# FUNCTIONAL REGULAR VARIATION OF LÉVY-DRIVEN MULTIVARIATE MIXED MOVING AVERAGE PROCESSES

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ABSTRACT. We consider the functional regular variation in the space $\mathbb{D}$ of càdlàg functions of multivariate mixed moving average (MMA) processes of the type $X_t = \int\int f(A, t-s)\Lambda(dA, ds)$. We give sufficient conditions for an MMA process $(X_t)$ to have càdlàg sample paths. As our main result, we prove that $(X_t)$ is regularly varying in $\mathbb{D}$ if the driving Lévy basis is regularly varying and the kernel function $f$ satisfies certain natural (continuity) conditions. Finally, the special case of supOU processes, which are used, e.g., in applications in finance, is considered in detail.

## 1. Introduction

In many applications of stochastic processes, the center of the distributions involved and related quantities (e.g. sample means, variances etc.) can be modeled quite well. In view of the central limit theorem, Gaussian distributions play an important role in that field. However, this needs not to be true for the tail of the distribution which is of great importance in many areas of application. Possible examples are severe crises in stock markets or extreme weather events which can cause huge losses to the financial industry, insurance companies and also to private people. Therefore, it is of great importance to model the distribution tail and related quantities (e.g. quantiles, exceedances, maxima etc.) correctly.

A very well established concept to model extreme values is regular variation. It has its origin in classical extreme value theory, where limit distributions for sample maxima are derived. The maximum domains of attraction of two of the three possible standard extreme value distributions (Fréchet and Weibull) can be described by regular variation of functions, meaning functions behaving like a power law in the limit, see also [19] and [39].

Moreover, regular variation can intuitively be extended to a multivariate setup. It is then formulated in terms of vague convergence of measures given by

$$nP(a_n X \in \cdot) \xrightarrow{v} \mu(\cdot), \tag{1.1}$$

where $X$ is a multivariate random vector, $(a_n)$ an increasing sequence and $\mu$ is a Radon measure. Since $\mu$ is homogeneous, multivariate regular variation of $X$ can be interpreted as convergence of the radial part $\|X\|$ to a univariate regularly varying random variable $Y$ and

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of the spherical part $X/\|X\|$ to a random variable $Z$ on the unit sphere, which is independent of $Y$ and can be described by the measure $\mu$. Detailed introductions to multivariate regular variation can be found in [39] and [26].

Finally, [24] extended the definition (1.1) to the space of multivariate stochastic processes with sample paths in the space $\mathbb{D}$ of càdlàg functions, i.e. right-continuous functions with limits from the left. The formulation of regular variation in such generality has the advantage that, in addition to functionals based on the values of a stochastic process at fixed time points, one can also analyze functionals acting on the complete sample paths of the process. This is a very powerful tool for the analysis of extremal properties of a process, especially in combination with methods for weak convergence of point processes which are closely related to regular variation (see Section 6). Despite the power of this technique, conditions ensuring regular variation in $\mathbb{D}$ have so far been given only for few classes of processes.

In this paper we apply the concept of regular variation on $\mathbb{D}$ to multivariate mixed moving average (MMA) processes with càdlàg sample paths. MMA processes have been first introduced by [45] in the univariate stable case and are given as integrals of the form

$$X_t = \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s) \Lambda(dA, ds),$$

where $\Lambda$ is a multivariate Lévy basis. The class of (multivariate) MMA processes covers a wide range of processes which are well known and extensively used in applications. Examples include Ornstein-Uhlenbeck processes (cf. [2] and [36]), superpositions of Ornstein-Uhlenbeck (supOU) processes (cf. [1] and [3]), (fractionally integrated) CARMA processes (cf. [12], [32]) and increments of fractional Lévy processes (cf. [31], [7] and references therein).

Regular variation of the finite-dimensional distributions of MMA processes has already been proved in [34], given that the underlying Lévy basis is regularly varying and the kernel function satisfies the integrability condition $f \in \mathbb{L}^\alpha$. In this paper we give additional integrability and continuity conditions on the kernel function $f$ such that the MMA process is functionally regularly varying on $\mathbb{D}$. Furthermore, we also analyze the special case of multivariate supOU processes given by

$$X_t = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)} \Lambda(dA, ds).$$

The paper is organized as follows. In Section 2.2 we introduce the notion of multivariate regular variation and related properties. In Section 2.3 we recall the definition of MMA processes and the related integration theory. Furthermore, we review the conditions for existence of MMA processes and for regular variation of their finite dimensional distributions. The sample path behavior of MMA processes is discussed in Section 3. We give an overview over the relevant literature and derive new sufficient conditions for MMA processes to have càdlàg sample paths in the case where the underlying Lévy process is of finite variation. In Section 4 we introduce the notion of functional regular variation and prove that MMA processes are regularly varying on $\mathbb{D}$, given certain conditions. In Section 5 we verify these conditions in the special case of supOU processes. Finally, in Section 6 we show the connection between functional regular variation and point process convergence and discuss the relevance of the results to the extremal analysis of MMA processes.

## 2. Preliminaries

2.1. **Notation.** Let $\mathbb{R}$ be the real numbers, $\mathbb{R}^+$ the positive and $\mathbb{R}^-$ the negative real numbers, both without 0. $\mathbb{N}$ is the set of positive integers and $\mathbb{Q}$ are the rational numbers. The Borel sets are denoted by $\mathcal{B}$ and $\mathcal{B}_b$ are the bounded Borel sets. $\lambda$ is the Lebesgue measure on $\mathbb{R}$ and $B_r(x) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$ is the closed ball of radius $r$ centered at $x$. $\mathbb{D}$ is the space of càdlàg (right-continuous with left limits) functions $x : [0,1] \to \mathbb{R}^d$ and $S_{\mathbb{D}} = \{x \in \mathbb{D} : \sup_{t\in[0,1]} \|x_t\| = 1\}$ is the unit sphere in $\mathbb{D}$.

For matrices, $M_{n,d}$ is the set of all $n \times d$ matrices and $M_d$ the set of all $d \times d$ matrices. $M_d^-$ is the set of all $d \times d$ matrices with all eigenvalues having strictly negative real part. $I_d$ is the $d \times d$ identity matrix. We write $A^T$ for the transposed of a matrix $A$ and $\|A\|$ for the matrix norm induced by the Euclidean norm.

If random variables, vectors, processes, measures etc. are considered, they are given as measurable mappings with respect to a complete probability space $(\Omega, \mathcal{F}, P)$.

Vague convergence is defined in terms of convergence of Radon measures and it is denoted by $\xrightarrow{v}$. It is defined on the one-point uncompactification $\overline{\mathbb{R}}^d\backslash\{0\}$, which assures that the sets bounded away from the origin can be referred to as the relatively compact sets in the vague topology. Similarly, $\hat{w}$-convergence is given by the convergence of boundedly finite measures and is defined on $\overline{\mathbb{D}}_0 = (0, \infty] \times S_{\mathbb{D}}$, which can be viewed as the one-point uncompactification in $\mathbb{D}$.

2.2. **Multivariate Regular Variation.** Regular variation on $\mathbb{R}^d$ is expressed in terms of vague convergence of measures and several different, but equivalent, definitions exist. For detailed and very good introductions to regular variation we refer to [9], [38], [39] and [29].

**Definition 2.1** (Multivariate Regular Variation)**.** A random vector $X \in \mathbb{R}^d$ is *regularly varying* if there exists a sequence $(a_n)_{n\in\mathbb{N}}$, $0 < a_n \nearrow \infty$, and a nonzero Radon measure $\mu$ on $\mathcal{B}(\overline{\mathbb{R}}^d\backslash\{0\})$ such that $\mu(\overline{\mathbb{R}}^d\backslash\mathbb{R}^d) = 0$ and, as $n \to \infty$,

$$nP(a_n^{-1}X \in \cdot) \xrightarrow{v} \mu(\cdot)$$

on $\mathcal{B}(\overline{\mathbb{R}}^d\backslash\{0\})$. Similarly, we call a Radon measure $\nu$ regularly varying if $(a_n)$ and $\mu$ exist as above such that, as $n \to \infty$,

$$n\,\nu(a_n\cdot) \xrightarrow{v} \mu(\cdot).$$

The limiting measure $\mu$ of the definition is homogeneous, i.e. it necessarily satisfies the condition

$$\mu(tB) = t^{-\alpha}\mu(B)$$

for all $B \in \mathcal{B}(\overline{\mathbb{R}}^d\backslash\{0\})$ and $t > 0$. Hence, we write $X \in RV(\alpha, (a_n), \mu)$ or $\nu \in RV(\alpha, (a_n), \mu)$, respectively. In the special case of an infinitely divisible random vector $X \in \mathbb{R}^d$ with Lévy measure $\nu$ we know that $X \in RV(\alpha, (a_n), \mu)$ if and only if $\nu \in RV(\alpha, (a_n), \mu)$ (see [25], Proposition 3.1). This result is very useful throughout this work, since MMA processes are infinitely divisible, just as the driving Lévy bases are. A detailed introduction to infinitely divisible distributions and Lévy processes can be found in [44], for instance.

2.3. **Multivariate Mixed Moving Average Processes.** In this section we shortly recall the definition and main properties of multivariate *mixed moving average processes* (short MMA processes).

A multivariate ($\mathbb{R}^n$-valued) MMA process $(X_t)$ can be defined as an integral over a measurable kernel function $f : M_d^- \times \mathbb{R} \to M_{n,d}$ with respect to an $\mathbb{R}^d$-valued Lévy basis $\Lambda$ on $M_d^- \times \mathbb{R}$, i.e.

$$X_t := \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s)\Lambda(dA, ds).$$

An $\mathbb{R}^d$-valued *Lévy basis* $\Lambda = (\Lambda(B))$ with $B \in \mathcal{B}_b(M_d^- \times \mathbb{R})$ is a random measure which is

- *infinitely divisible*, i.e. the distribution of $\Lambda(B)$ is infinitely divisible for all $B \in \mathcal{B}_b(M_d^- \times \mathbb{R})$,
- *independently scattered*, i.e. for any $n \in \mathbb{N}$ the random variables $\Lambda(B_1), \ldots, \Lambda(B_n)$ are independent for pairwise disjoint sets $B_1, \ldots, B_n \in \mathcal{B}_b(M_d^- \times \mathbb{R})$ and
- *$\sigma$-additive*, i.e. for any pairwise disjoint sets $(B_i)_{i\in\mathbb{N}} \in \mathcal{B}_b(M_d^- \times \mathbb{R})$ with $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}_b(M_d^- \times \mathbb{R})$ we have $\Lambda(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \Lambda(B_n)$ almost surely.

Thus, Lévy bases are also called infinitely divisible independently scattered random measures (i.d.i.s.r.m.). Following the relevant literature (cf. [20], [21], [22], [3] and [34]) we only consider time-homogeneous and factorisable Lévy bases, i.e. Lévy bases with characteristic function

$$\mathbb{E}\left(e^{iu^T\Lambda(B)}\right) = e^{\varphi(u)\Pi(B)} \tag{2.1}$$

for all $u \in \mathbb{R}^d$ and $B \in \mathcal{B}_b(M_d^- \times \mathbb{R})$, where $\Pi = \pi \times \lambda$ is the product of a probability measure $\pi$ on $M_d^-(\mathbb{R})$ and the Lebesgue measure $\lambda$ on $\mathbb{R}$ and

$$\varphi(u) = iu^T\gamma - \frac{1}{2}u^T\Sigma u + \int_{\mathbb{R}^d} \left(e^{iu^Tx} - 1 - iu^Tx\mathbb{1}_{[-1,1]}(\|x\|)\right)\nu(dx)$$

is the characteristic function of an infinitely divisible distribution with characteristic triplet $(\gamma, \Sigma, \nu)$. The distribution of the Lévy basis is then completely determined by $(\gamma, \Sigma, \nu, \pi)$ which is therefore called the *generating quadruple*. By $L$ we denote the *underlying Lévy process* which is given by $L_t = \Lambda(M_d^- \times (0,t])$ and $L_{-t} = \Lambda(M_d^- \times [-t, 0))$ for $t \in \mathbb{R}^+$. For more details on Lévy bases see [37] and [35].

