On the Relation Between the vec and BEKK Multivariate GARCH Models

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Abstract

The question which multivariate GARCH models in the vec form are representable in the BEKK form is addressed. Using results from linear algebra, it is established that all vec models not representable in the simplest BEKK form contain matrices as parameters which map the vectorised positive semi-definite matrices into a strict subset of themselves. Moreover, a general result from linear algebra is presented implying that in dimension two the models are equivalent and in dimension three a simple analytically tractable example for a vec model having no BEKK representation is given.

Keywords:
BEKK model, linear preserver problems, multivariate GARCH model, vec model

1 Introduction

Multivariate GARCH models have been studied intensively in recent years and many different specifications have been used in the literature (cf. Bauwens, Laurent & Rombouts (2006) for a comprehensive overview and Boussama (1998, 2006) for a detailed discussion on strict stationarity and geometric ergodicity). In this paper we present some results on the relationship between the vec and BEKK models. These models have been presented and analysed in detail in Engle & Kroner (1995). In that paper it has been noted that all BEKK models are representable as vec ones, but regarding the converse it has only

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been shown that all diagonal vec models are representable as diagonal BEKK ones and stated that the BEKK parametrisation "eliminates very few if any interesting models allowed by the vec representation". However, apart from the recent paper by Scherrer & Ribarits (2007), which came to our attention only after finishing the work on the present paper, no further results on the relationship between the two models seem to have been obtained since then, nor are there simple and tractable examples of vec models which are not representable in the BEKK form to be found in the literature.

Applying long known results from linear algebra in a straightforward manner, we show in this paper that in dimension two the models are actually equivalent and that all vec-models not representable in the simplest BEKK form with invertible parameter matrices exhibit necessarily some degeneracy, viz. that one of the matrices appearing in the vec model is degenerated in the sense that it maps the vectorised positive semi-definite matrices to a strict subset of themselves. Finally, we present an example of a vec model with no BEKK representation in dimension three. Comparing our results to those of Scherrer & Ribarits (2007) they have shown the equivalence in dimension two using semi-definite programming, whereas we note that it is an immediate consequence of a long known result in linear algebra. The linear algebra literature we are referring to seems not to have been used in connection to GARCH models before, but it is obviously intimately connected to multivariate GARCH models and should be useful to obtain other results as well. For an example of a vec model having no BEKK representation Scherrer & Ribarits (2007) refer to Ribarits (2006). The example presented in that thesis on page 61 (stated in a transformed way only) is of a rather complicated structure and it is argued by numerical optimisation and not an analytical proof that it gives an admissible vec term which cannot be represented in the BEKK form. In contrast to this we present a very simple example with interesting properties which is analysed completely analytically.

The remainder of this paper is organised as follows. We briefly state the necessary definitions of multivariate GARCH models in the next section and then present our results in Section 3.

Regarding notation we denote the set of real $d \times d$ matrices by $M_d(\mathbb{R})$, the group of invertible $d \times d$ matrices by $GL_d(\mathbb{R})$, the linear subspace of symmetric matrices by $\mathbb{S}_d$ and the positive semi-definite cone by $\mathbb{S}_d^+$. Finally, A^T is the transposed of a matrix $A \in M_d(\mathbb{R})$.

2 Multivariate GARCH processes

The well-known single dimensional GARCH(p, q) model introduced in Bollerslev (1986) is defined via an i.i.d. sequence $(\epsilon_n)_{n\in\mathbb{N}}$ and the equations

$$X_n = \sqrt{\sigma_n^2}\epsilon_n \tag{1}$$

$$\sigma_n^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i X_{n-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{n-j}^2 \tag{2}$$

for $n \in \mathbb{N}$. Moreover, the initial values $\sigma_0^2, \sigma_{-1}^2, \ldots, \sigma_{1-p}^2$ and the parameters $\alpha_1, \ldots, \alpha_q$, $\beta_1, \ldots, \beta_p$ are non-negative and $\alpha_0 > 0$. $X = (X_n)_{n\in\mathbb{N}}$ is referred to as a GARCH(p,q) process and σ^2 is its latent conditional variance process.

