Inheritance of strong mixing and weak dependence under renewal sampling.

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Abstract

Let X be a continuous-time strongly mixing or weakly dependent process and T a renewal process independent of X with inter-arrival times $\{\tau_i\}$. We show general conditions under which the sampled process $(X_{T_i}, \tau_i)^\top$ is strongly mixing or weakly dependent. Moreover, we explicitly compute the strong mixing or weak dependence coefficients of the renewal sampled process and show that exponential or power decay of the coefficients of X is preserved (at least asymptotically). Our results imply that essentially all central limit theorems available in the literature for strongly mixing or weakly dependent processes can be applied when renewal sampled observations of the process X are at disposal.

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Introduction

Time series are ubiquitous in many applications and it is often the case that the time separating successive observations is itself random. We approach the study of such times series by using a continuous time stationary process $X = (X_t)_{t\in\mathbb{R}}$ and a renewal process $T = (T_i)_{i\in\mathbb{Z}}$ which reflects the sampling scheme applied to X. We assume that X is strongly mixing or weakly dependent as defined, respectively, in Rosenblatt (1956) and Dedecker et al. (2008) and that T is a process independent of X with inter-arrival time sequence $\tau = (\tau_i)_{i\in\mathbb{Z}\setminus\{0\}}$. In this general model set-up, we show under which assumptions the renewal sampled process $Y = (Y_i)_{i\in\mathbb{Z}}$ defined as $Y_i = (X_{T_i}, \tau_i)^\top$ inherits strong mixing or weak dependence.

In the literature, the statistical inference methodologies based on renewal sampled data seldom make use of a strongly mixing or weakly dependent process Y. To the best of our knowledge, the only existing example of this approach can be found in Aït Sahalia and Mykland (2004) where it is shown that Y is ρ-strongly mixing and this property is used

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to study the consistency of maximum likelihood estimators for continuous time diffusion processes. On the contrary, there exist several statistical estimators whose asymptotic properties heavily rely on ad-hoc tailor made arguments in specific modeling set-ups. Examples of the kind are the estimators defined by Lii and Masry (1992), Masry (1978a), Masry (1978b), and Masry (1983). In these works, the authors study non-parametric and parametric estimators of the spectral density of X by means of an aliasing-free sampling scheme defined trough a renewal process, see Lii and Masry (1992) for a general definition of this set-up. Such schemes like the Poisson one allow to overcome the aliasing problem which is typically observed with a not band limited process sampled in $\mathbb{Z}$. Moreover, working with an aliasing-free sampling allows to show that the spectral density estimators are consistent and asymptotic normal distributed once assumed that X has finite moments of all orders. Renewal sampled data are also used to define kernel density estimators for strongly mixing processes in Masry (1988), non-parametric estimators of volatility and drift for scalar diffusion in Chorowski and Trabs (2016), and parametric estimators of the covariance function of X as in McDunnough and Wolfson (1979) and Brandes and Curato (2019). In the latter, an estimator of the covariance function of a Gauss-Markov process and a continuous-time Lévy driven moving average are respectively analyzed. In Brandes and Curato (2019), in particular, the asymptotic properties of the estimator are obtained by an opportune truncation of a Lévy driven moving average process X that is strongly mixing.

Determining conditions under which the process Y inherits the asymptotic dependence of X can enlarge the field of applicability of renewal sample data beyond the literature above. Just as indicative examples, our analysis could enable the use of renewal sampled data to study spectral estimators as in Rosenblatt (1984), Whittle estimators as in Bardet et al. (2008), and generalized method of moments estimators as in Curato and Stelzer (2019) and do Rego Sousa and Stelzer (2019). Moreover, the knowledge of the asymptotic dependence of Y allows to apply well-established asymptotic results for α-mixing processes like the ones in (Bradley, 2007, Chapter 10), Dedecker and Rio (2000) and Kanaya (2017). The latter are respectively functional and triangular array central limit theorems. The same argument can be applied to central limit theorems for weakly dependent processes like the ones presented in Bulinski and Sashkin (2005), Dedecker and Doukhan (2003), and Doukhan and Wintenberger (2007). To this end, we give results on the decay rate of the weak dependence or strong-mixing coefficients, cf. (Dedecker et al., 2008, Section 2.2) and (Bradley, 2007, Definiton 3.5), of the process Y which can be used to determine under which conditions central limit theorems for strongly mixing or weak dependent processes can be applied to renewal sampled data.

More specifically, we show the inheritance of η, λ, κ, ζ, θ-weak dependence and α-mixing which are extensively analyzed in the monographs Dedecker et al. (2008), Bradley (2007) and Doukhan (1994), respectively. Moreover, under the additional condition that X admits exponential or power decaying coefficients, we show that Y inherits strong mixing or weak dependence and its related coefficients preserve the exponential or power decay (at least asymptotically).

We present in Section 1 a unified formulation of weak dependence and α-strong mixing conditions. We achieve this by introducing Ψ-weak dependence which encompasses both.

In Section 2, we explicitly compute the weak dependence or strong mixing coefficients of the process Y. Finally, in Section 3, we show that if the underlying process X admits exponential or power decaying coefficients then the process Y is Ψ-weakly dependent and has coefficients with (at least asymptotically) the same decay rate. Therefore, the process Y inherits the asymptotic dependence structure of X. The last section includes several examples of renewal sampling in particular the Poisson one.

1 Weak dependence and strong mixing conditions

We assume that all random variables and processes are defined on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

In the following, we will refer by $\mathbb{N}^*$ to the set of positive integers, by $\mathbb{N}$ to the set of the non-negative integers, by $\mathbb{Z}$ to the set of all integers and by $\mathbb{R}_+$ to the set of the non-negative real numbers. We denote the Euclidean norm by $\|\cdot\|$. However, due to the equivalence of all norms, none of our results depends on the choice of the norm.