We should stress that the set $M_d^-$ in the definition of MMA processes can be replaced by $M_d$ or basically any other topological space. The choice of $M_d^-$ is motivated by the special case of supOU processes, where this is the canonical choice.

Necessary and sufficient conditions for the existence of MMA processes are given by the multivariate extension of Theorem 2.7 in [37].

**Theorem 2.2.** *Let $\Lambda$ be an $\mathbb{R}^d$-valued Lévy basis with characteristic function of the form* (2.1) *and let $f : M_d^- \times \mathbb{R} \to M_{n,d}$ be a measurable function. Then $f$ is $\Lambda$-integrable as a limit in probability in the sense of* [37] *if and only if*

$$\int_{M_d^-} \int_{\mathbb{R}} \left\| f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x\left(\mathbb{1}_{[0,1]}\left(\|f(A,s)x\|\right) - \mathbb{1}_{[0,1]}\left(\|x\|\right)\right)\nu(dx) \right\| ds\pi(dA) < \infty, \tag{2.2}$$

$$\int_{M_d^-} \int_{\mathbb{R}} \|f(A,s)\Sigma f(A,s)^T\| ds\pi(dA) < \infty \quad \textit{and} \tag{2.3}$$

$$\int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(A,s)x\|^2\right)\nu(dx)ds\pi(dA) < \infty. \tag{2.4}$$

*If $f$ is $\Lambda$-integrable, the distribution of $X_0 = \int_{M_d^-}\int_{\mathbb{R}^+} f(A,s)\Lambda(dA,ds)$ is infinitely divisible with characteristic triplet $(\gamma_{int}, \Sigma_{int}, \nu_{int})$ given by*

$$\gamma_{int} = \int\limits_{M_d^-}\int\limits_{\mathbb{R}} \left( f(A,s)\gamma + \int\limits_{\mathbb{R}^d} f(A,s)x\left(\mathbb{1}_{[0,1]}\left(\|f(A,s)x\|\right) - \mathbb{1}_{[0,1]}\left(\|x\|\right)\right)\nu(dx)\right) ds\pi(dA),$$

$$\Sigma_{int} = \int\limits_{M_d^-}\int\limits_{\mathbb{R}} f(A,s)\Sigma f(A,s)^T ds\pi(dA) \quad \textit{and}$$

$$\nu_{int}(B) = \int\limits_{M_d^-}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^d} \mathbb{1}_B(f(A,s)x)\nu(dx)ds\pi(dA) \quad \textit{for all Borel sets } B \subseteq \mathbb{R}^n.$$

However, since our focus is on regularly varying processes, we recall sufficient conditions derived in [34] which are intrinsically related to the conditions needed to ensure regular variation. To this end we define

$$\mathbb{L}^\delta(\pi\times\lambda) := \left\{ f : M_d^- \times \mathbb{R} \to M_{n,d} \text{ measurable}, \int\limits_{M_d^-}\int\limits_{\mathbb{R}} \|f(A,s)\|^\delta ds\pi(dA) < \infty,\, n\in\mathbb{N}\right\}.$$

**Theorem 2.3** ([34], Theorem 2.6)**.** *Let $\Lambda$ be a Lévy basis with values in $\mathbb{R}^d$ and characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$. Furthermore, let $\nu$ be regularly varying with index $\alpha$ and let $f : M_d^- \times \mathbb{R} \to M_{n,d}$ be a measurable function. Then $f$ is $\Lambda$-integrable in the sense of* [37] *and $X_t$ is well defined for all $t \in \mathbb{R}$, stationary and infinitely divisible with known characteristic triplet (see Theorem 2.2) if one of the following conditions is satisfied:*

(i) *$L_1$ is $\alpha$-stable with $\alpha \in (0,2)\backslash\{1\}$ and $f \in \mathbb{L}^\alpha \cap \mathbb{L}^1$.*
(ii) *$f$ is bounded and $f \in \mathbb{L}^\delta$ for some $\delta < \alpha$, $\delta \le 1$.*
(iii) *$f$ is bounded, $\mathbb{E}\,L_1 = 0$, $\alpha > 1$ and $f \in \mathbb{L}^\delta$ for some $\delta < \alpha$, $\delta \le 2$.*

Regular variation of $X_t$ for fixed $t \in \mathbb{R}$ as well as regular variation of the finite dimensional distributions of the process $(X_t)$ have been derived in [34] under similar conditions.

**Theorem 2.4** ([34], Th. 3.2 and Cor. 3.5)**.** *Let $\Lambda$ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ and let $\nu \in RV(\alpha, (a_n), \mu_\nu)$. If $X_t = \int_{M_d^-}\int_{\mathbb{R}} f(A, t-s)\Lambda(dA,ds)$ exists (in the sense of Theorem 2.2), $f \in \mathbb{L}^\alpha(\pi\times\lambda)$ and $\mu_\nu(f^{-1}(A,s)(\mathbb{R}^n\backslash\{0\})) = 0$ does not hold for $\pi\times\lambda$ almost-every $(A,s)$, then $X_0 \in RV(\alpha, (a_n), \mu_X)$ with*

$$\mu_X(B) := \int\limits_{M_d^-}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^d} \mathbb{1}_B\left(f(A,t-s)x\right)\mu_\nu(dx)ds\pi(dA)\ \forall B \in \mathcal{B}(\overline{\mathbb{R}^n}\backslash\{0\}).$$

*Furthermore, the finite dimensional distributions $(X_{t_1},\ldots,X_{t_k})$, $t_i \in \mathbb{R}$ and $k \in \mathbb{N}$, are also regularly varying with index $\alpha$ and limiting measure*

$$\mu_{t_1,\ldots,t_k}(B) := \int\limits_{M_d^-}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^d} \mathbb{1}_B\left(\begin{pmatrix} f(A,t_1-s)\\ \vdots \\ f(A,t_k-s)\end{pmatrix} x\right)\mu_\nu(dx)ds\pi(dA)\ \forall B \in \mathcal{B}(\overline{\mathbb{R}^{nk}}\backslash\{0\})$$

.

Comparable necessary conditions for regular variation do also exist, see [34], Theorem 3.4, for details. Note in particular that in the one-dimensional case $f \in \mathbb{L}^{\alpha}(\pi \times \lambda)$ is both sufficient and necessary.

Next we introduce a result which allows to decompose a Lévy basis into a drift, a Brownian part, a part with bounded jumps and a part with finite variation. This is the extension of the Lévy-Itô decomposition to Lévy bases.

**Theorem 2.5** ([3], Theorem 2.2)**.** *Let $\Lambda$ be a Lévy basis on $M_d^- \times \mathbb{R}$ with characteristic function of the form* (2.1) *and generating quadruple $(\gamma, \Sigma, \nu, \pi)$. Then there exists a modification $\widetilde{\Lambda}$ of $\Lambda$ which is also a Lévy basis with the same characteristic quadruple $(\gamma, \Sigma, \nu, \pi)$ such that there exists an $\mathbb{R}^d$-valued Lévy basis $\widetilde{\Lambda}^G$ on $M_d^- \times \mathbb{R}$ with generating quadruple $(0, \Sigma, 0, \pi)$ and an independent Poisson random measure $N$ on $\mathbb{R}^d \times M_d^- \times \mathbb{R}$ with intensity measure $\nu \times \pi \times \lambda$ such that*

$$\widetilde{\Lambda}(B) = \gamma(\pi \times \lambda)(B) + \widetilde{\Lambda}^G(B) + \int\limits_{\|x\| \leq 1} \int\limits_B x(N(dx, dA, ds) - \pi(dA) ds \nu(dx)) + \int\limits_{\|x\|>1} \int\limits_B xN(dx, dA, ds)$$

*for all $B \in \mathcal{B}_b(M_d^- \times \mathbb{R})$ and $\omega \in \Omega$. If, additionally, $\int_{\|x\| \leq 1} \|x\| \nu(dx) < \infty$, then*

$$\widetilde{\Lambda}(B) = \gamma_0(\pi \times \lambda)(B) + \widetilde{\Lambda}^G(B) + \int\limits_{\mathbb{R}^d} \int\limits_B xN(dx, dA, ds)$$

*for all $B \in \mathcal{B}_b(M_d^- \times \mathbb{R})$, where $\gamma_0 := \gamma - \int_{\|x\| \leq 1} x\nu(dx)$. Moreover, the Lebesgue integral exists with respect to $N$ for all $\omega \in \Omega$.*

Throughout the remainder of this paper we assume that all Lévy bases occurring are already modified such that they have the above Lévy-Itô decomposition. Moreover, for a Lévy basis $\Lambda$ we define two Lévy bases $\Lambda^{(1)}$ and $\Lambda^{(2)}$ by

$$\Lambda^{(1)}(B) = \gamma(\pi \times \lambda)(B) + \widetilde{\Lambda}^G(B) + \int\limits_{\|x\| \leq 1} \int\limits_B x(N(dx, dA, ds) - \pi(dA) ds \nu(dx)) \tag{2.5}$$

$$\Lambda^{(2)}(B) = \int\limits_{\|x\|>1} \int\limits_B xN(dx, dA, ds) \tag{2.6}$$

for all Borel sets $B$.

If the underlying Lévy process is of finite variation, the integral can be defined $\omega$-wise. Note that by Theorem 21.9 in [44] finite variation of $(L_t)$ is equivalent to $\Sigma = 0$ and $\int_{\|x\| \leq 1} \|x\| \, \nu(dx) < \infty$.

**Proposition 2.6** ([3], Prop. 2.4)**.** *Let $\Lambda$ be a Lévy basis on $M_d^- \times \mathbb{R}$ with characteristic function of the form* (2.1) *and generating quadruple $(\gamma, 0, \nu, \pi)$ such that $\int_{\|x\| \leq 1} \|x\| \, \nu(dx) < \infty$. Let $\gamma_0$ and $N$ be as defined in Theorem 2.5. If $f \in \mathbb{L}^1$ and*

$$\int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^d} (1 \wedge \|f(A, s)x\|) \, \nu(dx) \, ds \, \pi(dA) < \infty,$$

*then*

$$X = \int_{M_d^-} \int_{\mathbb{R}} f(A,s)\Lambda(dA,ds) = \int_{M_d^-} \int_{\mathbb{R}} f(A,s)\,\gamma_0\,ds\pi(dA) + \int_{\mathbb{R}^d} \int_{M_d^-} \int_{\mathbb{R}} f(A,s)\,x\,N(dx,dA,ds)$$

*and the integrals on the right hand side exist as Lebesgue integrals for every $\omega \in \Omega$. Moreover, the distribution of $X$ is infinitely divisible with characteristic function*

$$\mathbb{E}\left(e^{iu^T X}\right) = \exp\left(iu^T \gamma_{int,0} + \int_{\mathbb{R}^d} \left(e^{iu^T x} - 1\right) \nu_{int}(dx)\right),$$

*where* $\gamma_{int,0} = \int_{M_d^-} \int_{\mathbb{R}} f(A,s)\,\gamma_0\,ds\,\pi(dA)$ *and*

$\nu_{int}(B) = \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_B(f(A,s)x)\,\nu(dx)\,ds\,\pi(dA)$ *for all Borel sets* $B \subseteq \mathbb{R}^n$.

The condition $f \in L^1$ is obsolete if $\gamma_0 = 0$.

## 3. Sample Path Behavior

In Section 4 we review the concept of regular variation for càdlàg processes and apply it to MMA processes. Therefore, we first have to discuss the sample path behavior of MMA processes.

