Adaptive wavelet methods on unbounded domains

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Motivation

- Problems in physics often posed on unbounded domains (optics, astrophysics,...).
- Problem setting in option pricing (finance):

$$\partial_t u + \mathcal{A}[u](t,x) = g(x), \quad x \in \mathbb{R}^n,$$
$$u(0,x) = h(x), \quad x \in \mathbb{R}^n.$$

- *Domain truncation:* $\Omega := [-R, R]^n$

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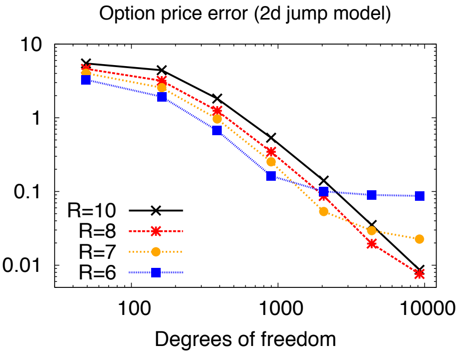
$$u(0,x) = h(x), \quad x \in \Omega.$$



⇝ *Discretization* vs. *truncation* error.
⇝ *Adaptive balancing*.

Outline

Consider a linear, elliptic operator equation for $f \in H^{-1}(\mathbb{R}^n)$ and $\mathcal{A}$ self-adjoint:

$$\mathcal{A}[u] = f \text{ in } H^{-1}(\mathbb{R}^n).$$

Requirements for numerical scheme

- Adaptive domain truncation and local refinement.
- Optimal convergence rate under weak smoothness assumptions.
- Linear complexity for a large class of operators $\mathcal{A}$.

New approach

Existing methods (Infinite Elements, BEM, ...) do either not cover *all* of these requirements or are only applicable for special classes of $\mathcal{A}$.

- *Equivalent formulation* in an infinite-dimensional sequence space ℓ_2 of *wavelet coefficients*,

$$\mathcal{A}[u] = f \text{ in } H^{-1}(\mathbb{R}^n) \quad \Longleftrightarrow \quad \mathbf{Au} = \mathbf{f} \text{ in } \ell_2.$$

- Approximation of $\mathbf{u}$ by means of *adaptive wavelet methods*.

Equivalent formulation as an ℓ_2-problem

Riesz basis: $\mathbf{\Psi} := \{\psi_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}$

- is a countable dense subset of $H^1(\mathbb{R}^n)$,
- $c_{\mathbf{\Psi}} \|\mathbf{v}\|_{\ell_2(\boldsymbol{\nabla})} \leq \| \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \mathbf{v}_{\boldsymbol{\lambda}} \psi_{\boldsymbol{\lambda}} \|_{H^1(\mathbb{R}^n)} \leq C_{\mathbf{\Psi}} \|\mathbf{v}\|_{\ell_2(\boldsymbol{\nabla})}, \quad \forall \mathbf{v} \in \ell_2(\boldsymbol{\nabla})$.

The solution u has a unique expansion in $\mathbf{\Psi}$, $u = \mathbf{u}^T \mathbf{\Psi} := \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \mathbf{u}_{\boldsymbol{\lambda}} \psi_{\boldsymbol{\lambda}}$:

$$\langle v, \mathcal{A}[u] \rangle = \langle v, f \rangle, \ \forall v \in H^1(\mathbb{R}^n) \iff \underbrace{\langle \mathbf{\Psi}, \mathcal{A}[\mathbf{\Psi}] \rangle}_{\mathbf{A}} \mathbf{u} = \underbrace{\langle \mathbf{\Psi}, f \rangle}_{\mathbf{f}}.$$

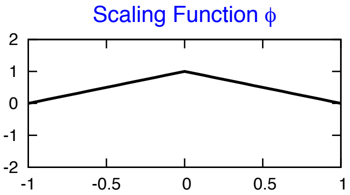
- *Infinite load vector* $\mathbf{f} \in \ell_2(\boldsymbol{\nabla})$.
- Boundedly invertible *bi-infinite stiffness matrix* $\mathbf{A} : \ell_2(\boldsymbol{\nabla}) \to \ell_2(\boldsymbol{\nabla})$.
- Known technique from [CDD01] ($\rightsquigarrow$ problems on *bounded* domains).
- Application of this principle to unbounded domains $\rightsquigarrow$ [KU11].

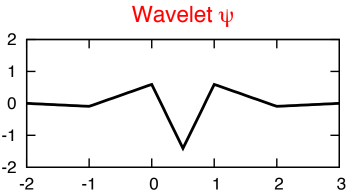
[CDD01] A. Cohen, W. Dahmen, R. DeVore. *Adaptive wavelet methods for elliptic operator equations: Convergence rates*. Mathematics of Computation, 2001

[KU11] S. Kestler and K. Urban. *Adaptive wavelet algorithms on unbounded domains*. Preprint (Submitted), 2011

Riesz wavelet bases

Riesz wavelet basis for $L_2(\mathbb{R})$ from [CDF92]:





$$\Psi_{L_2(\mathbb{R})} = \{2^{j_0/2}\phi(2^{j_0}\cdot - k) : k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^{j}\cdot - k) : j \geq j_0, k \in \mathbb{Z}\},\ j_0 \in \mathbb{Z}$$
$$= \{\psi_\lambda : \lambda := (j,k) \in \nabla\}$$

Notation: *Level* j and *minimal* level j_0

[CDF92] A. Cohen, I. Daubechies and J.C. Feauveau. *Biorthogonal bases of compactly supported wavelets.* Communications on Pure and Applied Mathematics, 1992.