Although the theory developed below is probably most relevant for sampling processes defined in continuous time, we work with a general index set $\mathcal{I}$ as this makes no difference and covers also other cases, like a sampling of discrete-time processes or random fields sampled along a walk, e.g. a self-avoiding walk that moves in positive coordinate directions. We refer the reader to Curato et al. (2020) and the references therein for an overview of strongly mixing and weakly dependent random fields. Even if our theory extends to sampling of random fields, we always refer to X as being a process to lighten the reading.

We assume throughout

Definition 1.1. *The index set $\mathcal{I}$ is denoting either $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}^m$ or $\mathbb{R}^m$. Given H and $J \subseteq \mathcal{I}$, we define $d(H,J) = min\{\|i-j\|, i \in H, j \in J\}$.*

Moreover, we consider

$$\mathcal{F} = \bigcup_{u \in \mathbb{N}} \mathcal{F}_u \quad \text{and} \quad \mathcal{G} = \bigcup_{v \in \mathbb{N}} \mathcal{G}_v \tag{1}$$

where $\mathcal{F}_u$ and $\mathcal{G}_v$ are respectively two classes of measurable functions from $(\mathbb{R}^d)^u$ to $\mathbb{R}$ and $(\mathbb{R}^d)^v$ to $\mathbb{R}$ that we specify individually later on. Finally, for a function that is unbounded or not Lipschitz, we set its $\|\cdot\|_\infty$ norm or Lipschitz constant to infinity.

Definition 1.2. *Let $\mathcal{I}$ be an index set as in Definition 1.1, $X = (X_t)_{t \in \mathcal{I}}$ be a process with values in $\mathbb{R}^d$ and Ψ a function from $\overline{\mathbb{R}_+^6}$ to $\mathbb{R}_+$ non-decreasing in all arguments. The process X is called Ψ-weakly dependent if there exists a sequence of coefficients $\iota = (\iota(r))_{r \in \mathbb{R}_+}$ converging to 0 and satisfying the following inequality*

$$|Cov(F(X_{i_1}, \ldots, X_{i_u}), G(X_{j_1}, \ldots, X_{j_v}))| \leq c\,\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v)\,\iota(r) \tag{2}$$

for all

$$\begin{cases} (u,v) \in \mathbb{N}^* \times \mathbb{N}^*; \\ r \in \mathbb{R}_+; \\ I_u = \{i_1, \dots, i_u\} \subseteq \mathcal{I} \text{ and } J_v = \{j_1, \dots, j_v\} \subseteq \mathcal{I}, \text{such that } d(I_u, J_v) \geq r; \\ \text{functions } F\colon (\mathbb{R}^d)^u \to \mathbb{R} \text{ and } G\colon (\mathbb{R}^d)^v \to \mathbb{R} \text{ belonging respectively to } \mathcal{F} \text{ and } \mathcal{G}, \end{cases}$$

where c is a constant independent of r.

W.l.o.g. we always choose the sequence of coefficients ι non-increasing.

Definition 1.2 encompasses the weak dependence conditions as described in Dedecker et al. (2008). For several choices of the function Ψ and $\mathcal{F}, \mathcal{G}$, the coefficients ι are already well-known.

- Let $\mathcal{F} = \mathcal{G}$ and $\mathcal{F}_u$ be the class of bounded Lipschitz functions from $(\mathbb{R}^d)^u$ to $\mathbb{R}$ with respect to the distance δ on $(\mathbb{R}^d)^u$ defined by

$$\delta(x^*, y^*) = \sum_{i=1}^{u} \|x_i - y_i\|, \tag{3}$$

  where $x^* = (x_1, \dots, x_u)$ and $y^* = (y_1, \dots, y_u)$ and $x_i, y_i \in \mathbb{R}^d$ for all $i = 1, \dots, u$. Then, $Lip(F) = \sup_{x \neq y} \frac{|F(x)-F(y)|}{\|x_1-y_1\|+\|x_2-y_2\|+\ldots+\|x_d-y_d\|}$. For

$$\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v) = uLip(F)\|G\|_\infty + vLip(G)\|F\|_\infty,$$

  ι corresponds to the η**-coefficients** as defined by Doukhan and Louhichi (1999). If instead

$$\begin{aligned}\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v) =& uLip(F)\|G\|_\infty + vLip(G)\|F\|_\infty \\ &+ uvLip(F)Lip(G),\end{aligned}$$

  ι corresponds to the λ**-coefficients** as defined by Doukhan and Wintenberger (2007). Moreover, for

$$\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v) = uvLip(F)Lip(G),$$

  ι corresponds to the κ**-coefficients**, and, for

$$\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v) = min(u,v)Lip(F)Lip(G),$$

  ι corresponds to the ζ**-coefficients** as defined by Doukhan and Louhichi (1999).

- Let $\mathcal{F}_u$ be the class of bounded measurable functions from $(\mathbb{R}^d)^u$ to $\mathbb{R}$ and $\mathcal{G}_v$ be the class of bounded Lipschitz functions from $(\mathbb{R}^d)^v$ to $\mathbb{R}$ with respect to the distance δ defined in (3). Then, for

$$\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v) = v\|F\|_\infty Lip(G),$$

  ι corresponds to the θ**-coefficients** as defined by Dedecker and Doukhan (2003).

