > 0$ such that $B \subseteq V_r := \{x : \|x\| > r\}$. For all x with $f(A,s)x \in B \subseteq V_r$ we have $r < \|f(A,s)x\| \leq \|f(A,s)\| \, \|x\|$. By using $f \in \mathbb{L}^\alpha(\lambda \times \pi)$ and the local finiteness of μ_ν, we get

$$\mu_X(B) \leq \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_{(r,\infty)}\left(\|f(A,s)\| \, \|x\|\right) \mu_\nu(dx) ds \pi(dA)$$

$$= \int_{M_d^-} \int_{\mathbb{R}} \mu_\nu\left(\left\{x : \|x\| \geq \|f(A,s)\|^{-1} r\right\}\right) \mathbb{1}_{\mathbb{R}\backslash\{0\}}(\|f(A,s)\|) \, ds \pi(dA)$$

$$= \mu_\nu(V_r) \int_{M_d^-} \int_{\mathbb{R}} \|f(A,s)\|^\alpha ds \pi(dA) < \infty.$$

□

The theorem shows that the tail behavior of the driving Lévy measure determines the tail behavior of the MMA process. Since the Lévy measure is related only to the jumps of the underlying Lévy process, we see that the regular variation of the MMA process is caused by the jumps of the underlying Lévy process. Furthermore, we intuitively have that the extremes of the MMA process are caused by a single extremely big jump in the Lévy basis.

Remark 3.3. Another important consequence of the theorem is that we know the concrete measure μ_X of regular variation. This is useful to describe the location or mass of the extremes in $\mathbb{R}^n$. It is similar to the spectral measure in an analogue definition of regular variation, see Theorem 1.15 (ii) in Lindskog (2004). See also Example 4.4 for some calculations of these measures in the Ornstein-Uhlenbeck case.

As mentioned before, Theorem 3.2 uses two crucial conditions. The first one is the regular variation of the driving Lévy measure, meaning that the tail behavior of the input determines the tail behavior of the resulting MMA process. The second condition $f \in \mathbb{L}^\alpha(\lambda \times \pi)$ is a restriction on the function f. We will now analyze its restrictiveness by looking at necessary conditions. Therefore, we define the set

$$\mathbb{J}^\alpha(\lambda \times \pi) := \left\{ f : M_d^- \times \mathbb{R} \mapsto M_{n,d} \text{ measurable}, \int\limits_{M_d^-} \int\limits_{\mathbb{R}} j(f(A,s))^\alpha ds \pi(dA) < \infty \right\},$$

where $j(A)$ is the modulus of injectivity of A.
The following theorem extends even the univariate work by Fasen (2005) and Hult and Lindskog (2005), where necessary conditions are not considered. Note that our focus is on necessary conditions on f whereas Jacobsen et al. (2009) considered whether regular variation of a moving average implies regular variation of the driving Lévy process in the univariate case.

Theorem 3.4. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ and let $\nu \in RV(\alpha, l, \mu_\nu)$. If $X_0 = \int_{M_d^-} \int_{\mathbb{R}+} f(A,s)\Lambda(dA, ds)$ exists and $\mu_\nu(f^{-1}(A,s)(\mathbb{R}^n \backslash \{0\})) = 0$ does not hold for $\pi \times \lambda$ almost-every (A,s), then $f \in \mathbb{J}^\alpha(\lambda \times \pi)$ is a necessary condition for $X_0 \in RV(\alpha, l, \mu_X)$ with*

$$\mu_X(B) := \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^d} \mathbb{1}_B\left(f(A,s)x\right) \mu_\nu(dx) ds \pi(dA).$$

Proof. We use a simple contradiction. Suppose $f \notin \mathbb{J}^\alpha(\lambda \times \pi)$, i.e.

$$\int\limits_{M_d^-} \int\limits_{\mathbb{R}} j(f(A,s))^\alpha ds \pi(dA) = \infty.$$

Since μ_ν is nonzero there is a positive number $r > 0$ such that $\mu_\nu(V_r) > 0$. Then we use the relation

$$j(f(A,s)) \leq \frac{\|f(A,s)x\|}{\|x\|}$$

for all $x \in \mathbb{R}^d$ and get

$$\begin{aligned} \mu_X(V_r) &= \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^d} \mathbb{1}_{V_r}\left(f(A,s)x\right) \mu_\nu(dx) ds \pi(dA) \\ &\geq \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^d} \mathbb{1}_{(r,\infty)}\left(j(f(A,s))\,\|x\|\right) \mu_\nu(dx) ds \pi(dA) \\ &= \mu_\nu(V_r) \int\limits_{M_d^-} \int\limits_{\mathbb{R}} j(f(A,s))^\alpha ds \pi(dA) = \infty \end{aligned}$$

and this is a contradiction to μ_X being a Radon measure.

□

Now we have necessary conditions as well as sufficient conditions and both lie close together. Since $j(f(A,s)) \leq \|f(A,s)\|$ we immediately have $\mathbb{L}^\alpha(\lambda \times \pi) \subseteq \mathbb{J}^\alpha(\lambda \times \pi)$. In the univariate case we even have $\mathbb{L}^\alpha(\lambda \times \pi) = \mathbb{J}^\alpha(\lambda \times \pi)$ and thus we get necessary and sufficient conditions.
Having proved the regular variation of the random vector, we can now easily get the regular variation of the process X_t.

Corollary 3.5. *Given the conditions of Theorem 3.2, the MMA process* $(X_t)_{t\in\mathbb{R}}$ *is also regularly varying with index* α *as a process.*

Proof. We have to show that the results also hold for the finite dimensional distributions of X_t. For $m \in \mathbb{N}$ and $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}^m$ we have

$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_m} \end{pmatrix} = \begin{pmatrix} \int\limits_{M_d^-}\int\limits_{\mathbb{R}} f(A, t_1 - s)\Lambda(dA, ds) \\ \vdots \\ \int\limits_{M_d^-}\int\limits_{\mathbb{R}} f(A, t_m - s)\Lambda(dA, ds) \end{pmatrix} = \int\limits_{M_d^-}\int\limits_{\mathbb{R}} \begin{pmatrix} f(A, t_1 - s) \\ \vdots \\ f(A, t_m - s) \end{pmatrix} \Lambda(dA, ds)$$
$$= \int\limits_{M_d^-}\int\limits_{\mathbb{R}} g(A, \mathbf{t}, s)\,\Lambda(dA, ds)$$

with the function $g : M_d^- \times \mathbb{R}^m \times \mathbb{R} \mapsto M_{nm,d}$ defined by

$$g(A, \mathbf{t}, s) := \begin{pmatrix} f(A, t_1 - s) \\ \vdots \\ f(A, t_m - s) \end{pmatrix}.$$