Many examples of results for MMA processes to have càdlàg sample paths exist in the special case where the underlying Lévy process has sample paths of finite variation, i.e. $\Sigma = 0$ and $\int_{\|x\| \leq 1} \|x\|\, \nu(dx) < \infty$. In this case, the sample path behavior of the driving Lévy process transfers to the sample paths of the MMA process. For example, define for any Lévy process $L_t$ the corresponding *filtered Lévy process* $X_t$ by

$$X_t = \int_0^t f(t,s)\,dL_s \tag{3.1}$$

for $t \in [0,1]$. If $X_t$ exists, $L_t$ is of finite variation and the kernel function $f$ is bounded and continuous, then $X_t$ has càdlàg sample paths (cf. [24], Lemma 28).

Another result for supOU processes is given by Theorem 3.12 in [3]. This result can be extended to the general case of MMA processes.

**Theorem 3.1.** *Let $\Lambda$ be a Lévy basis on $M_d^- \times \mathbb{R}$ with characteristic function of the form (2.1) and generating quadruple $(\gamma, 0, \nu, \pi)$ such that $\int_{\|x\|\leq 1} \|x\|\,\nu(dx) < \infty$. Suppose that the kernel function $f(A,s)$ is continuous and differentiable in $s$ for all $s \in \mathbb{R}\backslash\{0\}$ and $f(A,0^-) = \lim_{s \nearrow 0} f(A,s) = C_1 \in M_{n,d}$ as well as $f(A,0^+) = \lim_{s\searrow 0} f(A,s) = C_2 \in M_{n,d}$ for all $A \in M_d^-$. Set*

$$f'(A,s) := \begin{cases} \frac{d}{ds} f(A,s) & \text{if } s \neq 0, \\ \lim_{s\searrow 0} \frac{d}{ds} f(A,s) & \text{if } s = 0 \end{cases}$$

*and assume that for some $\delta > 0$ and for every $t_1, t_2 \in \mathbb{R}$ such that $t_1 \leq t_2$ and $t_2 - t_1 \leq \delta$ the function $\sup_{t\in[t_1,t_2]} \|f'(A, t-s)\|$ satisfies the conditions of Proposition 2.6, where $(\gamma, 0, \nu, \pi)$ is replaced by $(\|\gamma\|, 0, \nu_T, \pi)$ and the Lévy measure $\nu_T(\cdot) = \nu(T^{-1}(\cdot))$ is transformed by $T(x) =$*

$\|x\|$. *If the process* $X_t = \int_{M_d^-}\int_{\mathbb{R}} f(A,t-s)\Lambda(dA,ds)$ *exists (in the sense of Proposition 2.6), then setting* $Z_t := \int_{M_d^-}\int_{\mathbb{R}} f'(A,t-s)\Lambda(dA,ds)$ *we have*

$$X_t = X_0 + \int_0^t Z_u\,du + (C_1 - C_2)\,L_t \tag{3.2}$$

*and consequently* $X_t$ *has sample paths in* $\mathbb{D}$ *which are of finite variation on compacts.*

*Proof.* Obviously the process $Z_t$ exists (in the sense of Proposition 2.6).

We follow the steps of the proof of Theorem 3.12 in [3] and begin by showing that $Z_t$ is locally uniformly bounded on compacts. Note that by Proposition 2.6 the processes $X_t$ and $Z_t$ can be given as integrals with respect to a Poisson measure and $\pi \times \lambda$. For $\delta > 0$ and every $t_1, t_2 \in \mathbb{R}$ such that $t_1 \leq t_2$ and $t_2 - t_1 \leq \delta$ we obtain $\sup_{t\in[t_1,t_2]} \|Z_t\| = \sup_{t\in[t_1,t_2]} \left\| \int_{M_d^-}\int_{\mathbb{R}} f'(A,t-s)\Lambda(dA,ds)\right\| \leq \int_{M_d^-}\int_{\mathbb{R}} \sup_{t\in[t_1,t_2]} \|f'(A,t-s)\|\Lambda_T(dA,ds)$, where $T : \mathbb{R}^d \to \mathbb{R}$ is given by $T(x) = \|x\|$ and $\Lambda_T$ is the transformed Lévy basis with characteristic triplet $(\|\gamma\|, 0, \nu_T, \pi)$. Existence of the right hand side is covered by Proposition 2.6. Thus $Z_t$ is locally uniformly bounded and it follows by Fubini that

$$\begin{aligned}
\int_0^t Z_u du &= \int_0^t \int_{M_d^-} \int_{-\infty}^u f'(A,u-s)\Lambda(dA,ds)\,du + \int_0^t \int_{M_d^-} \int_u^\infty f'(A,u-s)\Lambda(dA,ds)\,du \\
&= \int_{M_d^-} \int_{-\infty}^t \int_{0\vee s}^t f'(A,u-s)\,du\,\Lambda(dA,ds) + \int_{M_d^-} \int_0^\infty \int_0^{t\wedge s} f'(A,u-s)\,du\,\Lambda(dA,ds) \\
&= \int_{M_d^-} \int_{-\infty}^t f(A,u-s)\Big|_{u=0\vee s}^t \Lambda(dA,ds) + \int_{M_d^-} \int_0^\infty f(A,u-s)\Big|_{u=0}^{t\wedge s} \Lambda(dA,ds) \\
&= X_t - X_0 + (C_1 - C_2)\,L_t.
\end{aligned}$$

□

**Remark 3.2.** (1) The inclusion of kernel functions with a discontinuity at $s = 0$ is motivated by the class of causal MMA processes where the kernel function is of the form $f(A,s)\mathbb{1}_{[0,\infty)}(s)$. For example, in the supOU case the kernel function is $e^{As}\,\mathbb{1}_{[0,\infty)}(s)$ and the limits at $s = 0$ can be given directly by $C_1 = \mathbf{0}$ and $C_2 = I_d$ yielding (see Theorem 3.12 in [3])

$$X_t = X_0 + \int_0^t Z_u\,du - L_t.$$

(2) The result can obviously be extended to the case where $f(A,0^-)$ and $f(A,0^+)$ exist for all $A$, but are not independent of $A$. Then (3.2) holds with the Lévy process $\tilde{L}_t := \int_{M_d^-}\int_0^t (f(A,0^+) - f(A,0^-))\Lambda(dA,ds)$ in place of $(C_1 - C_2)\,L_t$.

(3) Intuitively a necessary condition for a MMA process $X$ to have càdlàg paths should be that $f(A,\cdot)$ is càdlàg for ($\pi$ almost) all $A$. In the case of continuous paths the

analogous necessary condition follows from [41, Theorem 2.4]. We conjecture that also in the càdlàg case one can prove this necessary condition by extending the arguments of [14, 41], but this is beyond the scope of the present paper.

(4) The condition on $f$ being differentiable with respect to time everywhere except at 0 is clearly only sufficient. For example, for any Lévy process $L$ with finite logarithmic moment

$$X_t = \int_{\mathbb{R}} \left( e^{-(t-s)} 1_{\mathbb{R}^+}(t-s) + e^{-(t-1-s)} 1_{\mathbb{R}^+}(t-1-s) \right) dL_s$$

is càdlàg as the sum of two OU processes which are càdlàg. Obviously, one can extend Proposition 3.1 and its proofs to appropriate functions which are differentiable except at finitely many points and thus cover the above counterexample. However, it seems completely unclear whether one can extend it to functions which are differentiable except at countably many points.

Note that the references given below for other results on the càdlàg property usually demand absolute continuity of $f$ in time or something similar and so the above counterexample also violates their conditions. All particular examples of MMA processes considered so far in the literature have – to the best of our knowledge – kernels $f$ discontinuous and non-differentiable at most at 0.

If $C_1 - C_2 = 0$ in the above theorem, further properties of the sample paths of $X_t$ follow directly.

**Corollary 3.3.** *Assume that the conditions of Theorem 3.1 hold. If additionally $C_1 = C_2$, then the paths of $X_t = \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s)\Lambda(dA, ds)$ are absolutely continuous and almost everywhere differentiable.*

**Remark 3.4.** The condition $C_1 = C_2$ holds if and only if $f(A, s)$ is continuous in $s = 0$ and $f(A, 0)$ is constant for all $A \in M_d^-$. This is satisfied, for example, by two-sided supOU processes which are MMA processes with kernel function

$$f(A, s) = e^{As}\mathbb{1}_{[0,\infty)}(s) + e^{-As}\mathbb{1}_{(-\infty,0)}(s).$$

In the case of moving average processes, where $\pi$ is a one-point measure, the condition only requires that $f$ is continuous in $s = 0$. Processes of this class include, for example, two-sided CARMA and two-sided Ornstein-Uhlenbeck processes.

Similar results for the sample paths of MMA processes, where the driving Lévy process is not of finite variation, are in general not so easy to obtain. [4], Corollary 3.3, give necessary and sufficient conditions for filtered Lévy processes of the form (3.1) to have càdlàg sample paths of bounded variation even if the driving Lévy process itself has sample paths of unbounded variation. Furthermore, they also study *two-sided moving averages* of the form $X_t = \int_{-\infty}^{t} (f_1(t-s) - f_2(-s))\, dL_t$, where $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are measurable kernel functions such that $f_1(s) = f_2(s) = 0$ for all $s \in (-\infty, 0)$. They give necessary and sufficient conditions for such processes to have càdlàg sample paths of finite variation. These conditions also allow the underlying Lévy process to be of infinite variation. For MMA processes [6] gives necessary and sufficient conditions for finite variation and absolute continuity of sample paths for underlying Lévy processes of infinite variation. Moreover, [4] also consider the special case where the driving Lévy process is symmetric $\alpha$-stable with $\alpha \in (1, 2]$ (cf. [4], Lemma 5.2, Proposition 5.3 and Proposition 5.5). Conditions for $\alpha$-stable MMA processes, $\alpha \in (0,2)$, to have càdlàg sample paths are also given in [5], Section 4.

Additionally, there also exist some results for the stronger property of continuous sample paths. See [30], [13] and [41] for results on general MMA processes to have continuous sample paths. For the special case of $\alpha$-stable MMA processes, see also [43] and [40].

## 4. Functional Regular Variation

We follow [24] to introduce the notion of regular variation on $\mathbb{D}$. Let $\mathbb{D}$ be the space of càdlàg (right-continuous with left limits) functions $x : [0,1] \to \mathbb{R}^d$ equipped with the $J_1$ metric (equivalent to the $d_0$ metric of [8]) such that $\mathbb{D}$ is a complete and separable metric space. Using the supremum norm $\|x\|_\infty = \sup_{t\in[0,1]} \|x_t\|$ we can then introduce the unit sphere $S_\mathbb{D} = \{x \in \mathbb{D} : \|x\|_\infty = 1\}$, equipped with the relativized topology of $\mathbb{D}$. Next, we equip $(0,\infty]$ with the metric $\rho(x,y) = |1/x - 1/y|$ which makes it a complete separable metric space. Then also the space $\overline{\mathbb{D}}_0 = (0,\infty] \times S_\mathbb{D}$, equipped with the metric $\max\{\rho(x^*,y^*), d_0(\widetilde{x},\widetilde{y})\}$, is a complete separable metric space.

If we use the polar coordinate transformation $T : \mathbb{D}\backslash\{0\} \to \overline{\mathbb{D}}_0$, $x \mapsto (\|x\|_\infty, x/\|x\|_\infty)$, we see that the spaces $\mathbb{D}\backslash\{0\}$ and $(0,\infty) \times S_\mathbb{D}$ are homeomorphic. Thus, the Borel sets $\mathcal{B}(\overline{\mathbb{D}}_0)$ of interest can be viewed as the infinite dimensional extension of the one-point uncompactification that is used to introduce finite dimensional regular variation (cf. [9], [19] and [38]).