Remark 1.3. *The weak dependence conditions can all be alternatively formulated by further assuming that $F \in \mathcal{F}$ and $G \in \mathcal{G}$ are bounded by one. For more details on this issue see Doukhan and Louhichi (1999) and Dedecker and Doukhan (2003). Therefore, an alternative definition of Ψ-weak dependence exists where the function Ψ in Definition (2) does not depend on $\|F\|_\infty$ and $\|G\|_\infty$. In this case, $\|F\|_\infty$ and $\|G\|_\infty$ are always bounded by one and therefore omitted in the notation.*

We now show that Definition 1.2 also encompasses α-mixing introduced by Rosenblatt (1956).

We suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are sub-σ-fields of $\mathcal{A}$ and define

$$\alpha(\mathcal{A}_1, \mathcal{A}_2) := \sup_{\substack{A \in \mathcal{A}_1 \\ B \in \mathcal{A}_2}} |P(A \cap B) - P(A)P(B)|.$$

Let $\mathcal{I}$ a set as in Definition 1.1, then a process $X = (X_t)_{t \in \mathcal{I}}$ with values in $\mathbb{R}^d$ is said to be α-mixing if

$$\alpha(r) := \sup\{\alpha(\mathcal{A}_{\Gamma_1}, \mathcal{B}_{\Gamma_2}) : \Gamma_1, \Gamma_2 \subseteq \mathcal{I}, d(\Gamma_1, \Gamma_2) \geq r\}, \tag{4}$$

converges to zero as $r \to \infty$, where $\mathcal{A}_{\Gamma_1} = \sigma(X_i : i \in \Gamma_1)$ and $\mathcal{B}_{\Gamma_2} = \sigma(X_j : j \in \Gamma_2)$. Throughout the paper, $(\alpha(r))_{r \in \mathbb{R}_+}$ are called the α**-coefficients**.

Proposition 1.4. *Let $\mathcal{I}$ be a set as in Definition 1.1 and $X = (X_t)_{t \in \mathcal{I}}$ be a process with values in $\mathbb{R}^d$ and $\mathcal{F} = \mathcal{G}$ where $\mathcal{F}_u$ is the class of bounded measurable functions from $(\mathbb{R}^d)^u$ to $\mathbb{R}$. X is α-mixing if and only if there exists a sequence $(\iota(r))_{r \in \mathbb{R}_+}$ converging to 0 such that*

$$|Cov(F(X_{i_1}, \ldots, X_{i_u}), G(X_{j_1}, \ldots, X_{j_v}))| \leq c\, \Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v)\, \iota(r), \tag{5}$$

where

$$\Psi(\|F\|_\infty, \|G\|_\infty, Lip(F), Lip(G), u, v) = \|F\|_\infty \|G\|_\infty, \tag{6}$$

for all

$$\begin{cases} (u, v) \in \mathbb{N}^* \times \mathbb{N}^*; \\ r \in \mathbb{R}_+; \\ I_u = \{i_1, \ldots, i_u\} \subseteq \mathcal{I} \text{ and } J_v = \{j_1, \ldots, j_v\} \subseteq \mathcal{I}, \text{such that } d(I_u, J_v) \geq r \\ \text{functions } F \colon (\mathbb{R}^d)^u \to \mathbb{R} \text{ and } G \colon (\mathbb{R}^d)^v \to \mathbb{R} \text{ belonging respectively to } \mathcal{F} \text{ and } \mathcal{G} \end{cases}$$

and where c is a constant independent of r.

Proof. Set $\mathcal{A}_{I_u} = \sigma(X_i : i \in I_u)$ and $\mathcal{B}_{J_v} = \sigma(X_j : j \in I_v)$. For arbitrary $(u, v) \in \mathbb{N}^* \times \mathbb{N}^*$ and $r \in \mathbb{R}_+$, let $I_u = \{i_1, \ldots, i_u\}$ and $J_v = \{j_1, \ldots, j_v\}$ be arbitrary subsets of $\mathcal{I}$ such that $d(I_u, J_v) \geq r$. Moreover, choose arbitrary $F \in \mathcal{F}_u$ and $G \in \mathcal{G}_v$,

By Theorem 4.4(a) in Bradley (2007), it holds that

$$|Cov(F(X_{i_1}, \ldots, X_{i_u}), G(X_{j_1}, \ldots, X_{j_v}))| \leq 4\, \alpha(\mathcal{A}_{I_u}, \mathcal{B}_{I_v})\, \|F\|_\infty \|G\|_\infty.$$

Definition (4) immediately implies that the right hand side of the inequality above is smaller than or equal to $4\alpha(r)\|F\|_\infty\|G\|_\infty$. Hence, if X is α-mixing then (5) holds with $\iota(r)=\alpha(r)$ and $c=4$.

We assume now that the sequence X is Ψ-weakly dependent with Ψ given by (6). By Theorem 4.4(a) and Remark 3.17(ii) in Bradley (2007), we can rewrite the definition of the α-coefficients as

$$\alpha(r)=\sup_{\substack{\Gamma_1,\Gamma_2\subseteq\mathcal{I}\\ |\Gamma_1|<\infty,|\Gamma_2|<\infty\\ d(\Gamma_1,\Gamma_2)\geq r}}\alpha(\mathcal{A}_{\Gamma_1},\mathcal{A}_{\Gamma_2})$$

$$=\sup_{(u,v)\in\mathbb{N}\times\mathbb{N}}\sup_{\substack{I_u,J_v\subseteq\mathcal{I}\\ d(I_u,J_v)\geq r}}\sup_{\substack{F\in\mathcal{F}_u\\ G\in\mathcal{G}_v}}\left\{\frac{1}{4\|F\|_\infty\|G\|_\infty}|Cov(F(X_{i_1},\ldots,X_{i_u}),G(X_{j_1},\ldots,X_{j_v}))|\right\}. \tag{7}$$

Hence,

$$\alpha(r)\leq\frac{c}{4}\iota(r).$$

Thus, if X is Ψ-weakly dependent, X is α-mixing.