Next we show that $f \in \mathbb{L}^\beta(\lambda \times \pi)$ implies $g \in \mathbb{L}^\beta(\lambda \times \pi)$ for all $\beta > 0$. Therefore, we choose the matrix norm

$$\|A\| := \max_{i,j}\{|a_{ij}|\}.$$

We get

$$\int\limits_{M_d^-}\int\limits_{\mathbb{R}} \|g(A, \mathbf{t}, s)\|^\beta \; ds\,\pi(dA) =$$
$$= \int\limits_{M_d^-}\int\limits_{\mathbb{R}} \left\| \begin{pmatrix} f(A, t_1 - s) \\ \vdots \\ f(A, t_m - s) \end{pmatrix} \right\|^\beta ds\,\pi(dA)$$
$$= \int\limits_{M_d^-}\int\limits_{\mathbb{R}} \max\{\|f(A, t_1 - s)\|, \dots, \|f(A, t_m - s)\|\}^\beta \; ds\,\pi(dA)$$
$$\le \int\limits_{M_d^-}\int\limits_{\mathbb{R}} \|f(A, t_1 - s)\|^\beta + \ldots + \|f(A, t_m - s)\|^\beta \, ds\,\pi(dA) < \infty,$$

since $f \in \mathbb{L}^\beta(\lambda \times \pi)$.
If the existence of X_t is ensured by Theorem 2.6 (ii) or (iii), this implies that for the existence and regular variation of $(X_{t_1}^T, \dots, X_{t_m}^T)^T$ a simple application of Theorem 3.2 and Theorem 2.6 conclude. However, in general we note that assuming existence of X_t in the sense of Theorem 2.5 implies that each of the m individual integrals of $(X_{t_1}^T, \dots, X_{t_m}^T)^T$ exists as a limit of approximating sums in probability. From these individual approximating sums one easily constructs a sequence of approximating sums for $\int_{M_d^-}\int_{\mathbb{R}} \left\|\left(f(A, t_1 - s)^T, \dots, f(A, t_m - s)^T\right)^T\right\|^\beta \Lambda(dA, ds)$ converging in probability. Hence, the necessary and sufficient existence conditions of Theorem 2.5 are satisfied and Theorem 3.2 shows the regular variation of $(X_{t_1}^T, \dots, X_{t_m}^T)^T$.

□

A very important class of heavy tailed distributions are α-stable distributions with $\alpha \in (0, 2)$. See Samorodnitsky and Taqqu (1994) for a detailed introduction. In Theorem 2.6 we have already given a criterion for the existence of MMA processes with stable driving Lévy process. Similar to Theorem 3.2, there is also a well-known link between stability of the driving Lévy measure and stability of the MMA process.

Lemma 3.6. *If the driving Lévy process of an MMA process X_t is α-stable and its Lévy measure is non-degenerate, then X_t is also α-stable.*

Proof. From Theorem 14.3 in Sato (2002) we have the result that α-stability of an infinitely divisible distribution is equivalent to

$$\Sigma = 0 \quad \text{and} \quad \nu(\cdot) = b^{-\alpha}\, \nu(b^{-1}\, \cdot) \quad \text{for all } b > 0.$$

Using the assumption together with Theorem 2.5, we immediately have $\Sigma_{X_t} = 0$ and $\nu_{X_t}(\cdot) = b^{-\alpha}\, \nu_{X_t}(b^{-1}\, \cdot)$.

□

Now we apply this result to multivariate continuous-time autoregressive moving average (MCARMA) processes.

Example 3.7 (MCARMA Processes). *Univariate Lévy-driven CARMA processes have been introduced by Brockwell (2001) and they have been extended to multivariate CARMA (MCARMA) processes by Marquardt and Stelzer (2007). A d-dimensional MCARMA(p,q) process, $p > q$, driven by a two-sided square integrable Lévy process $(L_t)_{t\in\mathbb{R}}$ with $\mathbb{E}\,(L_1) = 0$ and $\mathbb{E}\,(L_1 L_1^T) = \Sigma_L$ can be formally interpreted as the stationary solution to the p-th order d-dimensional differential equation*

$$P(D)Y_t = Q(D)DL_t,$$

where D denotes the differentiation operator with respect to t. The autoregressive and moving average polynomials are given by

$$P(z) = I_d z^p + A_1 z^{p-1} + \ldots + A_p \quad \text{and} \quad Q(z) = B_0 z^q + B_1 z^{q-1} + \ldots + B_q$$

with $A_1, \ldots, A_p$, $B_0, \ldots, B_q \in M_d$ such that $B_q \neq 0$ and $\{z \in \mathbb{C} : det(P(z)) = 0\} \subset \mathbb{R}\backslash\{0\} + i\mathbb{R}$.

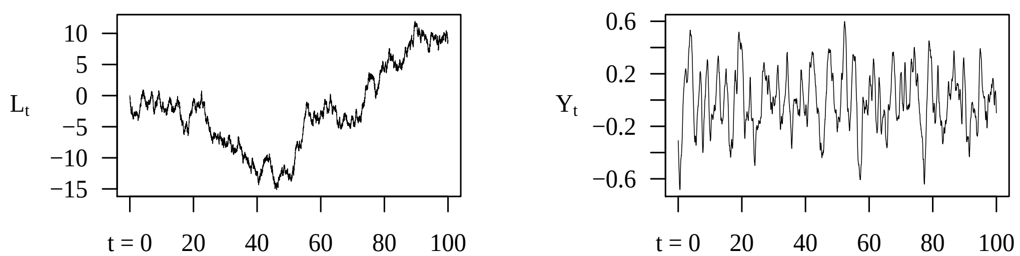

Figure 3.1: Simulations of one path of the driving Lévy process L_t and the CARMA(3,1) process Y_t in the α-stable case with $\alpha = 2$