When one moves from a scalar X to a d-dimensional X, the variance process σ^2 becomes a $d \times d$ covariance matrix process Σ and one uses the vec (or alternatively vech) transformation in order to specify the model. The vec transformation maps the $d \times d$ matrices bijectively to $\mathbb{R}^{d^2}$ by stacking the columns of a matrix below one another. This leads to the vec-model (Engle & Kroner (1995)) which is given by:

$$X_n = \Sigma_n^{1/2}\epsilon_n \tag{3}$$

$$\mathrm{vec}(\Sigma_n) = \mathrm{vec}(C) + \sum_{i=1}^{q} \tilde{A}_i \mathrm{vec}(X_{n-i}X_{n-i}^T) + \sum_{j=1}^{p} \tilde{B}_j \mathrm{vec}(\Sigma_{n-j}). \tag{4}$$

for $n \in \mathbb{N}$ where $(\epsilon_n)_{n\in\mathbb{N}}$ is now an $\mathbb{R}^d$-valued i.i.d. sequence and $\Sigma_n^{1/2}$ denotes the unique positive semi-definite matrix whose square is Σ_n, i.e. $\Sigma_n^{1/2} \in \mathbb{S}_d^+$ and $\Sigma_n^{1/2}\Sigma_n^{1/2} = \Sigma_n$. To ensure the positive semi-definiteness of the process Σ the initial values and C have to be positive semi-definite and $\tilde{A}_1, \ldots, \tilde{A}_q, \tilde{B}_1, \ldots, \tilde{B}_p$ need to be $d^2 \times d^2$ matrices mapping the vectorised positive semi-definite matrices into themselves.

For notational convenience we shall not only use the vec-model in the following, but also an obviously equivalent specification defined directly on the symmetric matrices. This model, referred to as the "general d-dimensional GARCH(p,q) model" in the following, is given by

$$X_n = \Sigma_n^{1/2}\epsilon_n \tag{5}$$

$$\Sigma_n = C + \sum_{i=1}^{q} A_i X_{n-i}X_{n-i}^T + \sum_{j=1}^{p} B_j \Sigma_{n-j}. \tag{6}$$

The only difference to the vec-model is that $A_1, \ldots, A_q$ and $B_1, \ldots, B_p$ are now linear operators from $\mathbb{S}_d$ to $\mathbb{S}_d$ that map the positive semi-definite $d \times d$ matrices into themselves, i.e. $A_i(\mathbb{S}_d^+) \subseteq \mathbb{S}_d^+$ and $B_j(\mathbb{S}_d^+) \subseteq \mathbb{S}_d^+$ for $i = 1, \ldots, q$ and $j = 1, \ldots, p$.

The restrictions on the linear operators A_i and B_j (or $\tilde{A}_i$ and $\tilde{B}_j$ in the vec-model) necessary to ensure positive semi-definiteness led to the introduction of the so-called BEKK model (see again Engle & Kroner (1995)), which automatically ensures positive semi-definiteness:

$$X_n = \Sigma_n^{1/2}\epsilon_n \tag{7}$$

$$\Sigma_n = C + \sum_{i=1}^{q}\sum_{k=1}^{l_i} \bar{A}_{i,k} X_{n-i}X_{n-i}^T \bar{A}_{i,k}^T + \sum_{j=1}^{p}\sum_{r=1}^{s_j} \bar{B}_{j,r}\Sigma_{n-j}\bar{B}_{j,r}^T, \tag{8}$$

where $\bar{A}_{i,k}, \bar{B}_{j,r}$ are now arbitrary elements of $M_d(\mathbb{R})$. It is standard that the BEKK model is equivalent to the vec model with $\tilde{A}_i = \sum_{k=1}^{l_i} \bar{A}_{i,k} \otimes \bar{A}_{i,k}$ and $\tilde{B}_j = \sum_{r=1}^{s_j} \bar{B}_{j,r} \otimes \bar{B}_{j,r}$ with $\otimes$ denoting the tensor (Kronecker) product.

3 The relationship between the vec and BEKK model

From the definitions of the models it is clear that studying the relationships between the vec (or general) multivariate GARCH and the BEKK model further is intrinsically related to characterising the linear operators on $\mathbb{S}_d$ that map the positive semi-definite matrices into themselves. The latter has been studied for a long time in linear algebra under the general topic "Linear Preserver Problems" (see, for instance, the overview articles Pierce, Lim, Loewy, Li, Tsing, McDonald & Beasley (1992) and Li & Pierce (2001)). From the results obtained there we need the following:

Proposition 3.1. *Let $\mathbf{A} : \mathbb{S}_d \to \mathbb{S}_d$ be a linear operator. Then:*