∎

2 Strong mixing and weak dependence coefficients under renewal sampling

We assume given a strictly stationary $\mathbb{R}^d$-valued process X, i.e. for all $n\in\mathbb{N}$ and all $t_1,\ldots,t_n\in\mathcal{I}$ it holds

$$\mathcal{L}(X_{t_1+h},\ldots,X_{t_n+h})=\mathcal{L}(X_{t_1},\ldots,X_{t_n})\ \ \forall h\in\mathcal{I}.$$

We want to investigate the asymptotic dependence of X sampled at a renewal sequence. We first need the following definitions.

Definition 2.1. *Let $\mathcal{I}\subseteq\mathbb{R}^m$ be a set as in Definition 1.1 and $\tau=(\tau_i)_{i\in\mathbb{Z}\setminus\{0\}}$ be an $\mathcal{I}$-valued sequence of non-negative (component-wise) i.i.d. random vectors with distribution function μ such that $\mu\{0\}<1$. For $i\in\mathbb{Z}$, we define an m-valued stochastic process $(T_i)_{i\in\mathbb{Z}}$ as*

$$T_0:=0\quad\text{and}\quad T_i:=\begin{cases}\sum_{j=1}^{i}\tau_j\,, & i\in\mathbb{N},\\ -\sum_{j=i}^{-1}\tau_j\,, & -i\in\mathbb{N}.\end{cases} \tag{8}$$

The sequence $(T_i)_{i\in\mathbb{Z}}$ is called a renewal sampling sequence. When $\mathcal{I}\subset\mathbb{R}$, we call τ the sequence of the inter-arrival times.

Definition 2.2. *Let* $X = (X_t)_{t\in\mathcal{I}}$ *be a process with values in* $\mathbb{R}^d$ *and let* $(T_i)_{i\in\mathbb{Z}}$ *be a renewal sampling sequence independent of* X*. We define the sequence* $Y = (Y_i)_{i\in\mathbb{Z}}$ *as the stochastic process with values in* $\mathbb{R}^{d+1}$ *given by*

$$Y_i = \begin{pmatrix} X_{T_i} \\ \tau_i \end{pmatrix}. \tag{9}$$

We call X *the underlying process and* Y *the renewal sampled process.*

Remark 2.3. *Definition 2.1 comes from Hunter (1974) and determines the sampling of a random field through self-avoiding paths that move in positive coordinate directions. However, there are other interesting walks in* $\mathbb{R}^m$ *that we could investigate by dropping the non-negativity of the sequence* τ *and using, for example, the definition of a renewal sequence as given in Stam (1969). This latter definition is also compatible with sampling a random field along a walk that moves in lexicographically increasing directions. The study of the asymptotic dependence of such samples is beyond the scope of the present paper but constitutes an interesting future research direction.*

In the following theorem, we work with the class of functions defined in (1) and

$$\tilde{\mathcal{F}} = \bigcup_{u\in\mathbb{N}} \tilde{\mathcal{F}}_u \quad \text{and} \quad \tilde{\mathcal{G}} = \bigcup_{v\in\mathbb{N}} \tilde{\mathcal{G}}_v, \tag{10}$$

where $\tilde{\mathcal{F}}_u$ and $\tilde{\mathcal{G}}_v$ are respectively two classes of measurable functions from $(\mathbb{R}^{d+1})^u$ to $\mathbb{R}$ and $(\mathbb{R}^{d+1})^v$ to $\mathbb{R}$ which can be either bounded or bounded Lipschitz.

Theorem 2.4. *Let* $Y = (Y_i)_{i\in\mathbb{Z}}$ *be a renewal sampled process with the underlying process* X *being strictly stationary and* Ψ*-weakly dependent with coefficients* ι*. Then, there exists a sequence* I *such that*

$$|Cov(\tilde{F}(Y_{i_1},\ldots,Y_{i_u}), \tilde{G}(Y_{j_1},\ldots,Y_{j_v}))| \leq C\,\Psi(\|\tilde{F}\|_\infty, \|\tilde{G}\|_\infty, Lip(\tilde{F}), Lip(\tilde{G}), u, v)\, I(n)$$

for all

$$\begin{cases} (u,v)\in\mathbb{N}^*\times\mathbb{N}^*; \\ n\in\mathbb{N}; \\ \{i_1,\ldots,i_u\}\subseteq\mathbb{Z} \text{ and } \{j_1,\ldots,j_v\}\subseteq\mathbb{Z}, \\ \text{with } i_1\leq\ldots\leq i_u < i_u+n \leq j_1\leq\ldots\leq j_v; \\ \text{functions } \tilde{F}\colon(\mathbb{R}^{d+1})^u\to\mathbb{R} \text{ and } \tilde{G}\colon(\mathbb{R}^{d+1})^v\to\mathbb{R} \text{ belonging to } \tilde{\mathcal{F}} \text{ and } \tilde{\mathcal{G}}, \end{cases}$$

where C *is a constant independent of* n*. Moreover,*

$$I(n) = \int_{\mathcal{I}} \iota(\|r\|)\, \mu^{*n}(dr), \tag{11}$$

where μ^{*0} *is the Dirac delta measure in zero, and,* μ^{*n} *is the n-fold convolution of* μ *for* $n\geq 1$*.*