Regular Variation on $\mathbb{D}$ can then be introduced in terms of the so-called $\hat{w}$-convergence of boundedly finite measures on $\overline{\mathbb{D}}_0$. A measure $\mu$ on a complete separable metric space $E$ is said to be *boundedly finite* if $\mu(B) < \infty$ for every bounded set $B \in \mathcal{B}(E)$. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of boundedly finite measures on $E$. Then $(\mu_n)$ converges to $\mu$ in the *$\hat{w}$-topology* if $\mu_n(B) \to \mu(B)$ for all bounded Borel sets $B \in \mathcal{B}(E)$ with $\mu(\partial B) = 0$. We write $\mu_n \xrightarrow{\hat{w}} \mu$. Note that for locally compact spaces $E$ the boundedly finite measures are called Radon measures and the notions of $\hat{w}$-convergence and vague convergence coincide. See [15] and [27] for details on $\hat{w}$-convergence and vague convergence.

We are now able to formulate regular variation for stochastic processes with sample paths in $\mathbb{D}$.

**Definition 4.1** (Regular Variation on $\mathbb{D}$)**.** A stochastic process $(X_t)$, $t \in [0,1]$, with sample paths in $\mathbb{D}$ is said to be *regularly varying* if there exists a positive sequence $(a_n)$, $n \in \mathbb{N}$, with $a_n \nearrow \infty$ and a nonzero boundedly finite measure $\mu$ on $\mathcal{B}(\overline{\mathbb{D}}_0)$ with $\mu(\overline{\mathbb{D}}_0\backslash\mathbb{D}) = 0$ such that, as $n \to \infty$,

$$nP(a_n^{-1}X \in \cdot) \xrightarrow{\hat{w}} \mu(\cdot) \quad \text{on } \mathcal{B}(\overline{\mathbb{D}}_0).$$

As in the finite dimensional case, direct calculation shows that the measure $\mu$ is homogeneous, i.e. there exists a positive index $\alpha > 0$ such that $\mu(uB) = u^{-\alpha}\mu(B)$ for all $u > 0$ and for every $B \in \mathcal{B}(\overline{\mathbb{D}}_0)$. Thus, we say that the process $(X_t)$ is *regularly varying with index $\alpha$* and write $X \in RV_{\overline{\mathbb{D}}_0}(\alpha, (a_n), \mu)$.

**Example 4.2** (Lévy Process)**.** Let $(L_t)$ be a Lévy process. Then by definition (or Theorem 11.5 in [44] resp.) $(L_t)$ has sample paths in $\mathbb{D}$. Furthermore, $(L_t)$ is also a strong Markov process (cf. [44], Theorem 10.5 and Corollary 40.11). Now the results of [24], Section 3, can be applied. If $L_t \in RV(\alpha, (a_n), t\mu)$ for one and thus all $t > 0$, then it follows by Theorem 13 of [24] that $(L_t) \in RV_{\overline{\mathbb{D}}_0}(\alpha, (a_n), \widetilde{\mu})$ for some measure $\widetilde{\mu}$. For details we refer to [24], Example 17.

The next theorem states some necessary and sufficient conditions for regular variation on $\mathbb{D}$. In the theorem, we use the notation

$$w(x,T_0) := \sup_{t_1,t_2\in T_0} \|x_{t_1} - x_{t_2}\| \quad \text{and}$$
$$w''(x,\delta) := \sup_{0\le t_1\le t\le t_2\le 1;\, t_2-t_1\le\delta} \min\{\|x_t - x_{t_1}\|, \|x_{t_2} - x_t\|\}$$

for $x \in \mathbb{D}$, $T_0 \subseteq [0,1]$ and $\delta \in [0,1]$.

**Theorem 4.3** ([24], Theorem 10). *Let $(X_t)$ be a stochastic process with sample paths in $\mathbb{D}$. Then the following statements are equivalent.*

*(i)* $X \in RV_{\overline{\mathbb{D}}_0}(\alpha, (a_n), \mu)$.

*(ii) There exists a set $T \subseteq [0,1]$ containing $0$, $1$ and all but at most countably many points of $[0,1]$, a positive sequence $a_n \nearrow \infty$ and a collection $\{\mu_{t_1,\dots,t_k} : t_i \in T, k \in \mathbb{N}\}$ of Radon measures on $\mathcal{B}(\overline{\mathbb{R}}^{dk}\backslash\{0\})$ with $\mu_{t_1,\dots,t_k}(\overline{\mathbb{R}}^{dk}\backslash\mathbb{R}^{dk}) = 0$ and $\mu_t$ is nonzero for some $t \in T$ such that*

$$nP(a_n^{-1}(X_{t_1},\dots,X_{t_k}) \in \cdot) \xrightarrow{v} \mu_{t_1,\dots,t_k}(\cdot) \quad \text{on } \mathcal{B}(\overline{\mathbb{R}}^{dk}\backslash\{0\}) \tag{4.1}$$

*holds for all $t_1,\dots,t_k \in T$. Furthermore, for every $\varepsilon, \eta > 0$, there exist $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,*

$$nP(a_n^{-1} w(X,[0,\delta)) \ge \varepsilon) \le \eta, \tag{4.2}$$
$$nP(a_n^{-1} w(X,[1-\delta,1)) \ge \varepsilon) \le \eta, \tag{4.3}$$
$$nP(a_n^{-1} w''(X,\delta) \ge \varepsilon) \le \eta. \tag{4.4}$$

**Remark 4.4.** The theorem links regular variation of the process $(X_t)_{t\in[0,1]}$ with sample paths in $\mathbb{D}$ to regular variation of the finite dimensional distributions $(X_{t_1},\dots,X_{t_k})$ of the the process. Key to that connection are the relative compactness criteria (4.2), (4.3) and (4.4) which restrict the oscillation of the process $(X_t)$ in small areas. See [24], Example 11, for a process satisfying conditions (4.2) and (4.3), but not (4.4).

Now we will extend the finite dimensional regular variation of MMA processes in the sense of Theorem 2.4 to regular variation in $\mathbb{D}$ by applying Theorem 4.3. Therefore, we need to restrict the MMA process $(X_t)$ as defined in Section 2.3 to the time interval $[0,1]$. Note that a restriction to any other compact time interval $[a,b]$, $a < b$, would not change any of the results. Furthermore, we assume that $(X_t)$ has sample paths in the space $\mathbb{D}$ of càdlàg functions. See Section 3 for possible conditions ensuring this. We start with the main theorem for functional regular variation of MMA processes.

In order not to overload the notation we from now on assume always that $t, t_1, t_2$ are restricted to the set $[0;1]$ when taking suprema without writing this explicitly. Furthermore, we are now using the decomposition (2.6) of our Lévy basis $\Lambda$ into $\Lambda^{(1)}$ and $\Lambda^{(2)}$.

**Theorem 4.5.** *Let $\Lambda$ be an $\mathbb{R}^d$-valued Lévy bases on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ such that $\nu \in RV(\alpha, (a_n), \mu_\nu)$. Assume that the kernel function $f$ is bounded, càdlàg in time, $f \in \mathbb{L}^\alpha(\pi \times \lambda)$, $\mu_\nu(f^{-1}(A,s)(\mathbb{R}^n\backslash\{0\})) = 0$ does not hold for $\pi \times \lambda$ almost-every $(A,s)$ and*

$$\int_{M_d^-} \int_{\mathbb{R}} \int_{\|x\|>1} (1 \wedge \|f(A,s)x\|)\, \nu(dx)\, ds\, \pi(dA) < \infty. \tag{4.5}$$

*Moreover, suppose that the MMA process* $X_t = \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s)\Lambda(dA, ds)$ *exists for* $t \in [0,1]$ *(in the sense of Theorem 2.2) and that the processes* $X_t$ *and* $X_t^{(2)} = \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s)\Lambda^{(2)}(dA, ds)$ *have càdlàg sample paths. If the function* $f_\delta$ *given by*

$$f_\delta(A, s) := \sup_{t_1 \le t_2;\, t_2 - t_1 \le \delta} \|f(A, t_2 - s) - f(A, t_1 - s)\| \, \mathbb{1}_{(t_1, t_2]^c}(s) \tag{4.6}$$

*satisfies* (4.5) *for all* $\delta > 0$ *small enough and, as* $\delta \to 0$,

$$\int_{M_d^-} \int_{\mathbb{R}} f_\delta(A, s)^\alpha \, ds\pi(dA) \to 0, \tag{4.7}$$

*then*

$$(X_t)_{t\in[0,1]} \in RV_{\overline{\mathbb{D}}_0}(\alpha, (a_n), \mu).$$

*The measure* $\mu$ *is uniquely determined by the measures* $\mu_{t_1,\dots,t_k}$ *in Theorem 2.4, concentrated on the càdlàg functions of the form* $[0;1] \to \mathbb{R}^n$, $s \mapsto f(A, s-t)x$ *where* $A \in supp(\pi), t \in \mathbb{R}$ *and* $x \in supp(\mu_\nu)$ *and we have that*

$$\mu\{f(A, \cdot - t)x : \, A \in \mathcal{A}, \, t \in T, \, x \in \mathcal{X}\} = \lambda(T)\pi(\mathcal{A})\mu_\nu(\mathcal{X}) \tag{4.8}$$

*for measurable sets* $\mathcal{A} \subseteq M_d^-$, $T \subseteq \mathbb{R}$ *and* $\mathcal{X} \subseteq \mathbb{R}^d$.

Before we prove this theorem in Section 4.1 we discuss its intuition and its assumptions. Let us stress that the aim of this paper is to derive *sufficient* conditions for MMA processes to be regularly varying in $\mathbb{D}$ which can be reasonably checked and are useful for the MMA processes typically employed.

**Remark 4.6.** There is a very simple intuition behind the limiting measure $\mu$. Asymptotically in the case of regular variation it is normally the largest event which alone matters. In our case it is the largest jump in our Lévy basis. The largest jump of size say $x_l$ occurs at a time $t_l$ and is "marked" with a matrix $A_l$ determining the function with which it propagates over time. In other words we have the function $s \mapsto f(A_l, s - t_l)x_l$ which determines the extremal behaviour. Now by the time homogeneity of the Lévy basis $t_l$ is uniformly chosen from $\mathbb{R}$ and by the factorisability $A_l$ has distribution $\pi$.

If $f$ is zero on an interval of length at least one, like for (sup)OU processes, $\mu$ has mass at the zero function which is not an extremal event. Our non-degeneracy condition "$\mu_\nu(f^{-1}(A, s)(\mathbb{R}^n \backslash \{0\})) = 0$ does not hold for $\pi \times \lambda$ almost-every $(A, s)$" ensures that $\mu$ is not concentrated on the zero function. For example in the case of (sup)OU processes (see the upcoming Section 5 for details) for the "true extremal" events the time $t_l$ is chosen uniformly on $(-\infty, 1]$. So for a (sup)OU process the extremal events are either functions on $[0,1]$ which decay exponentially or functions first identical zero, then having a jump and decaying exponentially thereafter.

Turning to the conditions it is clear that we need to demand that the process has càdlàg paths and in view of Remark 3.2 it is natural to demand $f$ càdlàg in time. However, in general this is a very problematic assumption, since as discussed in Section 3 the conditions known to ensure càdlàg paths are much too strong. It is easy to construct non-càdlàg regularly varying processes as the following example exhibits.

**Example 4.7.** Let $L$ be a compound Poisson process (or a Lévy process of finite variation) with regularly varying jumps of index $\alpha$. Then for any $\lambda > 0$

$$X_t = \int_{-\infty}^{t} \left( e^{-\lambda(t-s)} 1_{\mathbb{R}^+}(t-s) + 1_{\{0\}}(t-s) \right) dL_s$$

exists $\omega$-wise and has finite-dimensional distributions which are regularly varying. However, it is not càdlàg at all jump times of the underlying Lévy process. Thus it cannot be regularly varying on $\mathbb{D}$. Note, however, that conditions (4.5) and (4.7) are satisfied. Moreover, the finite-dimensional distributions are the same as those of $\int_{-\infty}^{t} e^{-\lambda(t-s)} 1_{\mathbb{R}^+}(t-s) dL_s$ which is regularly varying on $\mathbb{D}$.