The MCARMA process Y_t can be represented as a moving average process

$$Y_t = \int_{\mathbb{R}} f(t-s)dL_s$$

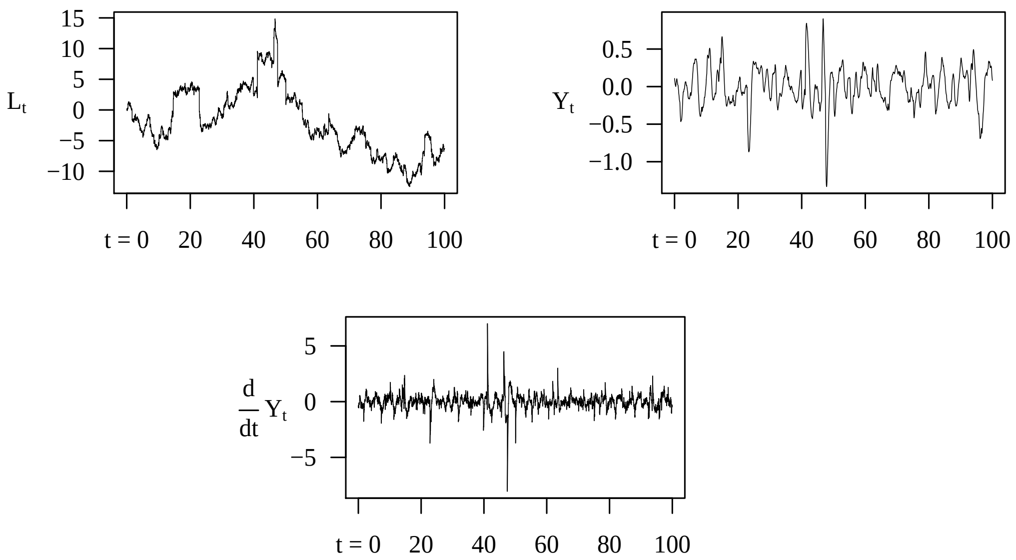

Figure 3.2: Simulations of one path of the driving Lévy process L_t, the CARMA(3,1) process Y_t and its derivative in the α-stable case with $\alpha = 1.5$

with kernel function $f : \mathbb{R} \mapsto M_d$ given by

$$f(t) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{iut} P(iu)^{-1} Q(iu) du.$$

Obviously, MCARMA processes are MMA processes and thus we can apply Lemma 3.6 to obtain an α-stable MCARMA process by using an α-stable driving Lévy process. Furthermore, by Proposition 3.32 of Marquardt and Stelzer (2007) we know that in the case $p > q+1$ MCARMA processes have continuous sample paths and these are $p-q-1$ times differentiable. This shows that in the case $\alpha \in (0,2)$ and $p > q+1$ we can get heavy tailed MCARMA processes, where the heavy tails come from the jumps of the underlying Lévy process, but the paths of the observed process are continuous and may even be differentiable.
To illustrate this, we simulated several univariate CARMA(3,1) processes. They are given by the autoregressive and moving average polynomials

$$p(z) = z^3 + 4.5z^2 + 6.5z + 3 \quad \textit{and} \quad q(z) = z.$$

The CARMA(3,1) process can then given in its state space representation (see Marquardt and Stelzer (2007), Theorem 3.12)

$$G(t) = \int\limits_{-\infty}^{t} e^{A(t-u)} \beta \, dL_u,$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -6.5 & -4.5 \end{pmatrix} \quad \textit{and} \quad \beta = \begin{pmatrix} 0 \\ 1 \\ -4.5 \end{pmatrix}.$$

This representation has the advantage that it applies also in the multivariate setting and it directly includes the derivatives of the CARMA process, as long as they exist. In our

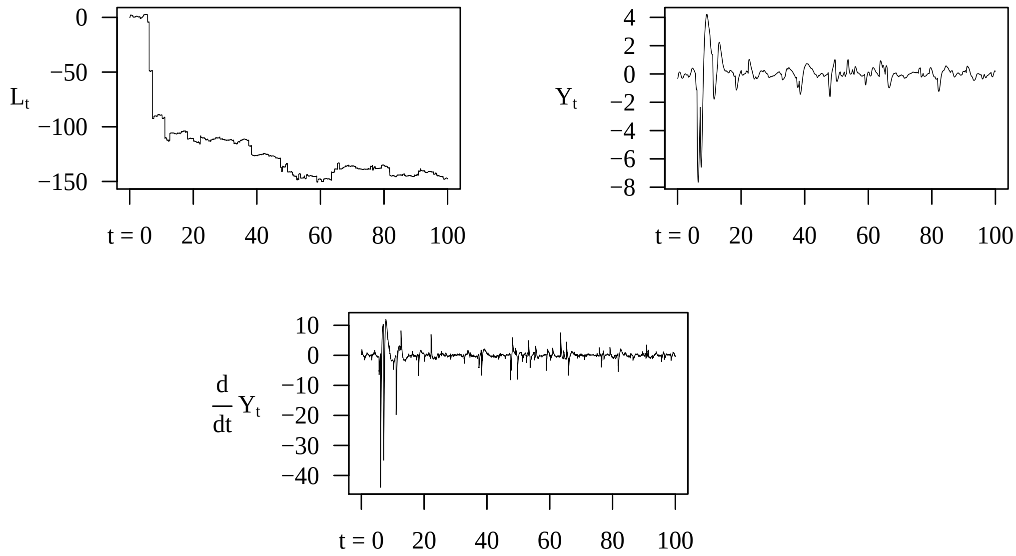

Figure 3.3: Simulations of one path of the driving Lévy process L_t, the CARMA(3,1) process Y_t and its derivative in the α-stable case with $\alpha = 1$

case, we have $Y_t = G_1(t)$ and $\frac{d}{dt}Y_t = G_2(t)$. Due to our foregoing results G is regularly varying with index α (resp. α-stable), if L is so.
For the driving Lévy process L_t we used a symmetric α-stable Lévy motion without skewness and with α-values of 2 (Brownian Motion), 1.5 and 1 (both heavy-tailed). Furthermore, we plotted the simulated values after a burn-in period of 1000 to ensure stationarity. In all three cases, one can see nicely, how the tail behavior of the driving Lévy process determines the tail behavior of the continuous CARMA(3,1) process.
In Figure 3.1 we see the case $\alpha = 2$, where the integrator is a light tailed Brownian Motion and the resulting CARMA process is also light tailed. In the cases $\alpha = 1.5$ (see Figure 3.2) and $\alpha = 1$ (see Figure 3.3) the driving process is heavy tailed and, as α is decreasing, the process is more and more determined by only a few very large jumps. The respective CARMA process is also heavy tailed and oscillates around the mean except for some large, but continuous shocks. For these two cases we also plotted the first derivatives of the paths of the CARMA process, which are not continuous anymore, but jointly α-stable together with the process itself.

4 Application to supOU Processes

One example of MMA processes are superpositions of Ornstein-Uhlenbeck processes, or supOU processes for short. They are especially useful in modeling the stochastic volatility in continuous time models or long range dependent time series. For an introduction to univariate supOU processes see Barndorff-Nielsen (2001) and for the extension to multivariate supOU processes we refer to Barndorff-Nielsen and Stelzer (2011a).