Proof. Y is a strictly stationary process by Proposition 2.1 in Brandes and Curato (2019). Consider arbitrary fixed $(u,v) \in \mathbb{N}^* \times \mathbb{N}^*$, $n \in \mathbb{N}$, $\{i_1,\dots,i_u\} \subseteq \mathbb{Z}$ and $\{j_1,\dots,j_v\} \subseteq \mathbb{Z}$ with $i_1 \leq \ldots \leq i_u \leq i_u + n \leq j_1 \leq \ldots \leq j_v$, and functions $\tilde{F} \in \tilde{\mathcal{F}}$ and $\tilde{G} \in \tilde{\mathcal{G}}$. By conditioning with respect to the sequence of the inter-arrival times τ and using the law of total covariance (cf. Proposition A.1 in Chan et al. (2019)), we obtain that

$$
\begin{aligned}
&|Cov(\tilde{F}(Y_{i_1},\dots,Y_{i_u}), \tilde{G}(Y_{j_1},\dots,Y_{j_v}))| \\
&\leq |\mathbb{E}(Cov(\tilde{F}(Y_{i_1},\dots,Y_{i_u}), \tilde{G}(Y_{j_1},\dots,Y_{j_v})|\tau_i : i=1,\dots,j_v))| \qquad (12)\\
&+|Cov(\mathbb{E}(\tilde{F}(Y_{i_1},\dots,Y_{i_u})|\tau_i : i=1,\dots,j_v), \mathbb{E}(\tilde{G}(Y_{j_1},\dots,Y_{j_v})|\tau_i : i=1,\dots,j_v))|. \qquad (13)
\end{aligned}
$$

Let us first discuss the summand (13). The term

$$
\mathbb{E}(\tilde{F}(Y_{i_1},\dots,Y_{i_u})|\tau_i : i=1,\dots,j_v) = \mathbb{E}(\tilde{F}(Y_{i_1},\dots,Y_{i_u})|\tau_i : i=1,\dots,i_u)
$$

because $\tilde{F}(Y_{i_1},\dots,Y_{i_u})$ is independent of $\{\tau_i : i = i_u+1,\dots,j_v\}$. On the other hand,

$$
\begin{aligned}
&\mathbb{E}(\tilde{G}(Y_{j_1},\dots,Y_{j_v})|\tau_i : i=1,\dots,j_v) \\
&= \mathbb{E}(\tilde{G}((X_{T_{i_u}+\sum_{i=i_u+1}^{j_1}\tau_i}, \tau_{j_1})',\dots,(X_{T_{i_u}+\sum_{i=i_u+1}^{j_v}\tau_i}, \tau_{j_v})')|\tau_i : i=1,\dots,j_v),
\end{aligned}
$$

and, by stationarity of the process X and the i.i.d property of $(\tau_i)_{i\in\mathbb{Z}\setminus\{0\}}$, it is equal to

$$
\begin{aligned}
&\mathbb{E}(\tilde{G}((X_{\sum_{i=i_u+1}^{j_1}\tau_i}, \tau_{j_1})',\dots,(X_{\sum_{i=i_u+1}^{j_v}\tau_i}, \tau_{j_v})')|\tau_i : i=1,\dots,j_v) \\
&= \mathbb{E}(\tilde{G}((X_{\sum_{i=i_u+1}^{j_1}\tau_i}, \tau_{j_1})',\dots,(X_{\sum_{i=i_u+1}^{j_v}\tau_i}, \tau_{j_v})')|\tau_i : i=i_u+1,\dots,j_v)
\end{aligned}
$$

because of the independence between $\{(X_{\sum_{i=i_u+1}^{j_1}\tau_i}, \tau_{j_1})',\dots,(X_{\sum_{i=i_u+1}^{j_v}\tau_i}, \tau_{j_v})'\}$ and $\{\tau_i : i=1,\dots,i_u\}$. Thus, the summand (13) is equal to zero, because

$$
\mathbb{E}(\tilde{F}(Y_{i_1},\dots,Y_{i_u})|\tau_i : i=1,\dots,i_u),
$$

and

$$
\mathbb{E}(\tilde{G}((X_{\sum_{i=i_u+1}^{j_1}\tau_i}, \tau_{j_1})',\dots,(X_{\sum_{i=i_u+1}^{j_v}\tau_i}, \tau_{j_v})')|\tau_i : i=i_u+1,\dots,j_v))
$$

are independent.

The summand (12) is less than or equal to

$$
\begin{aligned}
\int_{\mathcal{I}^{j_v}} \Big| &Cov(\tilde{F}((X_{\sum_{i=1}^{i_1} s_i}, s_{i_1})',\dots,(X_{\sum_{i=1}^{i_u} s_i}, s_{i_u})'), \tilde{G}((X_{\sum_{i=1}^{j_1} s_i}, s_{j_1})',\dots \\
&\dots,(X_{\sum_{i=1}^{j_v} s_i}, s_{j_v})'))\Big| \, d\mathbb{P}_{\{\tau_i : i=1,\dots,j_v\}}(s_1,\dots,s_{j_v}),
\end{aligned}
$$

where $\mathbb{P}_{\{\tau\}}$ indicates the joint distribution of the inter-arrival times sequence τ. For a given $(s_{i_1},\dots,s_{j_v}) \in \mathcal{I}^{j_v}$, we have that $\tilde{F}((\cdot,s_{i_1}),\dots,(\cdot,s_{i_u})) \in \mathcal{F}$ and $\tilde{G}((\cdot,s_{j_1}),\dots,(\cdot,s_{j_v})) \in \mathcal{G}$. X is a Ψ-weakly dependent process, then the above inequality is less than or equal to