An alternative example of a moving average process with regularly finite dimensional distributions not regularly varying on $\mathbb{D}$ is $\int_{-\infty}^{t} \left( e^{-\lambda(t-s)} 1_{\mathbb{R}^+}(t-s) + 1_{(0,1)}(t-s) \right) dL_s$.

The condition $f \in \mathbb{L}^\alpha(\pi \times \lambda)$ has already been discussed in [34]. It is close to necessary in general and necessary in the one-dimensional case.

The boundedness of the kernel function allows us to use results on the regular variation on $\mathbb{D}$ of the underlying Lévy process in the proof. Since $f$ has to be càdlàg, some local boundedness (in time) is already demanded and in applications no unbounded kernel functions are used to the best of our knowledge.

Note that an essential part of the proof of the theorem is going to be to show that under the above assumption it is essentially only $X^{(2)} = \int\limits_{\|x\| \geq 1} \int\limits_{M_d^-} \int\limits_{\mathbb{R}} f(A, t-s)\, x\, N(dx, dA, ds)$ that matters regarding the extremal behaviour. Condition (4.5) ensures that this integral is $\omega$-wise well defined. In the proof this is extremely important, because only the $\omega$-wise existence allows us to bound the probabilities occurring in Theorem 4.3 in the way we do. Intuitively condition (4.5) says that $f$ is integrable with respect to the big jumps of the Lévy bases. Actually, this is a severe restriction, since to have $X$ well-defined only square integrability is needed.

**Example 4.8.** Let $L$ be a stable Lévy process with index $\alpha \in (1, 2)$. Then the process

$$X_t = \int_{\mathbb{R}} \frac{e^{i(t-s)} - 1}{i(t-s)} dL_s$$

is well-defined and its finite dimensional distributions are regularly varying with index $\alpha$ (follows similar to [23, Proposition 3.9]). However, condition (4.5) is not satisfied (follows from [23, Proposition 3.9 (i)]. In this case condition (4.7) is satisfied due to Lemmata 4.11, 4.13 and Remark 4.12. Moreover, the conditions of [4, Proposition 5.5] are easily verified which allows us to take $X$ càdlàg.

So the only problem prohibiting us to show that $X$ is regularly varying on $\mathbb{D}$ is condition (4.5). Since it is essential for our proof to have the $\omega$-wise existence of the integral over the big jumps and we cannot think of an alternative approach at the moment, we do not know whether or not $X$ is regularly varying on $\mathbb{D}$.

Note that this example is related to the spectral representation of a Lévy process relevant in continuous time time series models (see [23]) but similar conclusions can always be drawn when the kernel function $f$ has norm proportional to $|t|^\gamma$ for all $t$ outside a compact set with $\gamma \in (1, 2)$.

We can also give sufficient conditions for a general function $f : M_d^- \times \mathbb{R} \to M_{n,d}$ to satisfy condition (4.5).

**Lemma 4.9.** *Let $f : M_d^- \times \mathbb{R} \to M_{n,d}$ be a measurable function. Then condition* (4.5) *holds if one of the following two conditions are satisfied:*

(i) $f \in \mathbb{L}^1(\pi \times \lambda)$ *and* $\alpha > 1$.
(ii) $f \in \mathbb{L}^{\alpha-\varepsilon}(\pi \times \lambda)$ *for one* $\varepsilon \in (0, \alpha)$ *and* $\alpha \leq 1$.

*Proof.* For (i) we calculate

$$\int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\|x\|>1} (1 \wedge \|f(A,s)x\|)\, \nu(dx)\, ds\, \pi(dA) \leq \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \|f(A,s)\|\, ds\, \pi(dA) \int\limits_{\|x\|>1} \|x\|\, \nu(dx) < \infty$$

by [44], Corollary 25.8, and similarly for (ii) we obtain

$$\int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\|x\|>1} (1 \wedge \|f(A,s)x\|)\, \nu(dx)\, ds\, \pi(dA) \leq \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \int\limits_{\|x\|>1} (1 \wedge \|f(A,s)x\|^{\alpha-\varepsilon})\, \nu(dx)\, ds\, \pi(dA)$$
$$\leq \int\limits_{M_d^-} \int\limits_{\mathbb{R}} \|f(A,s)\|^{\alpha-\varepsilon}\, ds\, \pi(dA) \int\limits_{\|x\|>1} \|x\|^{\alpha-\varepsilon}\, \nu(dx) < \infty.$$

□

**Remark 4.10.** (i) The conditions of Lemma 4.9 are only sufficient, not necessary, similar to the ones of Theorem 2.3. Thus in general we will only demand the weaker condition (4.5) which is also one of the existence conditions for MMA processes with driving Lévy process of finite variation in Proposition 2.6.

(ii) Note that this theorem can be also used to verify the condition of Theorem 4.5 that (4.5) needs to hold for $f_\delta$ with $\delta > 0$ small enough.

The condition $\int_{M_d^-} \int_{\mathbb{R}} f_\delta(A,s)^\alpha\, ds\pi(dA) \to 0$ is closely linked to the behaviour of the function $f_\delta$ over small intervals. It restricts the amplitude of jumps and continuous oscillations for arbitrarily small values of $\delta$.

We first show that provided we have $f_\delta \in L^\alpha$ for some $\delta > 0$ only pointwise convergence needs to be shown.

**Lemma 4.11.** *Let $\pi$ be a probability measure and $f : M_d^- \times \mathbb{R} \to M_{n,d}$ be a measurable kernel function. Assume that the function $f_\delta(A,s)$ given by* (4.6) *satisfies $f_\delta \in \mathbb{L}^\alpha$ for some $\delta > 0$. Then* (4.7) *holds if and only if $\lim_{\delta\to 0}\ f_\delta(A,s) \to 0$ for $\pi \times \lambda$ almost every $(A,s)$.*

*Proof.* Let $f_\delta(A,s) \to 0$ for $\pi \times \lambda$ almost every $(A,s)$. Then $\int\limits_{M_d^-} \int\limits_{\mathbb{R}} f_\delta(A,s)^\alpha\, ds\pi(dA) \to 0$ follows by dominated convergence and the assumption $f_\delta \in \mathbb{L}^\alpha$ for some $\delta > 0$.

On the other hand, suppose that the set $\tilde{B} := \left\{(A,s) \in \mathcal{B}_b(M_d^- \times \mathbb{R}) : f_\delta(A,s) \to 0 \text{ as } \delta \to 0\right\}$ satisfies $\pi \times \lambda\,(\tilde{B}^c) = C > 0$. Then the monotonicity of $f_\delta$ in $\delta$ implies $\lim_{\delta\to 0} f_\delta(A,s) > 0$ for every $(A,s) \in \tilde{B}^c$ and thus $\lim\limits_{\delta\to 0} \int\limits_{M_d^-} \int\limits_{\mathbb{R}} f_\delta(A,s)^\alpha\, ds\pi(dA) > 0$. □

**Remark 4.12.** From the definition of $f_\delta$ we see that the condition $\lim_{\delta\to 0}\ f_\delta(A,s) \to 0$ for $\pi \times \lambda$ almost every $(A,s)$ is equivalent to the kernel function $f(A,s)$ being continuous in $s$ for all $s \in \mathbb{R}\backslash\{0\}$. Now we also see the importance of the restriction $\mathbb{1}_{(t_1,t_2]^c}(s)$ in the definition of $f_\delta$ because it allows for $f(A,s)$ being discontinuous at $s = 0$. Without such a restriction, condition (4.7) would be violated by many examples of the class of causal MMA processes which have a kernel function of the type $f(A,s)\mathbb{1}_{[0,\infty)}(s)$. Causal MMA processes with $f(A,0)\neq 0$ include CARMA and supOU processes as well as other well-known examples of MMA processes.

Thus we know that as soon as we have the integrability condition ensured for $f_\delta$ then the kernel function $f$ needs to be continuous.

**Lemma 4.13.** *Assume $f: M_d^- \times \mathbb{R} \to M_{n,d}$ is a bounded measurable kernel function and that there exists a function $g: M_d^- \times \mathbb{R} \to \mathbb{R}^+$ which is monotonically decreasing on $[T,\infty)$ and monotonically increasing on $(-\infty,-T]$ for some $T>0$. If $g \in L^\alpha(\pi\times\lambda)$ and $\|f(A,s)\| \leq g(A,s)$ for all $A\in M_d^-$ and $s\in(-\infty,-T]\cup[T,\infty)$, then $f_\delta \in L^\alpha(\pi\times\lambda)$ for all $0<\delta\leq 1$.*

This result can also be used in connection with Lemma 4.9 to verify that (4.5) holds for $f_\delta$ with $\delta>0$ small enough.

*Proof.* From the definition it is easy to see that we have

$$\|f_1(A,s)\| \leq \begin{cases} 2g(A,-s-1) & \text{for all } s\in(-\infty,-T-1],\ A\in M_d^- \\ \sup_{s\in\mathbb{R}}\|f_\delta(A,s)\| & \text{for all } s\in(-T-1,T+1],\ A\in M_d^- \\ 2g(A,-s+1) & \text{for all } s\in[T+1,\infty)),\ A\in M_d^- \end{cases}$$

and this concludes as $f$ is bounded and $g\in L^\alpha(\pi\times\lambda)$. Moreover, note that $f_1\in L^\alpha(\pi\times\lambda)$ implies $f_\delta\in L^\alpha(\pi\times\lambda)$. □

From Lemmata 4.11, 4.13 and Remark 4.12 we see that if (4.7) is violated for a kernel function $f$ in $L^\alpha(\pi\times\lambda)$, then $f$ must either be discontinuous at some time different from zero or of a rather irregular behaviour such that it is in $L^\alpha(\pi\times\lambda)$ but we cannot find an ultimately monotone bound in $L^\alpha(\pi\times\lambda)$. Thus it seems very hard to construct a continuous function $f$ for which we know that it satisfies all conditions of Theorem 4.5 except (4.7), because the available sufficient conditions for càdlàg paths available seem to be much too demanding.

However, it s not hard to construct examples with discontinuities.

**Example 4.14.** For any regularly varying Lévy process $L$ with index $\alpha$ consider

$$X_t = \int_{\mathbb{R}} \left(e^{-(t-s)}1_{\mathbb{R}^+}(t-s) + e^{-(t-1-s)}1_{\mathbb{R}^+}(t-1-s)\right) dL_s.$$

We know from Remark 3.2 that $X$ has càdlàg paths and all other conditions of Theorem 4.5 except (4.7) are straightforwardly established. Since the conditions of Lemma 4.13 are satisfied we can conclude from Remark 4.12 and Lemma 4.11 that (4.7) is violated due to the discontinuity of the kernel function at one. Intuitively, $X$ is the sum of two OU processes which are regularly varying on $\mathbb{D}$ and thus should also be regulary varying on $\mathbb{D}$. Actually, one can establish this by looking at the details of the proof of Theorem 4.5.

**Remark 4.15.** By excluding also the other times of discontinuity in (4.6) and dealing with them as with the discontinuity at time zero in the proof of Theorem 4.5, one can extend all the previous results to kernel functions $f$ which have discontinuities at finitely many points in time.

Since this does not really add additional insight and seems not to be relevant in applications, but becomes notationally very cumbersome, we have decided not to state the theorem and the proof for this setting.

As soon as we try to extend Example 4.14 to countably many discontinuities we run into severe problems. For example, it is easy to show that $f(A,s) = \sum_{k=1}^{\infty} \frac{1}{k} e^{-k(s-k)} 1_{[k,\infty)}(s)$ is in $L^1(\pi \times \lambda)$, but $f_\delta \notin L^1(\pi \times \lambda)$ for any $\delta > 0$. Yet, the problem is that we do not know whether we can take $X$ to have càdlàg sample paths in the first place, since we have no sharp enough conditions for this.

## 4.1. **Proof of Theorem 4.5.**

In this subsection we now gradually prove Theorem 4.5.