Definition 4.1 ($\mathbb{R}^d$-valued supOU process). *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on*

$M_d^- \times \mathbb{R}$. *If the process*

$$X_t := \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)} \Lambda(dA, ds)$$

exists for all $t \in \mathbb{R}$*, it is called* $\mathbb{R}^d$-valued supOU process*.*

We easily see that supOU processes are MMA processes with special kernel function

$$f(A, s) = e^{As} \, \mathbb{1}_{[0,\infty)}(s).$$

Consequently, existence of supOU processes is covered by Theorem 2.5. But if we take the special properties of supOU processes into account, some more accessible sufficient conditions for the existence can be given.

Theorem 4.2 (Barndorff-Nielsen and Stelzer (2011a), Theorem 3.1). *Let* X_t *be an* $\mathbb{R}^d$*-valued supOU process as defined in Definition 4.1. If*

$$\int_{\|x\|>1} \ln(\|x\|)\nu(dx) < \infty$$

and there exist measurable functions $\rho : M_d^- \mapsto \mathbb{R}^+\backslash\{0\}$ *and* $\kappa : M_d^- \mapsto [1, \infty)$ *such that*

$$\left\|e^{As}\right\| \leq \kappa(A)e^{-\rho(A)s} \; \forall s \in \mathbb{R}^+ \; \pi\text{-almost surely and} \int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)}\pi(dA) < \infty,$$

then the supOU process $X_t = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)}\Lambda(dA, ds)$ *is well defined for all* $t \in \mathbb{R}$ *and stationary. Furthermore, the stationary distribution of* X_t *is infinitely divisible with characteristic triplet* $(\gamma_X, \Sigma_X, \nu_X)$ *given by Theorem 2.5.*

Now we want to go one step further and analyze the tail behavior, but regular variation of the supOU processes follows directly from Theorem 3.2.

Corollary 4.3. *Let* $\Lambda \in \mathbb{R}^d$ *be a Lévy basis on* $M_d^- \times \mathbb{R}$ *with generating quadruple* $(\gamma, \Sigma, \nu, \pi)$ *and let* $\nu \in RV(\alpha, l, \mu_\nu)$*. If the conditions of Theorem 4.2 hold and additionally*

$$\int_{M_d^-} \frac{\kappa(A)^\alpha}{\rho(A)} \, \pi(dA) < \infty,$$

then $X_0 = \int_{M_d^-} \int_{\mathbb{R}^+} e^{As}\Lambda(dA, ds) \in RV(\alpha, l, \mu_X)$ *with Radon measure*

$$\mu_X(\cdot) := \int_{M_d^-} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \mathbb{1}_{(\cdot)} \left(e^{As}x\right) \mu_\nu(dx)ds\pi(dA).$$

Proof. Using the given conditions, we have

$$\begin{aligned}\int_{M_d^-} \int_{\mathbb{R}^+} \left\|e^{As}\right\|^\alpha ds\pi(dA) &\leq \int_{M_d^-} \int_{\mathbb{R}^+} \kappa(A)^\alpha e^{-\alpha\rho(A)s} ds\pi(dA) \\ &= \alpha^{-1} \int_{M_d^-} \frac{\kappa(A)^\alpha}{\rho(A)}\pi(dA) < \infty\end{aligned}$$

and thus $e^{As} \in \mathbb{L}^\alpha(\lambda \times \pi)$. It is left to show that $\mu_\nu(f^{-1}(A,s)\mathbb{R}^n) = 0$ does not hold for $\pi \times \lambda$ almost-every (A,s), but since

$$\mu_\nu\left(e^{-As}\,\mathbb{R}^d\right) = \mu_\nu\left(\mathbb{R}^d\right)$$

for any (A,s), this follows simply from μ_ν being a nonzero measure.

□

For illustration, let us now calculate the measures μ_X of regular variation in some special cases.

Example 4.4 (Measure of regular variation of OU Processes). *SupOU processes with probability measure π being a one-point measure (i.e. $\pi(A) = 1$ for some $A \in M_d^-$) are called Ornstein-Uhlenbeck (OU) processes and their measure of regular variation is given by*

$$\mu_X(B) := \int\limits_{\mathbb{R}^+} \mu_\nu\left(e^{-As}B\right) ds.$$

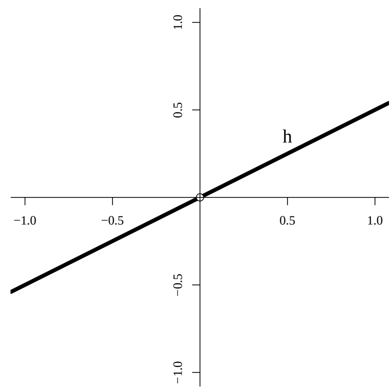

Figure 4.1: Mass of the measures μ_ν and μ_X in case *1*.

We consider several examples in the case $d = 2$. Let us first assume that the mass of the measure μ_ν is concentrated on a straight line, i.e. on the points of the form $h = (a(1,b)^T)_{a\in\mathbb{R}\setminus\{0\}}$ for $b \in \mathbb{R}$, see Figure 4.1 for an example with $b = 0.5$.

1. *If $A = c\,I_d, c \in \mathbb{R}^-$, is a multiple of the identity matrix, then*

$$\mu_X(B) = \int\limits_{\mathbb{R}^+} \mu_\nu\left(e^{-cs}B\right) ds = \int\limits_{\mathbb{R}^+} e^{c\alpha s}\mu_\nu\left(B\right) ds = -\frac{\mu_\nu(B)}{c\,\alpha}.$$

   Consequently, μ_X has its mass in the same directions as μ_ν and thus its mass is also concentrated on h.

2. *If $A = diag(a_1, a_2)$ is a diagonal matrix, then the mass of μ_X is concentrated on the cones between the straight line h and one of the two axes, see Figure 4.2. The mass is drawn to the horizontal axis, if $a_2 > a_1$, and to the vertical axis, if $a_1 > a_2$ (i.e. to the axis associated with the slower exponential decay rate). Intuitively this happens as follows. An extreme jump $(x_1,x_2)^T$ occurred at some time u in the past and had direction s. This causes an extreme value $(e^{d_1(t-u)}x_1, e^{d_2(t-u)}x_2)^T$ at a later time t. Since one of the components decays slower, this extreme event is now in a direction closer to the direction with the slowest exponential decay.*

3. If A is real diagonalizable, i.e. $A = UDU^{-1}$ with $D = diag(d_1, d_2)$, then the mass is drawn to the eigenspace e that belongs to the biggest eigenvalue $\max(d_1, d_2)$. This means that the mass is concentrated on the cone between e and h, see Figure 4.3. This follows immediately by a change of the basis from the last case.