$$
\int_{\mathcal{I}^{j_1-i_u}} C\,\Psi(\|\tilde{F}((\cdot,s_{i_1}),\dots,(\cdot,s_{i_u}))\|_\infty, \|\tilde{G}((\cdot,s_{j_1}),\dots,(\cdot,s_{j_v}))\|_\infty, Lip(\tilde{F}((\cdot,s_{i_1}),\dots,(\cdot,s_{i_u}))),
$$

$$Lip(\tilde{G}((\cdot, s_{j_1}), \dots, (\cdot, s_{j_v}))), u, v) \, \iota\Big(\Big\| \sum_{i=i_u+1}^{j_1} s_i \Big\|\Big) d\mathbb{P}_{\{\tau_i : i=1,\dots,j_v\}}(s_1, \dots, s_{j_v}),$$

and, because the sequence $\{\tau_i : i = i_u + 1, \dots, j_1\}$ is independent of the sequence $\{\tau_i : i = 1, \dots, i_u, j_2, \dots, j_v\}$, it is less than or equal to

$$\int_{\mathcal{I}^{j_1 - i_u}} C\, \Psi(\|\tilde{F}\|_\infty, \|\tilde{G}\|_\infty, Lip(\tilde{F}), Lip(\tilde{G}), u, v) \, \iota\Big(\Big\| \sum_{i=i_u+1}^{j_1} s_i \Big\|\Big) d\mathbb{P}_{\{\tau_i : i=i_u+1,\dots,j_1\}}(s_{i_u+1}, \dots, s_{j_1}).$$

We have that $j_1 - i_u \geq n$ and that w.l.o.g. the coefficients ι are non increasing. Thus, we can conclude that the above integral is less than or equal to

$$C\, \Psi(\|\tilde{F}\|_\infty, \|\tilde{G}\|_\infty, Lip(\tilde{F}), Lip(\tilde{G}), u, v) \int_{\mathcal{I}} \iota(\|r\|) \mu^{*n}(dr).$$

∎

Note that, if the coefficients (11) converge to zero as n goes to infinity then Y inherits the asymptotic dependence structure of X.

3 Ψ-weakly dependent renewal sampled processes

In this section, we consider renewal sampling of $X = (X_t)_{t \in \mathbb{R}}$. Therefore, the inter-arrival times are a sequence of non-negative i.i.d random variables with values in $\mathbb{R}$.

We first show that if X is Ψ-weakly dependent and admits exponential or power decaying coefficients ι then Y is in turn Ψ-weakly dependent and its coefficients I preserve (at least asymptotically) the decay behavior of ι. This result directly enables the application of the limit theory for a vast class of Ψ-weakly dependent processes Y of which we present several examples throughout the section.

In fact, central limit theorems for a Ψ-weakly dependent process X typically hold under sufficient conditions of the following type: $\mathbb{E}[\|X_0\|^\delta] < \infty$ for some $\delta > 0$ and the coefficients ι satisfy a condition

$$\sum_{i=1}^{\infty} \iota(n)^{A(\delta)} < \infty, \tag{14}$$

where $A(\delta)$ is a certain function of δ. If X admits coefficients ι with exponential or sufficiently fast power decay then conditions of type (14) are satisfied. If in turn, Y is Ψ-weakly dependent with coefficients having exponential or sufficiently fast power decay, then conditions of type (14) are satisfied also under renewal sampling.

3.1 Exponential decay

In terms of the Laplace transform of the inter-arrival times, we can obtain a general formula for the coefficients $(I(n))_{n\in\mathbb{N}}$.

Proposition 3.1. *Let $X = (X_t)_{t\in\mathbb{R}}$, $Y = (Y_i)_{i\in\mathbb{Z}}$ and $(T_i)_{i\in\mathbb{Z}}$ be as in Theorem 2.4. Let us assume that $\iota(r) \leq C\mathrm{e}^{-\gamma r}$ for $\gamma > 0$ and denote the Laplace transform of the distribution function μ by*

$$\mathcal{L}_\mu(t) = \int_{\mathbb{R}^+} \mathrm{e}^{-tr}\,\mu(dr),\ t \in \mathbb{R}_+.$$

Then, the process Y admits coefficients

$$I(n) \leq C\Big(\frac{1}{\mathcal{L}_\mu(\gamma)}\Big)^{-n}$$

which converge to zero as n goes to infinity.

Proof. We notice that $\mathcal{L}_\mu(t) < 1$ for $t > 0$ and that $\mathcal{L}_{\mu^{*n}}(t) = (\mathcal{L}_\mu(t))^n$, cf. (Sato, 2013, Proposition 2.6).

Using the result obtained in Theorem 2.4, we have that

$$\begin{aligned} I(n) = \int_{\mathbb{R}_+} \iota(r)\,\mu^{*n}(dr) \leq C\int_{\mathbb{R}_+} \mathrm{e}^{-\gamma r}\,\mu^{*n}(dr) = \mathcal{L}_{\mu^{*n}}(\gamma) \\ = C(\mathcal{L}_\mu(\gamma))^n. \end{aligned}$$

∎

To summarize, if X is η, λ, κ, ζ, θ-weakly dependent or α-mixing then Y inherits the same kind of asymptotic dependence structure under renewal sampling as long as the η, λ, κ, θ, ζ or α-coefficients of X are exponentially decaying.