Let $(X_t)$ be an MMA process as given in Theorem 4.5, i.e. $(X_t)$ exists for $t \in [0,1]$ (in the sense of Theorem 2.2), the kernel function $f$ is bounded by $C \in \mathbb{R}^+$ and the regular variation conditions of Theorem 2.4 hold. Then there exists a positive sequence $a_n \nearrow \infty$ and a collection $\{\mu_{t_1,\dots,t_k} : t_i \in T, k \in \mathbb{N}\}$ of Radon measures on $\mathcal{B}(\overline{\mathbb{R}}^{dk} \backslash \{0\})$ with $\mu_{t_1,\dots,t_k}(\overline{\mathbb{R}}^{dk} \backslash \mathbb{R}^{dk}) = 0$ and $\mu_t$ is nonzero for some $t \in T$ such that

$$nP(a_n^{-1}(X_{t_1}, \dots, X_{t_k}) \in \cdot) \xrightarrow{v} \mu_{t_1,\dots,t_k}(\cdot) \quad \text{on } \mathcal{B}(\overline{\mathbb{R}}^{dk} \backslash \{0\}).$$

Applying Theorem 4.3, it is left to show that the conditions (4.2), (4.3) and (4.4) hold.

Using the Lévy-Itô decomposition we have two independent Lévy bases $\Lambda^{(1)}$ and $\Lambda^{(2)}$ such that $\Lambda^{(1)}$ has generating quadruple $(\gamma, \Sigma, \nu_1, \pi)$ and $\Lambda^{(2)}$ has generating quadruple $(0, 0, \nu_2, \pi)$, where $\nu_1 = \nu|_{B_1(0)}$ and $\nu_2 = \nu|_{B_1(0)^c}$. This yields

$$X_t = X_t^{(1)} + X_t^{(2)}, \tag{4.9}$$

where

$$X_t^{(1)} = \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s) \Lambda^{(1)}(dA, ds) \tag{4.10}$$

and

$$X_t^{(2)} = \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s) \Lambda^{(2)}(dA, ds). \tag{4.11}$$

Note that $X_t^{(2)}$ can be written in the form $X_t^{(2)} = \int_{\|x\| \geq 1} \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s)\, x\, N(dx, dA, ds)$, where $N$ is a Poisson random measure with mean measure $\nu \times \pi \times \lambda$. Before we proceed, we need to ensure the existence of $X_t^{(1)}$ and $X_t^{(2)}$. Therefore, we give conditions for $\omega$-wise existence of $X_t^{(2)}$ as a Lebesgue integral. Then the existence of $X_t^{(1)} = X_t - X_t^{(2)}$ follows from the existence of $X_t$ and $X_t^{(2)}$.

**Proposition 4.16.** *Let $X_t^{(2)}$ be the process given by* (4.11)*, where $\Lambda^{(2)}$ is a Lévy basis with generating quadruple* $(0, 0, \nu_2, \pi)$*. If $\int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} (1 \wedge \|f(A,s)x\|)\, \nu_2(dx)\, ds\, \pi(dA) < \infty$, then $X_t^{(2)}$ exists as a Lebesgue integral for all $\omega \in \Omega$.*

*Proof.* By definition, $X_t^{(2)}$ has no Gaussian component and $\int_{\|x\| \leq 1} \|x\| \nu_2(dx) = 0$ and thus we have an underlying Lévy process of finite variation. Now the result follows as a special case of Proposition 2.6, where the condition $f \in \mathbb{L}^1$ is obsolete due to the absence of a drift. $\square$

Like for $X_t$, we also assumed that $X_t^{(2)}$ has càdlàg sample paths. Then also $X_t^{(1)} = X_t - X_t^{(2)}$ has càdlàg sample paths. Appropriate conditions for MMA processes to have càdlàg sample paths have been given in Section 3.

Now we continue the proof of Theorem 4.5 by verifying the relative compactness conditions (4.2), (4.3) and (4.4). The essential point is to relate the conditions back to the analogous conditions on the underlying Lévy process.

For the first condition (4.2) we obtain

$$\sup_{t_1,t_2\in[0,\delta)} \|X_{t_1} - X_{t_2}\| \le \sup_{t_1,t_2\in[0,\delta)} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| + \sup_{t_1,t_2\in[0,\delta)} \|X_{t_1}^{(2)} - X_{t_2}^{(2)}\|$$

and hence

$$\begin{aligned} &nP\Big(a_n^{-1} \sup_{t_1,t_2\in[0,\delta)} \|X_{t_1} - X_{t_2}\| \ge \varepsilon\Big) \le \\ &\quad \le nP\Big(a_n^{-1} \sup_{t_1,t_2\in[0,\delta)} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| \ge \varepsilon/2\Big) + nP\Big(a_n^{-1} \sup_{t_1,t_2\in[0,\delta)} \|X_{t_1}^{(2)} - X_{t_2}^{(2)}\| \ge \varepsilon/2\Big). \end{aligned}$$

The analogue result for the second condition (4.3) can be obtained likewise. For the third condition (4.4) we estimate

$$\begin{aligned} &\sup_{t_1\le t\le t_2;\, t_2-t_1\le\delta} \min\Big\{\|X_{t_2} - X_t\|, \|X_t - X_{t_1}\|\Big\} \le \\ &\quad \le \sup_{t_1\le t_2;\, t_2-t_1\le\delta} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| + \sup_{t_1\le t\le t_2;\, t_2-t_1\le\delta} \min\Big\{\|X_{t_2}^{(2)} - X_t^{(2)}\|, \|X_t^{(2)} - X_{t_1}^{(2)}\|\Big\}, \end{aligned}$$

and

$$\begin{aligned} &nP\Big(a_n^{-1} \sup_{t_1\le t\le t_2;\, t_2-t_1\le\delta} \min\Big\{\|X_{t_2} - X_t\|, \|X_t - X_{t_1}\|\Big\} \ge \varepsilon\Big) \le \\ &\qquad \le nP\Big(a_n^{-1} \sup_{t_1\le t_2;\, t_2-t_1\le\delta} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| \ge \varepsilon/2\Big) \\ &\qquad + nP\Big(a_n^{-1} \sup_{t_1\le t\le t_2;\, t_2-t_1\le\delta} \min\Big\{\|X_{t_2}^{(2)} - X_t^{(2)}\|, \|X_t^{(2)} - X_{t_1}^{(2)}\|\Big\} \ge \varepsilon/2\Big). \end{aligned}$$

For every $\varepsilon, \eta > 0$ we have to show that there exists $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for $n \ge n_0$ these quantities can be bounded by $\eta$. Regarding the quantities based on $X_t^{(1)}$ we observe

$$nP\Big(a_n^{-1} \sup_{t_1,t_2\in[0,\delta)} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| \ge \varepsilon/2\Big) \le nP\Big(a_n^{-1} \sup_{t_1\le t_2;\, t_2-t_1\le\delta} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| \ge \varepsilon/2\Big)$$

and for (4.3)

$$nP\Big(a_n^{-1} \sup_{t_1,t_2\in[1-\delta,1)} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| \ge \varepsilon/2\Big) \le nP\Big(a_n^{-1} \sup_{t_1\le t_2;\, t_2-t_1\le\delta} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| \ge \varepsilon/2\Big)$$

and thus it is sufficient to prove the bound only for the right hand side of the inequality.

**Proposition 4.17.** *Let $\Lambda^{(1)}$ be the $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ determined by the generating quadruple $(\gamma, \Sigma, \nu_1, \pi)$, where $\nu_1 = \nu|_{B_1(0)}$. Assume that the kernel function $f$ is bounded, that the MMA process $X_t^{(1)}$ given by (4.10) exists for $t \in [0,1]$ and that $X_t^{(1)}$ has càdlàg sample paths. Moreover, suppose that $\nu \in RV(\alpha, (a_n), \mu_\nu)$. Then $X_t^{(1)}$ satisfies*

$$\lim_{n\to\infty} nP(a_n^{-1} \sup_{t_1\le t_2;\, t_2-t_1\le\delta} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| \ge \varepsilon) = 0$$

*for all $\delta \in (0,1)$ and $\varepsilon > 0$.*

*Proof.* We start by observing that $X_t^{(1)}$ is càdlàg and thus also separable and hence we can estimate $\sup_{t_1 \le t_2;\, t_2 - t_1 \le \delta} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| \le 2 \sup_{t \in [0,1]} \|X_t^{(1)}\| = 2 \sup_{t \in [0,1]\cap\mathbb{Q}} \|X_t^{(1)}\|$. Due to the equivalence of norms, we can now choose the matrix norm $\|A\| := \max\{|a_{ij}| : 1 \le i \le n \text{ and } 1 \le j \le d\}$ for $A \in M_{n,d}$ and denote by $X_{t,i}^{(1)} \in \mathbb{R}$, $1 \le i \le n$, the $i$-th component of $X_t^{(1)}$, i.e. $X_t^{(1)} = \left(X_{t,1}^{(1)}, X_{t,2}^{(1)}, \ldots, X_{t,n}^{(1)}\right)^T$. Furthermore, define the (countable) set $\widetilde{T} := \{(t,i) : t \in [0,1]\cap\mathbb{Q} \text{ and } i \in \{1,\ldots,n\}\}$. Then we obtain $\sup_{t\in[0,1]\cap\mathbb{Q}} \|X_t^{(1)}\| = \sup_{t\in[0,1]\cap\mathbb{Q}} \max_{1\le i\le n} \|X_{t,i}^{(1)}\| = \sup_{s\in\widetilde{T}} \|X_s^{(1)}\|$, where $\sup_{s\in\widetilde{T}}$ is a subadditive functional on $\mathbb{R}^{\widetilde{T}}$. Furthermore, by Theorem 2.2 the processes $X_{t,i}^{(1)}$ are infinitely divisible with specified characteristic triplet $(\gamma_{t,i}, \Sigma_{t,i}, \nu_{t,i})$ and Lévy measure

$$\nu_{t,i}(B) = \int_{M_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_B(f_i(A, t-s)x)\nu_1(dx)ds\pi(dA)$$

for all $B \in \mathcal{B}(\mathbb{R})$, where $f_i$ denotes the $i$-th row of $f$, i.e.

$$f(A, t-s) = \begin{pmatrix} f_1(A, t-s) \\ \vdots \\ f_d(A, t-s) \end{pmatrix}.$$

It follows that $\mathbf{X}^{(1)} = \{X_s^{(1)} : x \in \widetilde{T}\}$ is infinitely divisible with characteristic triplet $(\widetilde{\gamma}, \widetilde{\Sigma}, \widetilde{\nu})$, where $\widetilde{\gamma}$, $\widetilde{\Sigma}$ and $\widetilde{\nu}$ are given as projective limits of the corresponding finite dimensional characteristics described by $(\gamma_{t,i}, \Sigma_{t,i}, \nu_{t,i})$ (cf. [33]). Moreover, the boundedness $\|f\| \le C$ implies $\|f_i\| \le C$ and this, together with the definition of $\nu_1 = \nu|_{B_1(0)}$, yields that the support of the Lévy measures $\nu_{t,i}$ and $\widetilde{\nu}$ can be bounded by $C$. Now we are able to apply Lemma 2.1 of [11] to obtain $\mathbb{E}\left(\exp\left(\varepsilon \sup_{s\in\widetilde{T}} \|X_s^{(1)}\|\right)\right) < \infty$ for all $\varepsilon > 0$. Finally, the finite exponential moments in combination with Lemma 1.32 of [29] yield

$$\lim_{n\to\infty} nP\Big(a_n^{-1} \sup_{t_1 \le t_2;\, t_2 - t_1 \le \delta} \|X_{t_1}^{(1)} - X_{t_2}^{(1)}\| \ge \varepsilon\Big) \le \lim_{n\to\infty} nP\Big(a_n^{-1} \sup_{s\in\widetilde{T}} \|X_s^{(1)}\| \ge \varepsilon/2\Big) = 0$$

for all $\varepsilon > 0$. $\square$

Next we check the process $X_t^{(2)}$ with respect to the relative compactness conditions (4.2), (4.3) and (4.4).