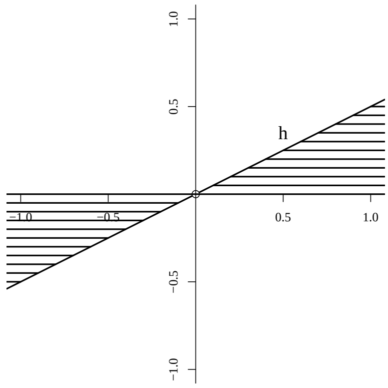

Figure 4.2: Mass of the measure μ_X in case *2*.

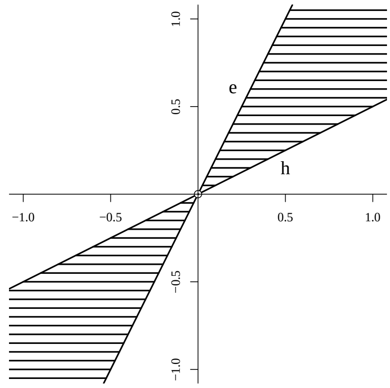

Figure 4.3: Mass of the measure μ_X in case *3*.

However, if the support of μ_ν is the whole space $\mathbb{R}^d$, then the support of μ_X is also $\mathbb{R}^d$, regardless of the choice of A in any of the three cases above.

Like in the general MMA case, we shall again have a closer look at the essential condition

$$\int_{M_d^-} \frac{\kappa(A)^\alpha}{\rho(A)} \pi(dA) < \infty.$$

Using the modulus of injectivity, we can derive necessary conditions similar to the previous chapter, see also Barndorff-Nielsen and Stelzer (2011a), Proposition 3.3, where comparable necessary conditions are given for the existence of supOU processes.

Corollary 4.5. *Let $\Lambda \in \mathbb{R}^d$ be a Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$, let $\nu \in RV(\alpha, l, \mu_\nu)$ and let $X_0 = \int_{M_d^-} \int_{\mathbb{R}^+} e^{As} \Lambda(dA, ds)$ exist following Theorem 4.2. Furthermore, assume there exist measurable functions $\tau : M_d^- \mapsto \mathbb{R}^+ \backslash \{0\}$ and $\vartheta : M_d^- \mapsto [1, \infty)$ such that*

$$j\left(e^{As}\right) \geq \vartheta(A) e^{-\tau(A)s} \; \forall s \in \mathbb{R}^+ \; \pi\text{-almost surely.}$$

Then

$$\int\limits_{M_d^-} \frac{\vartheta(A)^\alpha}{\tau(A)} \pi(dA) < \infty$$

is a necessary condition for $X_0 \in RV(\alpha, l, \mu_X)$ with

$$\mu_X(\cdot) := \int\limits_{M_d^-} \int\limits_{\mathbb{R}^+} \int\limits_{\mathbb{R}^d} \mathbb{1}_{(\cdot)}\left(e^{As}x\right) \mu_\nu(dx) ds \pi(dA).$$

Proof. Suppose

$$\int_{M_d^-} \frac{\vartheta(A)^\alpha}{\tau(A)}\,\pi(dA) = \infty.$$

Then

$$\begin{aligned}\int_{M_d^-}\int_{\mathbb{R}+} j\left(e^{As}\right)^\alpha ds\pi(dA) &\geq \int_{M_d^-}\int_{\mathbb{R}+} \vartheta(A)^\alpha e^{-\alpha\tau(A)s} ds\pi(dA)\\ &= \alpha^{-1}\int_{M_d^-}\frac{\vartheta(A)^\alpha}{\tau(A)}\pi(dA) = \infty.\end{aligned}$$

Consequently $e^{As} \notin \mathbb{J}^\alpha(\lambda \times \pi)$ and Theorem 3.4 yields the result.

□

Finally, as a consequence of Theorem 3.5, we also have regular variation of the process.

Corollary 4.6. *Given the conditions of Corollary 4.3, the supOU process* $(X_t)_{t\in\mathbb{R}}$ *is also regularly varying with index* α *as a process.*

5 Stochastic Volatility Model

5.1 The Model

In this section we review and analyze the supOU type stochastic volatility model introduced in Barndorff-Nielsen and Stelzer (2011b). We consider a d-dimensional logarithmic stock price process $(X_t)_{t\in\mathbb{R}}$ given by an equation of the form

$$\begin{aligned} dX_t &= \Sigma_t^{1/2} dW_t \\ X_0 &= 0, \end{aligned} \tag{5.1}$$

where W is a d-dimensional Brownian motion and $\Sigma^{1/2}$ denotes the unique positive semidefinite square root. The stochastic volatility process $(\Sigma_t)_{t\in\mathbb{R}}$ is given by an $\mathbb{S}_d^+$-valued supOU process that is independent of the Brownian Motion W.

Definition 5.1 (Positive semi-definite supOU process). *Let* Λ *be a Lévy basis on* $M_d^- \times \mathbb{R}$ *with values in* $\mathbb{S}_d$*. If the process*

$$\Sigma_t := \int_{M_d^-}\int_{-\infty}^{t} e^{A(t-s)}\Lambda(dA, ds)e^{A^T(t-s)}$$

exists for all $t \in \mathbb{R}$*, it is called a* positive semi-definite (or $\mathbb{S}_d^+$-valued) supOU process.

The process $(X_t)_{t\in\mathbb{R}^+}$ being given by equation (5.1) with volatility process $(\Sigma_t)_{t\in\mathbb{R}}$ given by a positive semi-definite supOU process is called *multivariate supOU type stochastic volatility model* or *SVsupOU*.
The introduced model is of course only the most basic version of a SVsupOU. We can easily enhance the model by adding a stochastic or deterministic drift a and a leverage term Ψ, see Barndorff-Nielsen and Stelzer (2011b) for details. The model is then given by the equation

$$dX_t = a_t dt + \Sigma_{t-}^{1/2} dW_t + \Psi(dL_t)$$

$$X_0 = 0,$$

where a is an $\mathbb{R}^d$-valued predictable process, W is the d-dimensional Brownian motion, L is the Lévy process associated with Λ and $\Psi : \mathbb{S}_d \mapsto \mathbb{R}^d$ is a linear operator. The stochastic volatility process $(\Sigma_t)_{t\in\mathbb{R}}$ is again a matrix-valued supOU process.
However, the drift term and the leverage term are usually dominating the tail behavior, if they are non-vanishing. The leverage term is determined by the behavior of the Lévy process L and as we always assume the driving Lévy measure to be regularly varying with index α, the leverage term is also regularly varying with index α. A popular choice for the drift term is

$$a_t = \mu + \beta\Sigma_t$$

with $\beta : \mathbb{S}_d \mapsto \mathbb{R}^d$ being a linear operator and in this case a_t is regularly varying with index α, as we will show below. This means that if such a drift or leverage term exists, they dominate the Brownian term, which will turn out to be regularly varying with index 2α. For this reason, we will only consider the simple model in this chapter.
Let us start with analyzing the volatility process. Existence of the positive semi-definite supOU processes is given similarly to the existence of $\mathbb{R}^d$-valued supOU processes.