Example 3.2. *If we have a renewal sampling with $\Gamma(\alpha,\beta)$-distributed inter-arrival times for $\alpha,\beta > 0$, then μ^{*n} is the distribution function of a $\Gamma(n\alpha,\beta)$ distributed random variable. By Proposition 3.1, we obtain the coefficients*

$$I(n) = \int_{\mathbb{R}_+} \iota(r)\,\mu^{*n}(dr) \leq C\int_{(0,+\infty)} \mathrm{e}^{-\gamma r}\frac{\beta^{n\alpha}}{\Gamma(n\alpha)}r^{n\alpha-1}\mathrm{e}^{-\beta r}\,dr = C\Big(\frac{\gamma+\beta}{\beta}\Big)^{-n\alpha}.$$

A special case of the coefficients above is obtained in the case of Poisson sampling, i.e. $\mu = Exp(\lambda)$ with $\lambda > 0$. In this instance, μ^{*n} is the distribution function of a $\Gamma(n,\lambda)$ distributed random variable. We then obtain the coefficients

$$I(n) = \int_{\mathbb{R}_+} \iota(r)\,\mu^{*n}(dr) \leq C\int_{(0,+\infty)} \mathrm{e}^{-\gamma r}\frac{\lambda^n}{\Gamma(n)}r^{n-1}\mathrm{e}^{-\lambda r}\,dr = C\Big(\frac{\lambda+\gamma}{\lambda}\Big)^{-n}.$$

3.2 Power decay

We now assume that the underlying process X is Ψ-weakly dependent with coefficients $\iota(r) \le Cr^{-\gamma}$ for $\gamma > 0$.

We start with some concrete examples of inter-arrival time distributions μ which preserve the power decay of the coefficients ι.

Example 3.3. *Let us consider renewal sampling with $\Gamma(\alpha, \beta)$ distributed inter-arrival times for $\alpha, \beta > 0$. Then, μ^{*n} is a $\Gamma(n\alpha, \beta)$ distribution. Thus,*

$$I(n) = \int_{\mathbb{R}_+} \iota(r)\, \mu^{*n}(dr) \le C \int_{(0,+\infty)} r^{-\gamma} \frac{\beta^{n\alpha}}{\Gamma(n\alpha)} r^{n\alpha-1} \mathrm{e}^{-\beta r}\, dr = C\beta^{\gamma} \frac{\Gamma(n\alpha-\gamma)}{\Gamma(n\alpha)}$$

$$\sim Cn^{-\gamma}(1 + O(n^{-1})) \sim Cn^{-\gamma} + O(n^{-\gamma-1})$$

by using Stirling's series, see Tricomi and Erdélyi (1951), for n to infinity.

In the special case of Poisson sampling, μ^{*n} is a $\Gamma(n, \lambda)$ distribution and

$$I(n) = \int_{\mathbb{R}_+} \iota(r)\, \mu^{*n}(dr) \le C \int_{(0,+\infty)} r^{-\gamma} \frac{\lambda^{n}}{\Gamma(n)} r^{n-1} \mathrm{e}^{-\lambda r}\, dr = C\lambda^{\gamma} \frac{\Gamma(n-\gamma)}{\Gamma(n)}$$

$$\sim Cn^{-\gamma}(1 + O(n^{-1})) \sim Cn^{-\gamma} + O(n^{-\gamma-1}).$$

Example 3.4. *We denote by Levy$(0, c)$ a Lévy distribution, cf. pg. 28 Zolotarev (1986), with location parameter 0 and scale parameter c (a completely skewed $\frac{1}{2}$-stable distribution). This distribution has infinite mean and variance. For Levy$(0, c)$ distributed inter-arrival times, we have that μ^{*n} is Levy$(0, cn)$. Thus,*

$$I(n) = \int_{\mathbb{R}_+} \iota(r)\, \mu^{*n}(dr) \le C \int_{\mathbb{R}_+} r^{-\gamma} \frac{(\frac{cn}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} r^{-\frac{3}{2}} \mathrm{e}^{-\frac{cn}{2r}}\, dr = \frac{\Gamma(\frac{1}{2}+\gamma)}{(\frac{cn}{2})^{\gamma}\Gamma(\frac{1}{2})} = \frac{\Gamma(\frac{1}{2}+\gamma)}{\Gamma(\frac{1}{2})} \Big(\frac{c}{2}\Big)^{-\gamma} n^{-\gamma}.$$

Example 3.5. *We consider now the case where μ is an inverse Gaussian distribution with mean m and shape parameter λ (short $IG(m, \lambda)$). We have that μ^{*n} is a $IG(nm, n^2\lambda)$ distribution and*

$$I(n) = \int_{\mathbb{R}_+} \iota(r)\, \mu^{*n}(dr) \le C \int_{(0,+\infty)} r^{-\gamma} \Big(\frac{n^2\lambda}{2\pi r^3}\Big)^{\frac{1}{2}} \mathrm{e}^{-\frac{n^2\lambda(r-nm)^2}{2n^2m^2r}}\, dr$$

$$= nC\Big(\frac{\lambda}{2\pi}\Big)^{\frac{1}{2}} \mathrm{e}^{\frac{\lambda n}{m}} \int_{(0,+\infty)} r^{-\gamma-\frac{3}{2}} \mathrm{e}^{-\frac{\lambda n}{2m}\left(\frac{r}{nm}+\frac{nm}{r}\right)}\, dr$$

$$= C\Big(\frac{\lambda}{2\pi}\Big)^{\frac{1}{2}} m^{-\gamma-\frac{1}{2}}\, n^{-\gamma+\frac{1}{2}}\, \mathrm{e}^{\frac{\lambda n}{m}}\, 2\,\mathcal{K}_{-\gamma-\frac{1}{2}}\Big(\frac{\lambda n}{m}\Big)$$

after applying the substitution $x := \frac{r}{nm}$ and where $\mathcal{K}_{-\gamma-\frac{1}{2}}$ denotes a modified Bessel function of the third kind with order $-\gamma - \frac{1}{2}$. By (Jørgensen, 1982, pg. 171), we have that $\mathcal{K}_v(x) \sim (\frac{\pi}{2})^{\frac{1}{2}} x^{-\frac{1}{2}} \mathrm{e}^{-x}$ for $x \to \infty$. Thus