**Proposition 4.18.** *Let $\Lambda$ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ and let $\nu \in RV(\alpha, (a_n), \mu_\nu)$. Assume that the kernel function $f$ is bounded, the MMA process $X_t^{(2)} = \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s)\Lambda^{(2)}(dA, ds)$ satisfies the existence conditions of Proposition 4.16 and that the regular variation conditions of Theorem 2.4 hold. If the function $f_\delta$ given by (4.6) satisfies the existence condition of Proposition 4.16 and, as $\delta \to 0$, $\int_{M_d^-} \int_{\mathbb{R}} f_\delta(A, s)^\alpha \, ds\pi(dA) \to 0$, then $X_t^{(2)}$ given by (4.11) satisfies the relative compactness conditions (4.2), (4.3) and (4.4).*

*Proof.* We define the difference function $g_{t_1,t_2}(A,s) := f(A,t_1-s) - f(A,t_2-s)$ and mention that for every $t_1,t_2 \in [0,1]$ the random vector

$$X^{(2)}_{t_1} - X^{(2)}_{t_2} = \int\limits_{\|x\|\geq 1}\int\limits_{M_d^-}\int\limits_{\mathbb{R}} g_{t_1,t_2}(A,s)\; x\; N(dx,dA,ds)$$

is again an integral similar to an MMA and by Theorem 2.3 and Theorem 2.4 it exists and is regularly varying with index $\alpha$.

*Condition* (4.2): We verify the condition by showing that, as $\delta \to 0$, $\lim\limits_{n\to\infty} nP(a_n^{-1} \sup\limits_{t_1,t_2\in[0,\delta)} \|X^{(2)}_{t_1} - X^{(2)}_{t_2}\| \geq \varepsilon) \to 0$ for every $\varepsilon > 0$. We use the decomposition

$$\begin{aligned}
X^{(2)}_{t_1} - X^{(2)}_{t_2} &= \\
&= \int\limits_{\|x\|\geq 1}\int\limits_{M_d^-}\int\limits_{(t_1,t_2]} g_{t_1,t_2}(A,s)\; x\; N(dx,dA,ds) + \int\limits_{\|x\|\geq 1}\int\limits_{M_d^-}\int\limits_{(t_1,t_2]^c} g_{t_1,t_2}(A,s)\; x\; N(dx,dA,ds) \\
&=: Z^{(1)}_{t_1,t_2} + Z^{(2)}_{t_1,t_2} \qquad (4.12)
\end{aligned}$$

which yields

$$\begin{aligned}
nP\Big(a_n^{-1} &\sup_{t_1,t_2\in[0,\delta)} \|X^{(2)}_{t_1} - X^{(2)}_{t_2}\| \geq \varepsilon\Big) \leq \\
&\leq nP\Big(a_n^{-1} \sup_{t_1,t_2\in[0,\delta)} \|Z^{(1)}_{t_1,t_2}\| \geq \varepsilon/2\Big) + nP\Big(a_n^{-1} \sup_{t_1,t_2\in[0,\delta)} \|Z^{(2)}_{t_1,t_2}\| \geq \varepsilon/2\Big). \qquad (4.13)
\end{aligned}$$

With $\nu_2 = \nu|_{B_1(0)^c}$ and using the transformation $T : \mathbb{R}^d \to \mathbb{R}$ given by $T(x) = \|x\|$ together with the boundedness $f(A,s) \leq C$ for all $(A,s) \in M_d^- \times \mathbb{R}$ we can now calculate

$$\begin{aligned}
\|Z^{(1)}_{t_1,t_2}\| &\leq \int\limits_{\|x\|\geq 1}\int\limits_{M_d^-}\int\limits_{(t_1,t_2]} \|g_{t_1,t_2}(A,s)\|\, \|x\|\, N(dx,dA,ds) \\
&\leq 2\; C\; \Lambda^{(2,T)}(M_d^- \times (t_1,t_2]) = 2\; C\; (L^{(2,T)}_{t_2} - L^{(2,T)}_{t_1}), \qquad (4.14)
\end{aligned}$$

where $\Lambda^{(2,T)}$ is a Lévy basis with generating quadruple $(0,0,{\nu_2}^T,\pi)$ and the transformed Lévy measure ${\nu_2}^T$ is given by ${\nu_2}^T(\cdot) = \nu_2(T^{-1}(\cdot))$. By $(L^{(2,T)}_t)$ we denote the underlying Lévy process given by $L^{(2,T)}_t = \Lambda^{(2,T)}(M_d^- \times (0,t])$ for $t > 0$. Using a continuous mapping argument similar to [24, Theorem 6], we see that $\nu \in RV(\alpha,(a_n),\mu_\nu)$ implies ${\nu_2}^T \in RV(\alpha,(a_n),\mu_{\nu^T})$ with $\mu_{\nu^T}$ defined respectively. Thus by Proposition 3.1 in [25] $L^{(2,T)}_1 \in RV(\alpha,(a_n),\mu_{\nu^T})$ and then by Example 4.2 also $(L^{(2,T)}_t) \in RV_{\overline{\mathbb{D}}_0}(\alpha,(a_n),\widetilde{\mu})$ for some measure $\widetilde{\mu}$. Now another application of Theorem 4.3 yields that condition (4.2) holds for the process $(L^{(2,T)}_t)$ and hence, as $\delta \to 0$, $\lim\limits_{n\to\infty} nP\Big(a_n^{-1} \sup\limits_{t_1,t_2\in[0,\delta)} \|Z^{(1)}_{t_1,t_2}\| \geq \varepsilon/2\Big) \leq \lim\limits_{n\to\infty} nP\Big(a_n^{-1} \sup\limits_{t_1,t_2\in[0,\delta)} (L^{(2,T)}_{t_2} - L^{(2,T)}_{t_1}) \geq \varepsilon/(4C)\Big) \to 0$.

Similarly, the supremum of the second term $Z^{(2)}_{t_1,t_2}$ can be bounded by

$$\sup_{t_1,t_2\in[0,\delta)} \|Z^{(2)}_{t_1,t_2}\| \leq \int_{\|x\|\geq 1}\int_{M_d^-}\int_{\mathbb{R}} \sup_{t_1,t_2\in[0,\delta)} \|g_{t_1,t_2}(A,s)\|\, \mathbb{1}_{(t_1,t_2]^c}(s)\,\|x\|\, N(dx,dA,ds)$$
$$\leq \int_{\|x\|\geq 1}\int_{M_d^-}\int_{\mathbb{R}} f_\delta(A,s)\,\|x\|\, N(dx,dA,ds) = \int_{M_d^-}\int_{\mathbb{R}} f_\delta(A,s)\Lambda^{(2,T)}(dA,ds) =: Y.$$

Then assumption (4.7) implies $f_\delta \in \mathbb{L}^\alpha$ for some $\delta > 0$ sufficiently small and another application of Theorem 2.4 yields $Y \in RV(\alpha,(a_n),\mu_Y)$ with

$$\mu_Y(B) := \int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathbb{1}_B\left(f_\delta(A,s)\|x\|\right)\mu_\nu(dx)ds\pi(dA).$$

Finally, as $n\to\infty$, we obtain

$$\begin{aligned} nP\Big(a_n^{-1}\sup_{t_1,t_2\in[0,\delta)} \|Z^{(2)}_{t_1,t_2}\| \geq \varepsilon/2\Big) &\leq nP(a_n^{-1}\,Y \geq \varepsilon/2) \\ &\overset{n\to\infty}{\to} \int_{M_d^-}\int_{\mathbb{R}} \mu_\nu(x : f_\delta(A,s)\|x\| \geq \varepsilon/2)\, ds\,\pi(dA) \\ &= \mu_\nu(x : \|x\| \geq \varepsilon/2)\int_{M_d^-}\int_{\mathbb{R}} f_\delta(A,s)^\alpha\, ds\,\pi(dA) \overset{\delta\to 0}{\to} 0 \end{aligned}$$

and since $\mu_\nu$ is a Radon measure the result follows by the assumption.

*Condition* (4.3)*:* The condition follows likewise to condition (4.2) (note also that the MMA process $(X_t)$ is stationary).

*Condition* (4.4)*:* For the third condition we use (4.12) again and obtain

$$\begin{aligned} &\sup_{t_1\leq t\leq t_2;\, t_2-t_1\leq\delta} \min\left\{\|X^{(2)}_t - X^{(2)}_{t_2}\|, \|X^{(2)}_{t_1} - X^{(2)}_t\|\right\} \leq \\ &\qquad\leq \sup_{t_1\leq t\leq t_2;\, t_2-t_1\leq\delta} \min\left\{\|Z^{(1)}_{t,t_2}\|, \|Z^{(1)}_{t_1,t}\|\right\} + \sup_{t_1\leq t_2;\, t_2-t_1\leq\delta} \|Z^{(2)}_{t_1,t_2}\| \end{aligned}$$

and

$$\begin{aligned} &nP\Big(a_n^{-1}\sup_{t_1\leq t\leq t_2;\, t_2-t_1\leq\delta} \min\left\{\|X^{(2)}_t - X^{(2)}_{t_2}\|, \|X^{(2)}_{t_1} - X^{(2)}_t\|\right\} \geq \varepsilon\Big) \leq \\ &\leq nP\Big(a_n^{-1}\sup_{t_1\leq t\leq t_2; t_2-t_1\leq\delta} \min\left\{\|Z^{(1)}_{t,t_2}\|, \|Z^{(1)}_{t_1,t}\|\right\} \geq \frac{\varepsilon}{2}\Big) + nP\Big(a_n^{-1}\sup_{t_1\leq t_2; t_2-t_1\leq\delta} \|Z^{(2)}_{t_1,t_2}\| \geq \frac{\varepsilon}{4}\Big). \end{aligned}$$

Applying (4.14) this implies, as $\delta\to 0$,

$$\begin{aligned} &\lim_{n\to\infty} nP\Big(a_n^{-1}\sup_{t_1\leq t\leq t_2;\, t_2-t_1\leq\delta} \min\left\{\|Z^{(1)}_{t,t_2}\|, \|Z^{(1)}_{t_1,t}\|\right\} \geq \varepsilon/2\Big) \\ &\quad\leq \lim_{n\to\infty} nP\Big(a_n^{-1}\sup_{t_1\leq t\leq t_2;\, t_2-t_1\leq\delta} \min\left\{\|L^{(2,T)}_{t_2} - L^{(2,T)}_t\|, \|L^{(2,T)}_t - L^{(2,T)}_{t_1}\|\right\} \geq \varepsilon/(4\,C)\Big) \to 0, \end{aligned}$$

since this is exactly condition (4.4) for the Lévy process $L_t^{(2,T)}$ which is regularly varying in $\mathbb{D}$ and thus by Theorem 4.3 satisfies (4.4). Furthermore,

$$\sup_{t_1\le t_2;\, t_2-t_1\le\delta}\|Z^{(2)}_{t_1,t_2}\| \le \int_{M_d^-}\int_{\mathbb{R}} f_\delta(A,s)\Lambda^{(2,T)}(dA,ds) = Y$$

and consequently, as $\delta\to 0$, $\lim_{n\to\infty} nP\Big(a_n^{-1}\sup_{t_1\le t_2;\, t_2-t_1\le\delta}\|Z^{(2)}_{t_1,t_2}\|\ge\varepsilon/4\Big)\le nP\Big(a_n^{-1}Y\ge \varepsilon/4\Big)\to 0$ as shown for condition (4.2). □

Finally, the results on which càdlàg functions $\mu$ is concentrated and (4.8) follows from Theorem 2.4 and the fact that its "finite dimensional distributions" uniquely determine a measure on $\overline{\mathbb{D}}_0$, since $\mu$ as defined in (4.8) gives the finite dimensional $\mu_{t_1,\dots,t_k}$ of Theorem 2.4. This concludes the proof of Theorem 4.5.