Theorem 5.2 (Barndorff-Nielsen and Stelzer (2011a), Theorem. 4.1). *Let Λ be an $\mathbb{S}_d$-valued Lévy basis with generating quadruple $(\gamma, 0, \nu, \pi)$ and with $\gamma_0 := \gamma - \int_{\|x\|\le 1} x\nu(dx) \in \mathbb{S}_d^+$, $\nu(\mathbb{S}_d \backslash \mathbb{S}_d^+) = 0$,*

$$\int_{\|x\|>1} \ln(\|x\|)\,\nu(dx) < \infty \quad \text{and} \quad \int_{\|x\|\le 1} \|x\|\nu(dx) < \infty.$$

Furthermore, assume the existence of measurable functions $\rho : M_d^- \mapsto \mathbb{R}^+$ and $\kappa : M_d^- \mapsto [1, \infty)$ such that

$$\left\|e^{As}\right\| \le \kappa(A)e^{-\rho(A)s}\ \forall s \in \mathbb{R}^+\ \pi\text{-almost surely and} \int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)}\pi(dA) < \infty.$$

Then the positive-semidefinite supOU process

$$\Sigma_t = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)}\Lambda(dA, ds)e^{A^T(t-s)}$$

is well-defined for all $t \in \mathbb{R}$, has values in $\mathbb{S}_d^+$ for all $t \in \mathbb{R}$ and its distribution is stationary and infinitely divisible. Moreover, the vector representation has the form

$$vec\,(\Sigma_t) = \int_{M_d^-} \int_{-\infty}^{t} e^{(A\otimes I_d + I_d\otimes A)(t-s)} vec(\Lambda)(dA, ds).$$

Note that in the above theorem $vec(\Lambda)$ is defined by $vec(\Lambda)(A) := vec((\Lambda(A))$ and it is a Lévy basis in $\mathbb{R}^{d^2}$.
Based on this theorem, we can now analyze the tail behavior of the volatility process. Therefore, we have to define regular variation in a matrix-valued setting, which is just a translation of $\mathbb{R}^d$-valued regular variation. A random matrix $X \in M_d$ is said to be regularly varying with index $\alpha > 0$, if there exists a slowly varying function $l : \mathbb{R} \mapsto \mathbb{R}$

and a nonzero Radon measure μ defined on $\mathcal{B}(\overline{M}_d \backslash \{0\})$ with $\mu(\overline{M}_d \backslash M_d) = 0$ such that, as $u \to \infty$,

$$u^\alpha l(u) P(u^{-1} X \in \cdot) \xrightarrow{v} \mu(\cdot)$$

on $\mathcal{B}(\overline{M}_d \backslash \{0\})$ and we write $X \in RV(\alpha, l, \mu)$. Of course, for a random matrix $X \in M_d$ there exists the straightforward connection that $X \in RV(\alpha, l, \mu)$ if and only if $vec(X) \in RV(\alpha, l, \mu^v)$, where $\mu^v(vec(A)) = \mu(A)$. Given this relationship, we can then analyze the tail behavior of the volatility process, where regular variation can be derived using the results of the previous chapters.

Corollary 5.3. *Let $\Lambda \in \mathbb{S}_d$ be a Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, 0, \nu, \pi)$ and let $\nu \in RV(\alpha, l, \mu_\nu)$. If the conditions of Theorem 5.2 hold and additionally*

$$\int_{M_d^-} \frac{\kappa(A)^{2\alpha}}{\rho(A)} \pi(dA) < \infty,$$

then $\Sigma_0 = \int_{M_d^-} \int_{\mathbb{R}^+} e^{As} \Lambda(dA, ds) e^{A^T s} \in RV(\alpha, l, \mu_\Sigma)$ with Radon measure

$$\mu_\Sigma(\cdot) := \int_{M_d^-} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \mathbb{1}_{(\cdot)} \left(e^{As} x e^{A^T s} \right) \mu_\nu(dx) ds \pi(dA).$$

Furthermore, the supOU process $(\Sigma_t)_{t \in \mathbb{R}}$ is also regularly varying with index α as a process.

Proof. From Theorem 5.2 we have

$$vec(\Sigma_t) = \int_{M_d^-} \int_{-\infty}^{t} e^{(A \otimes I_d + I_d \otimes A)(t-s)} vec(\Lambda)(dA, ds)$$

and thus the vectorized volatility process $vec(\Sigma_t)$ is an MMA process with kernel function $f(A, s) = e^{(A \otimes I_d + I_d \otimes A)s} \mathbb{1}_{[0,\infty)}(s)$. In order to apply Theorem 3.2, it is left to show that $f \in \mathbb{L}^\alpha(\lambda \times \pi)$. For that purpose, we make use of the relations $e^{(A \otimes I_d + I_d \otimes A)s} = e^{As} \otimes e^{As}$ and $\|e^{As} \otimes e^{As}\| = \|e^{As}\|^2$ (see Horn and Johnson (1991), Chapter 4.2, Problem 28, and Chapter 6.2, Problem 14) and obtain

$$\begin{aligned}
\int_{M_d^-} \int_{\mathbb{R}^+} \|e^{(A \otimes I_d + I_d \otimes A)s}\|^\alpha ds \pi(dA) &\le \int_{M_d^-} \int_{\mathbb{R}^+} \|e^{As}\|^{2\alpha} ds \pi(dA) \\
&\le \int_{M_d^-} \int_{\mathbb{R}^+} \kappa(A)^{2\alpha} e^{-2\alpha\rho(A)s} ds \pi(dA) \\
&= \frac{1}{2\alpha} \int_{M_d^-} \frac{\kappa(A)^{2\alpha}}{\rho(A)} \pi(dA) < \infty.
\end{aligned}$$

□

Now we want to go one step further and analyze the tail behavior of the logarithmic stock price process. Therefore, we use the independence between W and Λ, which yields the equality

$$\int_0^t \Sigma_s^{1/2} dW_s \stackrel{d}{=} \left(\int_0^t \Sigma_s ds \right)^{1/2} W_t. \tag{5.2}$$

We immediately see that it is necessary to analyze the integrated volatility

$$\Sigma_t^+ := \int_0^t \Sigma_s ds$$

in order to obtain regular variation of the stock price process. We start with its existence.