$$I(n) \sim m^{-\gamma}\, n^{-\gamma}.$$

Example 3.6. *Let the inter-arrival times follow a Bernoulli distribution with parameter $p > 0$. Then, μ^{*n} is a $Bin(n,p)$ distribution. If X admits coefficients $\iota(r) = C(1 \wedge r^{-\gamma})$ for $\gamma > 0$, we have that $I(n)$*

$$I(n) = \int_{\mathbb{R}_+} \iota(r)\, \mu^{*n}(dr) = C\Big((1-p)^n + \sum_{j=1}^{n} j^{-\gamma}\binom{n}{j} p^j (1-p)^{n-j}\Big)$$
$$\sim C((1-p)^n + \tilde{C}(np)^{-\gamma}(1 + O(n^{-1}))) \sim \hat{C}(np)^{-\gamma} + O(n^{-\gamma-1})$$

as $n \to \infty$, by using Theorem 1 in Wuyungaowa and Wang (2008).

Example 3.7. *Let us consider inter-arrival times such that $\mu([0,k)) = 0$ for a fixed $k > 0$. Then, straightforwardly*

$$I(n) = \int_{\mathbb{R}_+} \iota(r)\, \mu^{*n}(dr) \leq C(nk)^{-\gamma}.$$

In Examples 3.3, 3.4, 3.5, 3.6 we obtain an exact (asymptotic) expression for the coefficients I. For a general inter-arrival time distribution we can just show that the coefficients I decay at least with the same power.

Proposition 3.8. *Let $X = (X_t)_{t\in\mathcal{T}}$, $Y = (Y_i)_{i\in\mathbb{Z}}$ and $(T_i)_{i\in\mathbb{Z}}$ be as in Theorem 2.4. Let us assume that $\iota(r) \leq Cr^{-\gamma}$ for $\gamma > 0$. Then, the process Y admits coefficients $I(n) \leq \tilde{C}n^{-\gamma}$ as $n \to \infty$.*

Proof. Let us assume w.l.o.g. that $\mu \neq \delta_a$ (otherwise Example 3.7 applies for any $a \in \mathbb{R}_+$), where δ_a denotes the Dirac-delta measure in $a \in \mathbb{R}_+$. We choose an $a > 0$, such that $\mu([0,a)) > 0$ and $\mu([a,\infty)) = p > 0$, and set $\nu = p\delta_a + (1-p)\delta_0$. The latter is a Bernoulli distribution that assigns probability p to the inter-arrival time a and $(1-p)$ to the one 0. It follows that $\mu([0,\epsilon)) \leq \nu([0,\epsilon))$ for all $\epsilon > 0$. Then, by using Lemma 3.9, the result in Example 3.6 and Theorem 1 in Wuyungaowa and Wang (2008)

$$I(n) \leq C\int_{\mathbb{R}_+} r^{-\gamma}\mu^{*n}(dr) \leq \int_{\mathbb{R}_+} r^{-\gamma}\nu^{*n}(dr) \sim C\Big((1-p)^n + \sum_{j=1}^{n}(aj)^{-\gamma}\binom{n}{j}p^j(1-p)^{n-j}\Big)$$
$$\sim C((1-p)^n + \tilde{C}(nap)^{-\gamma}(1+O(n^{-1}))) \sim \hat{C}(nap)^{-\gamma} + O(n^{-\gamma-1})$$

as $n \to \infty$.

∎

The proposition above relies on the following lemma.

Lemma 3.9. *Let μ, ν be two probability measures on $\mathbb{R}^+$ such that $\mu([0,\epsilon)) \leq \nu([0,\epsilon))$ for all $\epsilon > 0$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be non-increasing. Then*

$$\int_{\mathbb{R}_+} f(r)\mu^{*n}(dr) \leq \int_{\mathbb{R}_+} f(r)\, \nu^{*n}(dr).$$

Proof. The proof follows by simply applying measure theoretic induction. ∎

Remark 3.10. *The result in Proposition 3.8 allows for the exact decay being faster in general than $n^{-\gamma}$, but so far we have found no example where this applies. In fact, even for extremely heavily tailed inter-arrival time distributions as in Example 3.4, we have a power decay of the same order of the coefficients ι.*

Proposition 3.8 summarizes the results given in this section. In fact, as long as X is η, λ, κ, ζ, θ-weakly dependent or α-mixing such that there exists a $\gamma > 0$ with $\iota(r) \leq Cr^{-\gamma}$ then Y is a Ψ-weakly dependent process. Note that Proposition 3.8 assures that Y is Ψ-weakly dependent also when, for example, $\iota(r) = C\frac{1}{rlog(r)}$ and then $\iota(r) \leq Cn^{-1}$. Therefore, caution has to be exercised when checking conditions of type (14) for the process Y.

Example 3.11. *Let us consider, the sufficient condition for the applicability of the central limit theorem for κ-weakly dependent processes, see Doukhan and Wintenberger (2007), where (14) holds with $A(\delta) = 1$. If X is a Ψ-weakly dependent process with coefficients $\iota(r) = C\frac{1}{rlog(r)}$, then Y is a Ψ-weakly dependent process with coefficients $I(n) \leq \tilde{C}n^{-1}$ by applying Proposition 3.8. We have that the coefficients $\iota(r)$ are summable and then satisfy (14) but we have no knowledge about the summability of the coefficients $I(n)$ as Proposition 3.8 just gives an upper bound of their value which is not summable.*

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