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# Erratum: Multivariate CARMA processes, continuous-time state space models and complete regularity of the innovations of the sampled processes

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A serious flaw in the proof of the equivalence of continuous time state space models and MCARMA processes spotted in Fasen-Hartmann and Schenk (*J. Time Series Anal.* **46** (2025) 692–726) is corrected. We point out that likewise an issue in the proof of Theorem 3.2 in Brockwell and Schlemm (*J. Multivariate Anal.* **115** (2013) 217–251) can be resolved and, hence, any MCARMA process and linear state space model has both a controller and an observer canonical representation. Equivalently, the transfer function has both a left and right matrix fraction representation.

*Keywords:* Multivariate CARMA process; state space representation

## 1. Introduction

Multivariate CARMA processes as introduced in Marquardt and Stelzer (2007) are used in various applications and have also been implemented in R packages (see e.g. Tómasson (2018)). The statistical inference theory developed in Fasen-Hartmann and Kimmig (2020), Fasen and Kimmig (2017), Fasen-Hartmann and Mayer (2022), Fasen-Hartmann and Scholz (2019), Schlemm and Stelzer (2012a), for instance, hinges crucially on the equivalence of the class of Lévy-driven MCARMA processes to the class of Lévy-driven linear state space models, as identifiability is typically ensured by considering state space models in echolon form.

But as observed in Fasen-Hartmann and Schenk (2025) the proof of this equivalence (Corollary 3.4) in Schlemm and Stelzer (2012b) is incorrect, since Appendix 2 of Caines (2018) does not guarantee that the leading coefficient of the "denominator" in the left matrix fraction representation of the transfer function can be chosen to be the identity. The same problem - likewise spotted by Fasen-Hartmann and Schenk (2025) - with right matrix fractions arises in the proof of Theorem 3.2 in Brockwell and Schlemm (2013), where the authors refer to Lemma 6.3-8 of Kailath (1980), which also does not establish that the leading coefficient in the "denominator" can be taken to be the identity.

Hence, we shall prove below that any Lévy-driven linear state space model can be represented in both observer and controller canonical form and that its transfer function always has both a left and a right matrix fraction representation with the "denominator" having the identity as the leading coefficient. This in particular implies that Corollary 3.4 in Schlemm and Stelzer (2012b) and Theorem 3.2 in Brockwell and Schlemm (2013) are correct.

In the following we refer the reader to our original paper Schlemm and Stelzer (2012b) for any unexplained notions and notation. We will refer to equations, definitions, theorems etc. in that paper by putting the letter "P" in front of the respective number.

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## 2. Results

Note first that there is a typo in Equation (P3.4b). In line with Marquardt and Stelzer (2007, Theorem 3.12) it should read

$$\beta_{p-j} = I_{\{0,\dots,q\}}(j)\left[-\sum_{i=1}^{p-j-1} A_i \beta_{p-j-i} + B_{q-j}\right]. \tag{P3.4b}$$

**Proposition 2.1 (Observer canonical form/MCARMA process representation).** *For the output process* $\mathbf{Y}$ *of a Lévy-driven state space model of the form (P3.5) there exist a* $p \in \mathbb{N}$ *and matrices* $\mathcal{A}, \mathcal{B}, C$ *of the form (P3.4) such that* $\mathbf{Y}$ *is the output process of an MCARMA state space representation (linear state space model in observer canonical form) of the form (P3.3).*

**Proof.** Denote by $H(z) = C(z\mathbb{I}_N - A)^{-1}B$ the transfer function of the linear state space model (P3.5). Two linear state space models driven by the same Lévy process produce the same output process $\mathbf{Y}$ if they have the same transfer function (see e.g. Lemma 3.2 of Schlemm and Stelzer (2012a)).

Necessarily, every element of the matrix $H(z)$ is a rational function in $z$ with the degree of the denominator exceeding the degree of the numerator (see p. 106 in Brockett (2015)). Now Proof 2 of Theorem 17.1 of Brockett (2015) gives matrices $\mathcal{A}, \mathcal{B}, C$ of the form (P3.4) with $H(z) = C(z\mathbb{I}_N - \mathcal{A})^{-1}\mathcal{B}$. These matrices define a model of the form (P3.3) with the given output process $\mathbf{Y}$. □