Theorem 5.4 (Barndorff-Nielsen and Stelzer (2011a), Theorem 4.3). *Let Σ be a positive semi-definite supOU process as given in Definition 5.1 that exists according to Theorem 5.2. Then $\Sigma_t(\omega)$ is measurable as a function of $t \in \mathbb{R}$ and $\omega \in \Omega$. If also*

$$\int_{M_d^-} \kappa(A)^2 \pi(dA) < \infty,$$

then the paths of Σ are uniformly bounded in t and the integrated process Σ_t^+ exists for all $t \in \mathbb{R}^+$.

Another important and closely related characteristic of a time series are observed log returns over given time periods of length $\Delta \in \mathbb{R}^+$ (representing for example observation intervals, trading periods etc.) given by

$$Z_n := X_{n\Delta} - X_{(n-1)\Delta} = \int_{(n-1)\Delta}^{n\Delta} \Sigma_s^{1/2} dW_s \overset{d}{=} \left(\int_{(n-1)\Delta}^{n\Delta} \Sigma_s ds \right)^{1/2} W_\Delta. \tag{5.3}$$

Existence of the related integrated volatilities

$$\Sigma_n^+ := \int_{(n-1)\Delta}^{n\Delta} \Sigma_s ds$$

is given by the previous theorem and conditions for regular variation of Σ_n^+ and of the integrated volatility Σ_t^+ can be derived simultaneously.

Corollary 5.5. *Let $\Lambda \in \mathbb{S}_d$ be a Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, 0, \nu, \pi)$ and let $\nu \in RV(\alpha, l, \mu_\nu)$. If the conditions of Theorem 5.4 hold and additionally*

$$\int_{M_d^-} \frac{\kappa(A)^{2\alpha}}{\rho(A)^{\alpha+1}} \pi(dA) < \infty,$$

then $\Sigma_n^+ \in RV(\alpha, l, \mu_{\Sigma_n^+})$ with Radon measure

$$\mu_{\Sigma_n^+}(B) := \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_B \left(\int_{u \vee (n-1)\Delta}^{n\Delta} e^{A(s-u)} x e^{A^T(s-u)} \mathbb{1}_{(-\infty, n\Delta]}(u) ds \right) \mu_\nu(dx) du \pi(dA)$$

and $\Sigma_t^+ \in RV(\alpha, l, \mu_{\Sigma_t^+})$ with

$$\mu_{\Sigma_t^+}(B) := \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_B \left(\int_{u \vee 0}^{t} e^{A(s-u)} x e^{A^T(s-u)} \mathbb{1}_{(-\infty, t]}(u) ds \right) \mu_\nu(dx) du \pi(dA).$$

Furthermore, the process $(\Sigma_t^+)_{t \in \mathbb{R}^+}$ is also regularly varying with index α as a process.

Proof. Again, we use the vector representation of the process and get from the proof of Theorem 3.12 in Barndorff-Nielsen and Stelzer (2011a)

$$\begin{aligned} vec\left(\Sigma_n^+\right) &= \int_{(n-1)\Delta}^{n\Delta} \int_{M_d^-} \int_{-\infty}^{s} e^{(A\otimes I_d + I_d \otimes A)(s-u)} vec(\Lambda)(dA, du) ds \\ &= \int_{\mathbb{R}} \int_{M_d^-} g(A,t,u) vec(\Lambda)(dA, du) \end{aligned}$$

with

$$g(A,t,u) := \int_{u\vee(n-1)\Delta}^{n\Delta} e^{(A\otimes I_d + I_d\otimes A)(s-u)} \mathbb{1}_{(-\infty,n\Delta]}(u) ds.$$

As before, it is left to show that $g(A,t,u) \in \mathbb{L}^{\alpha}(\lambda \times \pi)$ in order to apply Theorem 3.2. Therefore, we estimate

$$\begin{aligned} \|g(A,t,u)\| &\le \mathbb{1}_{(-\infty,n\Delta]}(u) \int_{u\vee(n-1)\Delta}^{n\Delta} \left\| e^{(A\otimes I_d + I_d\otimes A)(s-u)} \right\| ds \\ &\le \mathbb{1}_{(-\infty,n\Delta]}(u) \int_{u\vee(n-1)\Delta}^{n\Delta} \kappa(A)^2 e^{-2\rho(A)(s-u)} ds \\ &= \frac{\kappa(A)^2}{-2\rho(A)} \Big(e^{-2\rho(A)(n\Delta-u)} \mathbb{1}_{(-\infty,n\Delta]}(u) - \mathbb{1}_{((n-1)\Delta,n\Delta)}(u) \\ &\qquad - e^{-2\rho(A)((n-1)\Delta-u)} \mathbb{1}_{(-\infty,(n-1)\Delta]}(u) \Big). \end{aligned}$$

For the first term of the sum we get

$$\int_{M_d^-} \int_{-\infty}^{n\Delta} \left| \frac{\kappa(A)^2 e^{-2\rho(A)(n\Delta-u)}}{-2\rho(A)} \right|^{\alpha} du\, \pi(dA) = \frac{1}{2^{\alpha+1}\alpha} \int_{M_d^-} \frac{\kappa(A)^{2\alpha}}{\rho(A)^{\alpha+1}} \pi(dA) < \infty.$$

The second summand is in $\mathbb{L}^{\alpha}(\lambda \times \pi)$, since the function has bounded support, and for the last term in the sum we simply substitute n by $(n-1)$ in the first term. The result for Σ_t^+ follows directly setting $\Delta = t$ and $n = 1$.

□

Note that this result also gives us regular variation with index α of a possible drift term $a_t = \mu + \beta\Sigma_t$.
The next step is to derive the tail behavior of the square root $(\Sigma^+)^{1/2}$ of the integrated volatility process.

Lemma 5.6. *Let Σ be a random variable with values in $\mathbb{S}_d^+$ and let $\Sigma^{1/2}$ be its square root. Then $\Sigma \in RV(\alpha, l, \mu_\Sigma)$, if and only if $\Sigma^{1/2} \in RV(2\alpha, l^{1/2}, \mu_\Sigma^{1/2})$ with $l^{1/2}(x) := l(x^2)$ and $\mu_\Sigma^{1/2}(B) := \mu(B^2)$.*

Proof. Note that the square root of a matrix in $\mathbb{S}_d^+$ is a bijective mapping and is thus well defined. Since both functions, the square as well as the square root, map compacts to compacts, we can apply Proposition 3.18 of Resnick (1987).