## 5. Application to SupOU Processes

Superpositions of Ornstein-Uhlenbeck processes (supOU processes) have useful properties and a wide range of applications. A supOU process $(X_t)$ can be defined as an MMA process with kernel function

$$f(A,s) = e^{As}\mathbb{1}_{[0,\infty)}(s). \tag{5.1}$$

We will shortly recall the main results of [3] and [34]. Sufficient conditions for the existence of supOU processes are given in the following theorem which takes the special properties of supOU processes into account.

**Theorem 5.1** ([3], Theorem 3.1). *Let $X_t$ be an $\mathbb{R}^d$-valued supOU process as defined by (5.1). If $\int_{\|x\|>1}\ln(\|x\|)\nu(dx)<\infty$ and there exist measurable functions $\rho: M_d^-\to\mathbb{R}^+\backslash\{0\}$ and $\kappa: M_d^-\to[1,\infty)$ such that*

$$\left\|e^{As}\right\|\le\kappa(A)e^{-\rho(A)s}\ \forall s\in\mathbb{R}^+\ \pi\text{-almost surely and}\ \int_{M_d^-}\frac{\kappa(A)^2}{\rho(A)}\pi(dA)<\infty,$$

*then the supOU process $X_t=\int_{M_d^-}\int_{-\infty}^t e^{A(t-s)}\Lambda(dA,ds)$ is well defined for all $t\in\mathbb{R}$ and stationary. Furthermore, the stationary distribution of $X_t$ is infinitely divisible with characteristic triplet $(\gamma_X,\Sigma_X,\nu_X)$ given by Theorem 2.2.*

Conditions for regular variation of $X_t$ and of the finite-dimensional distributions of $(X_t)$ are given by the following result.

**Corollary 5.2** ([34], Cor. 4.3 and Cor. 4.6). *Let $\Lambda\in\mathbb{R}^d$ be a Lévy basis on $M_d^-\times\mathbb{R}$ with generating quadruple $(\gamma,\Sigma,\nu,\pi)$ and let $\nu\in RV(\alpha,(a_n),\mu_\nu)$. If the conditions of Theorem 5.1 hold and additionally $\int_{M_d^-}(\kappa(A)^\alpha/\rho(A))\,\pi(dA)<\infty$, then $X_0=\int_{M_d^-}\int_{\mathbb{R}^+}e^{As}\Lambda(dA,ds)\in RV(\alpha,(a_n),\mu_X)$ with Radon measure*

$$\mu_X(\cdot) := \int_{M_d^-}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}\mathbb{1}_{(\cdot)}\left(e^{As}x\right)\mu_\nu(dx)ds\pi(dA).$$

*Furthermore, the finite dimensional distributions $(X_{t_1},\dots,X_{t_k})$, $t_i\in\mathbb{R}$ and $k\in\mathbb{N}$, are also regularly varying with index $\alpha$ and given limiting measure $\mu_{t_1,\dots,t_k}$.*

In order to apply Theorem 4.5 to obtain conditions for regular variation of supOU processes in $\mathbb{D}$, we state some useful sufficient conditions for the function

$$f_\delta(A,s) = \sup_{t_1 \le t_2;\, t_2 - t_1 \le \delta} \|f(A, t_2 - s) - f(A, t_1 - s)\| \, \mathbb{1}_{(t_1,t_2]^c}(s)$$

to be an element of $\mathbb{L}^\alpha$ for $\alpha > 0$.

**Proposition 5.3.** *Let $f(A,s) = e^{As}\mathbb{1}_{[0,\infty)}(s)$ be the kernel function of a supOU process satisfying the conditions of Theorem 5.1 and let $f_\delta$ be given by (4.6). If $f$ is bounded on supp$(\pi) \times \mathbb{R}^+$ and for some $\alpha > 0$*

$$\int_{M_d^-} \frac{\kappa(A)^\alpha}{\rho(A)} \, \pi(dA) < \infty,$$

*then $f_\delta \in \mathbb{L}^\alpha$ for every $0 \le \delta \le 1$.*

It should be noted that we require $f$ to be bounded, but not $\kappa$. However, in contrast to Corollary 5.1 we demand $(A,s) \mapsto e^{As}1_{\mathbb{R}^+}(s)$ to be bounded at least for all $A$ that may possibly occur. The reason is that if $\pi$ has mass on the non-unitarily diagonalizable or even non-diagonalisable elements of $M_d^-$ this is not clear, although $e^{A0} = I_d$ for all $A \in M_d^-$.

*Proof.* This follows from Lemma 4.13, since the assumptions imply that $(s,A) \to \kappa(A)e^{-\rho(A)s}$ is in $L^\alpha$. Furthermore, $f_1 \in \mathbb{L}^\alpha$ implies $f_\delta \in \mathbb{L}^\alpha$ for every $0 \le \delta \le 1$. □

Now we can use Proposition 5.3 to obtain conditions for functional regular variation of supOU processes with sample paths in $\mathbb{D}$. Therefore, we restrict the time interval to $t \in [0,1]$ and assume the supOU process to have càdlàg sample paths, see Section 3 and [3], Theorem 3.12, for details on the sample path behavior of supOU processes.

**Theorem 5.4.** *Let $\Lambda$ and $\Lambda_2$ be $\mathbb{R}^d$-valued Lévy bases on $M_d^- \times \mathbb{R}$ with generating quadruples $(\gamma, \Sigma, \nu, \pi)$ and $(0, 0, \nu|_{B_1(0)^c}, \pi)$ respectively such that $\nu \in RV(\alpha, (a_n), \mu_\nu)$. Assume that the supOU process $(X_t)$ given by $X_t = \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)}\Lambda(dA, ds)$ exists for $t \in [0,1]$ (in the sense of Theorem 5.1) and that the processes $X_t$ and $X_t^{(2)} = \int_{M_d^-} \int_{\mathbb{R}} f(A, t-s)\Lambda_2(dA, ds)$ have càdlàg sample paths. Furthermore, suppose that $f(A,s) = e^{As}1_{\mathbb{R}^+}(s)$ is bounded on supp$(\pi) \times \mathbb{R}^+$ and that*

$$\int_{M_d^-} \frac{\kappa(A)^\alpha}{\rho(A)} \, \pi(dA) < \infty.$$

*If $f(A,s) = e^{As}\mathbb{1}_{[0,\infty)}(s)$ and the function $f_\delta$ given by (4.6) satisfy condition (4.5), then*

$$(X_t)_{t \in [0,1]} \in RV_{\overline{\mathbb{D}}_0}(\alpha, (a_n), \mu).$$

*The measure $\mu$ is uniquely determined by the measures $\mu_{t_1,\dots,t_k}$ in Corollary 5.2, concentrated on the càdlàg functions of the form $[0;1] \to \mathbb{R}^n$, $s \mapsto e^{A(s-t)}x\mathbb{1}_{[t,1]}(s)$ where $A \in$ supp$(\pi), t \in (-\infty, 1]$ and $x \in$ supp$(\mu_\nu)$ and we have that*

$$\mu\left\{e^{A(\cdot - t)}x\mathbb{1}_{[t,1]}(s) :\, A \in \mathcal{A},\, t \in T,\, x \in \mathcal{X}\right\} = \lambda(T)\pi(\mathcal{A})\mu_\nu(\mathcal{X}) \tag{5.2}$$

*for measurable sets $\mathcal{A} \subseteq M_d^-$, $T \subseteq (-\infty, 1]$ and $\mathcal{X} \subseteq \mathbb{R}^d$.*

The condition that $f(A,s) = e^{As}1_{\mathbb{R}^+}(s)$ is bounded on $\operatorname{supp}(\pi) \times \mathbb{R}^+$ is, of course, satisfied if $\kappa$ can be taken bounded.

*Proof.* Combine Theorem 4.5, Lemmata 4.11, 4.13 and Remark 4.12. □

Conditions for $f$ and $f_\delta$ to satisfy the existence condition (4.5), i.e.

$$\int_{M_d^-} \int_{\mathbb{R}} \int_{\|x\|>1} (1 \wedge \|f(A,s)x\|)\, \nu(dx)\, ds\, \pi(dA) < \infty$$

can be obtained by combining Proposition 5.3 with Lemma 4.9 and [3, Proposition 3.5].

**Corollary 5.5.** *Let $f(A,s) = e^{As}\mathbb{1}_{[0,\infty)}(s)$ be the kernel function of a supOU process satisfying the conditions of Theorem 5.1 and let $f_\delta$ be given by* (4.6)*. Then $f$ and $f_\delta$ satisfy condition* (4.5) *if one of the following two conditions are satisfied:*

(i) $\alpha > 1$ *as well as*

$$\int_{M_d^-} \frac{\kappa(A)}{\rho(A)}\, \pi(dA) < \infty.$$

(ii) $\alpha \leq 1$ *and there exists* $\varepsilon \in (0, \alpha)$ *such that*

$$\int_{M_d^-} \frac{\kappa(A)^{\alpha-\varepsilon}}{\rho(A)}\, \pi(dA) < \infty.$$

## 6. Point Process Convergence

In this section we discuss the use of the results of the previous two sections in combination with point process results for stochastic processes with sample paths in $\mathbb{D}$. Therefore, let $M_p(\overline{\mathbb{D}}_0)$ denote the space of all point measures on $\overline{\mathbb{D}}_0$ equipped with the $\hat{w}$-topology and let $\varepsilon_x$ be the Dirac measure at the point $x$. Furthermore, let $X_i$, $i \in \mathbb{N}$, be a sequence of iid copies of a regularly varying stochastic process $X \in RV_{\overline{\mathbb{D}}_0}(\alpha, (a_n), \mu)$ with values in $\mathbb{D}$.

We start by stating the main result that links regular variation of $X$ to weak convergence of the point processes

$$N_n = \sum_{i=1}^{n} \varepsilon_{a_n^{-1} X_i}, \quad n \in \mathbb{N}.$$

The following theorem is the extension of the classical result of Proposition 3.21 in [38] to a state space which is not locally compact. Similar results have also been proved by [18], Theorem 2.4, in the case of real-valued processes which are regularly varying with index 1 and by [16] for $\mathbb{D}$-valued random fields.

**Theorem 6.1.** *Let $(X_i)_{i\in\mathbb{N}}$ be an iid sequence of stochastic processes with values in $\mathbb{D}$. Then $X_1 \in RV_{\overline{\mathbb{D}}_0}(\alpha, (a_n), \mu)$ if and only if $N_n \xrightarrow{d} N$ in $M_p(\overline{\mathbb{D}}_0)$, where $N$ is a Poisson random measure with mean measure $\mu$ (short PRM($\mu$)).*

*Proof.* The proof can be obtained by changing from vague-topology to the $\hat{w}$-topology in the proof of Proposition 3.21 in [38]. This change of topology does not affect the proof which is based on the Laplace functionals of the point processes involved (cf. [16], Proof of Lemma 2.2). □

This result can now be combined with the results of Sections 4 and 5 to obtain functional point process convergence for MMA and supOU processes. Point processes of that kind include full information of the complete paths of the process $X$. In combination with the continuous mapping theorem (cf. [15], Proposition A2.3.V) this is an extremely powerful tool to analyze the extremal behavior of MMA and supOU processes. Using such methods, one gets a better understanding of the structure of the extreme values and their properties, e.g. the extremal clustering behavior or long memory effects.

In contrast to finite-dimensional point process results, functional point process convergence does not only allow to analyze, for example, the behavior of maxima at fixed time points, but also of functionals acting on the paths of the process in compact time intervals. Examples of such functionals are the subadditive functionals (e.g. suprema) studied by [42] for a subexponential, by [11] for an exponential, and by [10] for a univariate regularly varying setting. Moreover, since point processes of suprema do not incorporate the directions of the extremes, it is also possible to include the directions into the analyzed point processes. Regarding the above issues introductions to the use of point processes in extreme value theory can be found in [19], [38], [39], [28] and [17] and for the exemplary use of functional point processes, see [18] and [16].

A thorough investigation of all these issues for mixed moving average processes is beyond the scope of the present paper and the topic of future research.

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