**Proposition 2.2 (Controller canonical form).** *For the output process* $\mathbf{Y}$ *of a Lévy-driven state space model of the form (P3.5) there exist a* $p \in \mathbb{N}$ *and matrices* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *of the form*

$$\mathfrak{A} = \begin{pmatrix} 0 & \mathbb{I}_m & 0 & \dots & 0 \\ 0 & 0 & \mathbb{I}_m & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \mathbb{I}_m \\ -\widetilde{A}_p & -\widetilde{A}_{p-1} & \dots & \dots & -\widetilde{A}_1 \end{pmatrix} \in M_{pm}(\mathbb{R}), \tag{2.1}$$

$$\mathfrak{B} = (0_m, \dots, 0_m, \mathbb{I}_m)^T \in M_{pm,m}(\mathbb{R}) \text{ and} \tag{2.2}$$

$$\mathfrak{C} = \left(\widetilde{B}_0, \dots, \widetilde{B}_{p-1}\right) \in M_{d,pm}(\mathbb{R}) \tag{2.3}$$

*such that* $\mathbf{Y}$ *is the output process of a linear state space model in controller canonical form*

$$\mathrm{d}\widetilde{\boldsymbol{G}}(t) = \mathfrak{A}\widetilde{\boldsymbol{G}}(t)\mathrm{d}t + \mathfrak{B}\mathrm{d}\boldsymbol{L}(t), \quad \boldsymbol{Y}(t) = \mathfrak{C}\widetilde{\boldsymbol{G}}(t), \quad t \in \mathbb{R}. \tag{2.4}$$

**Proof.** The proof is analogous to the one of Proposition 2.1 only that one now uses Proof 1 of Theorem 17.1 of Brockett (2015). □

**Proposition 2.3 (Matrix fraction representations).** *Let* $H$ *be the transfer function of a Lévy-driven state space model of the form (P3.5). There exist* $p, q, \tilde{q} \in \mathbb{N}_0$, $p > q, p \geq \tilde{q}$ *and polynomials* $P \in M_d(\mathbb{R}[z]), Q \in M_{d,m}(\mathbb{R}[z])$ *(left matrix fraction description) as well as* $\widetilde{P} \in M_m(\mathbb{R}[z]), \widetilde{Q} \in M_{d,m}(\mathbb{R}[z])$ *(right matrix fraction description) such that*

$$H(z) = C(z\mathbb{I}_N - A)^{-1}B = P^{-1}(z)Q(z) = \widetilde{Q}(z)\widetilde{P}^{-1}(z)\ \forall z \in \mathbb{C}.$$

*with*

$$P(z) = \mathbb{I}_d z^p + A_1 z^{p-1} + \ldots + A_p, \qquad Q(z) = B_0 z^q + B_1 z^{q-1} + \ldots + B_q,$$
$$\widetilde{P}(z) = \mathbb{I}_m z^p + \widetilde{A}_1 z^{p-1} + \ldots + \widetilde{A}_p, \qquad \widetilde{Q}(z) = \widetilde{B}_0 z^{\tilde{q}} + \widetilde{B}_1 z^{\tilde{q}-1} + \ldots + \widetilde{B}_{\tilde{q}}.$$

**Proof.** *Left matrix fraction:* Let $\mathcal{A}, \mathcal{B}, C$ be the matrices obtained by Proposition 2.1, define $P, p$ by the elements/dimensions of $\mathcal{A}$, set $q = p - \min(\{i = 1, \ldots, p : \beta_i \neq 0\}$ and $B_{q-j} = \beta_{p-j} + \sum_{i=1}^{p-j-1} A_i \beta_{p-j-i}$, $j = 0, \ldots, q$. Defining $Q$ accordingly the arguments in the first step of the proof of Theorem P3.3 combined with Marquardt and Stelzer (2007), Theorem 3.12, establish that $H(z) = P^{-1}(z) Q(z)$.

*Right matrix fraction:* Likewise let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be the matrices obtained by Proposition 2.2 and define $\widetilde{P}, \widetilde{Q}, p, \tilde{q}$ via their elements/dimensions. Then the arguments in the proof of Brockwell and Schlemm (2013), Theorem 3.2, show $H(z) = \widetilde{Q}(z) \widetilde{P}^{-1}(z)$.

That the degrees of the polynomials $P, \widetilde{P}$ agree follows from comparing Proof 1 and 2 of Theorem 17.1 of Brockett (2015). □

Whereas in the original paper and in Brockwell and Schlemm (2013), respectively, the existence of the observer and controller canonical form, respectively, was incorrectly a consequence of the existence of the matrix left and right factorization with the leading coefficient of the "denominator" being the identity, in our above flow of arguments it is now actually the other way round.

**Remark 2.4.** Our results imply also that Proposition 4.3 of Fasen-Hartmann and Scholz (2020), which states the equivalence of the classes of cointegrated MCARMA and linear state space models and which has likewise been proven using Appendix 2 of Caines (2018) without noticing that it does not guarantee the required leading coefficient, remains fully valid.

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