□

Now we can consider the log-returns and the logarithmic stock price process regarding their tail behavior.

Theorem 5.7. *Let $(X_t)_{t\in\mathbb{R}}$ be the stock price process given by equation* (5.1)*, let Z_n be the log-returns given by* (5.3) *and let Σ_n^+ be the increments of a positive semi-definite supOU process $(\Sigma_t)_{t\in\mathbb{R}^+}$. Furthermore, let the conditions of Corollary 5.5 hold. Then $Z_n \in RV(2\alpha, l^{1/2}, \mu_Z)$ with Radon measure*

$$\mu_Z(B) := \mathbb{E}\left(\mu_{\Sigma_n^+}^{1/2}\left(W_\Delta^{-1}(B)\right)\right)$$

and $X_t \in RV(2\alpha, l^{1/2}, \mu_X)$ with

$$\mu_X(B) := \mathbb{E}\left(\mu_{\Sigma_t^+}^{1/2}\left(W_t^{-1}(B)\right)\right),$$

where $W_\Delta : M_d^- \mapsto \mathbb{R}^d$ is considered to be a random linear mapping with $W_\Delta(x) := x \cdot W_\Delta \stackrel{d}{=} \Delta x N(0, I_d)$ (likewise for W_t). Furthermore, $(X_t)_{t\in\mathbb{R}}$ is also regular varying with index 2α as a process.

Proof. Since W and Λ are independent, we have

$$Z_n \stackrel{d}{=} \left(\int_{(n-1)\Delta}^{n\Delta} \Sigma_s ds\right)^{1/2} W_\Delta \quad \text{and} \quad \int_0^t \Sigma_s^{1/2} dW_s \stackrel{d}{=} \left(\int_0^t \Sigma_s ds\right)^{1/2} W_t.$$

From Corollary 5.5 we know that

$$\Sigma_n^+ \in RV(\alpha, l, \mu_{\Sigma_n^+}) \quad \text{and} \quad \Sigma_t^+ \in RV(\alpha, l, \mu_{\Sigma_t^+})$$

and, together with Lemma 5.6, this yields

$$(\Sigma_n^+)^{1/2} \in RV(2\alpha, l^{1/2}, \mu_{\Sigma_n^+}^{1/2}) \quad \text{and} \quad (\Sigma_t^+)^{1/2} \in RV(2\alpha, l^{1/2}, \mu_{\Sigma_t^+}^{1/2}).$$

Finally, we use the fact that the Brownian Motion has finite moments to apply the multivariate version of Breiman's Lemma (see Basrak et al. (2002), Proposition A.1), which yields the result.

□

5.2 Relevance and Applications in Finance

Let us conclude with some final remarks. First, we easily see that the model allows for heavy tails, in the volatility as well as in logarithmic stock prices and log-returns. This is a useful fact, since observed marked data often shows heavy tails. Furthermore, we see that there is a direct connection between the indexes of regular variation of the driving Lévy measure on the one hand and the volatility, log-prices and log-returns on the other hand. We can also calculate the concrete measure μ of regular variation in order to describe the spatial structure of the extremes.
Second, we note that all the results given above hold also in the case of an Ornstein-Uhlenbeck type stochastic volatility model, where the volatility is modeled by an $\mathbb{S}_d^+$-valued OU process. This is obvious, since OU processes are special cases of supOU processes with π being a Dirac measure.
In a financial context, the results can now be used for a statistical analysis of observed data. We can use one of the well established estimators (see Embrechts et al. (1997) or

Resnick (2007)) to estimate the index of regular variation of the given data (logarithmic stock prices or log-returns). The result can then be compared with the estimated index of regular variation of the integrated volatility. If they do not match by the factor of 2, this is a hint for the existence of a leverage or drift term of the form specified in this paper. Yet, there is still some future work to be done to analyze and estimate the (index of regular variation of the) integrated volatility, since it cannot be observed directly. If we make additional assumptions on the different terms (leverage, drift) to exist or not, we can calculate the index of regular variation of the log-prices or log-returns from the index of the volatility and vice versa.
It would also be very interesting to generalize the stochastic volatility model by substituting the Brownian motion W_t by a more general Lévy process $\widetilde{L}_t$. However, as there is then no analogue of (5.2) available, it will be much more difficult to get results for this case and different methods will be needed.

Modelling the correlation breakdown

Applied research in financial mathematics and econometrics has often noted that one typically encounters what has been dubbed "correlation breakdown" in times of severe crisis. This notion means that when extreme negative events potentially affecting the whole (or large parts of the) economy occur, basically all traded stocks are loosing tremendously in value simultaneously and the correlations between them are seemingly more or less one. Moreover, after such an event the variances are typically extremely high. Models employed in financial institutions (for risk management) clearly need to include this feature in order to be realistic and provide accurate predictions.
Our results on the (sup)OU stochastic volatility model allow us to understand how to incorporate such effects into the model. Clearly, the most extreme movements in crisis can typically not come from the Brownian term but have to come from jumps. Hence, Ψ needs to be chosen non-zero and such that when all variances have a jump upwards all prices have jumps downwards. Assume now that the positive semidefinite Lévy basis is regularly varying with index α and the measure μ_ν is concentrated on the rank one matrices with all diagonal elements being non-zero and all correlations being 1. The extremes of Σ are now caused by a single big jump which will be such that it is (almost) a rank one matrix with all diagonal elements being non-zero. After the occurrence of the jump the process Σ will be almost equal to the value of this jump and hence all correlations will be very close to one for quite some time afterwards. Moreover, by the choice of parameters the prices will simultaneously have a huge jump downward. Clearly, this would model the "correlation breakdown". Our results actually show that Σ would be regularly varying with index α and μ_Σ would be concentrated on the rank one matrices with all diagonal elements being non-zero as this class of matrices is preserved by the mappings $X \mapsto e^{As} X e^{A^T s}$ for all $A \in M_d(\mathbb{R})$ and $s \in \mathbb{R}^+$. Likewise the log prices would be regularly varying with index α (unless we had a drift with heavier tails which seems not reasonable) and the measure of regular variation for the log prices follows easily, because what matters is only the linear transformation Ψ of the driving positive semidefinite Lévy basis. Note that the measure of regular variation of the log prices will in general still have a completely non-degenerate support (in the positive d-dimensional cone).
In practice the above explained model can only form an important building block of a realistic and suitable model, since not all extreme events affect the whole economy, some only affect individual sectors of industry or companies. However, also for such extensions (which basically demand regular variation of Λ with appropriate μ_Λ) our results provide the necessary insight into the resulting tail behavior.

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