1. *$\mathbf{A}(\mathbb{S}_d^+) = \mathbb{S}_d^+$, if and only if there exists a matrix $A \in GL_d(\mathbb{R})$ such that $\mathbf{A}$ can be represented as $X \mapsto AXA^T$.*

2. *For $d = 2$, $\mathbf{A}(\mathbb{S}_d^+) \subseteq \mathbb{S}_d^+$, if and only if there is an $r \in \mathbb{N}$ and $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_r \in M_d(\mathbb{R})$ such that $\mathbf{A}$ can be represented as*

$$X \mapsto \sum_{i=1}^{r} \bar{A}_i X \bar{A}_i^T.$$

Proof. (a) was initially proved in Schneider (1965), alternatively a more general proof in a Hilbert space context may be found in Li, Rodman & Semrl (2003). (b) was established in Størmer (1963) (cf. also Loewy (1992)). ■

From this we can immediately infer the relations between the general (or equivalently vec) multivariate GARCH model and the BEKK model:

Theorem 3.2. 1. *For $d \leq 2$ the general (or* vec*) multivariate GARCH model and the BEKK model are equivalent.*

2. *Every general multivariate GARCH(p,q) model satisfying $A_i(\mathbb{S}_d^+) = \mathbb{S}_d^+$ and $B_j(\mathbb{S}_d^+) = \mathbb{S}_d^+$ for $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, p$ can be represented as a BEKK GARCH(p,q) model with $l_i = s_j = 1 \; \forall \, i, j$ and $\bar{A}_{i,1}, \bar{B}_{j,1} \in GL_d(\mathbb{R})$.*

For the vec model $A_i(\mathbb{S}_d^+) = \mathbb{S}_d^+$ and $B_j(\mathbb{S}_d^+) = \mathbb{S}_d^+$ for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$ translates into demanding that $\tilde{A}_i$ and $\tilde{B}_j$ map the vectorised positive semi-definite matrices onto themselves.

The above result means that when a general (or vec) multivariate GARCH model does not have a BEKK representation with $l_i = s_j = 1 \; \forall \, i, j$ and invertible $\bar{A}_{i,1}, \bar{B}_{j,1}$ it has to be the case that A_i or B_j map the positive semi-definite matrices into a strict subset of themselves for some $i = 1, 2, \ldots, q$ or $j = 1, 2, \ldots, p$. This is a somehow degenerated situation, since it may imply that the distribution of $\Sigma_n - C$ for all $n \in \mathbb{N}$ (and thus any limiting or stationary distribution) is concentrated on a subset of the positive semi-definite matrices.

Let us now turn to providing an example for a vec model that cannot be represented in the BEKK form. Consider $d = 3$ and the linear operator $D : \mathbb{S}_d \to \mathbb{S}_d$ given by

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} + 2x_{22} & -x_{12} & -x_{13} \\ -x_{12} & x_{22} + 2x_{33} & -x_{23} \\ -x_{13} & -x_{23} & x_{33} + 2x_{11}. \end{pmatrix} \tag{9}$$

It has been shown by Choi (1975) that $D(\mathbb{S}_d^+) \subseteq \mathbb{S}_d^+$ and that there exist no $r \in \mathbb{N}$ and $E_1, E_2, \ldots, E_r \in M_3(\mathbb{R})$ such that

$$DX = \sum_{i=1}^{r} E_i X E_i^T \text{ for all } X \in \mathbb{S}_d.$$

Hence, using D as some A_i or B_j gives a general three-dimensional multivariate GARCH model having no BEKK representation. Clearly this means that the corresponding vec models have no BEKK representation. Since the operator D was defined on $\mathbb{S}_d$ only and not on $M_d(\mathbb{R})$, the corresponding 9×9 matrix in the vec model is not unique, but the corresponding vec models are unique. One of the possible 9×9 matrices the operator D corresponds to is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It should be noted that D is an invertible linear operator, as can easily be seen, and that Proposition 3.1 implies $D(\mathbb{S}_d^+) \subset \mathbb{S}_d^+$. An example for a positive semi-definite matrix not being the image of another positive semi-definite matrix under D is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = D \begin{pmatrix} 1/9 & 0 & 0 \\ 0 & 4/9 & 0 \\ 0 & 0 & -2/9 \end{pmatrix}.$$

So we have given an example showing the following:

Proposition 3.3. *For $d \geq 3$ there exist general (or* vec*) multivariate GARCH models that cannot be represented in the BEKK form.*

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