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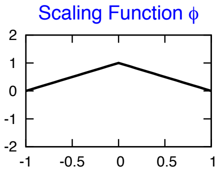
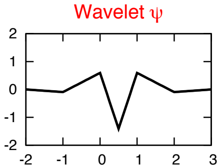
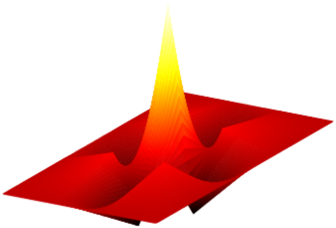




$$\Psi_{L_2(\mathbb{R})} = \{\psi_\lambda : \lambda := (j,k) \in \nabla\}.$$

Riesz wavelet basis for $H^1(\mathbb{R}^n)$:

- Multi-index: $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_n)$
- *Tensor wavelets*: $\psi_{\boldsymbol{\lambda}} := \psi_{\lambda_1} \otimes \cdots \otimes \psi_{\lambda_n}$.



$$\Psi_{H^1(\mathbb{R}^n)} = \{\psi_{\boldsymbol{\lambda}} / \|\psi_{\boldsymbol{\lambda}}\|_{H^1(\mathbb{R}^n)} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}.$$

Adaptive approximation of $\mathbf{A}\mathbf{u} = \mathbf{f}$

Basic principle of the adaptive scheme from [GHS07]

Solve *finite dimensional* Galerkin systems on nested index sets $\boldsymbol{\Lambda}^{(k)} \subsetneq \boldsymbol{\Lambda}^{(k+1)} \subsetneq \cdots \subset \boldsymbol{\nabla}$.

IDEALIZED–ADWAV$[\varepsilon]$

for $k = 0$; $\|\mathbf{r}^{(k)}\|_{\ell_2} \leq \varepsilon$; $k = k + 1$ **do**

Solve the Galerkin system

(1) $\mathbf{A}_{\boldsymbol{\Lambda}^{(k)}} \mathbf{u}_{\boldsymbol{\Lambda}^{(k)}} = \mathbf{f}_{\boldsymbol{\Lambda}^{(k)}}$.

Compute the *infinite* residual

(2) $\mathbf{r}^{(k)} := \mathbf{A}\mathbf{u}_{\boldsymbol{\Lambda}^{(k)}} - \mathbf{f}$.

Compute $\boldsymbol{\Lambda}^{(k+1)} \supset \boldsymbol{\Lambda}^{(k)}$ minimal s.t.

(3) $\|\mathbf{r}^{(k)}|_{\boldsymbol{\Lambda}^{(k+1)}}\|_{\ell_2} \geq \mu \, \|\mathbf{r}^{(k)}\|_{\ell_2}$.

end for

[GHS07] T. Gantumur, R. Stevenson and H. Harbrecht. *An optimal adaptive wavelet method without coarsening of the iterands*. Mathematics of Computation, 2007.

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Adaptive domain truncation

In (3), significant wavelet indices $\boldsymbol{\lambda} \in \mathbf{\nabla}$ are added for

- domain extension,
- singularities.

Saturation property (cf. [CDD01])

$$\|u - u_{\mathbf{\Lambda}^{(k)}}\|_{H^1(\mathbb{R}^n)} \lesssim \theta(\mu)^k \, \|u - u_{\mathbf{\Lambda}^{(0)}}\|_{H^1(\mathbb{R}^n)},$$

with $\theta(\mu) < 1$, $u_{\mathbf{\Lambda}} := \mathbf{u}_{\mathbf{\Lambda}}^T \mathbf{\Psi}$.

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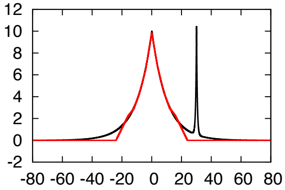
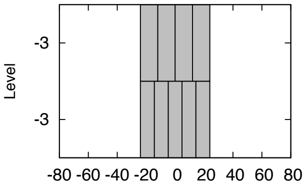
Exact / numerical solution



Index set Λ



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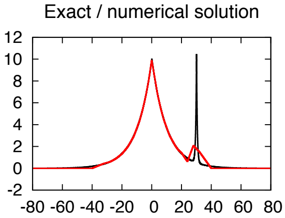

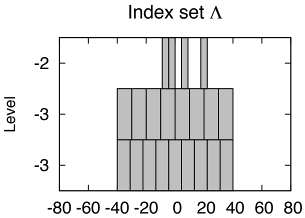

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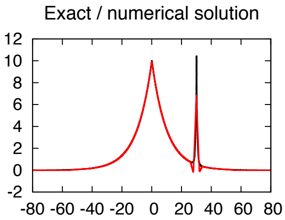

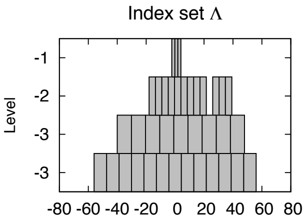

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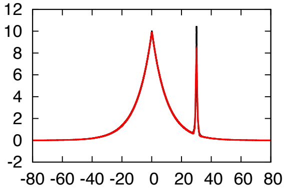

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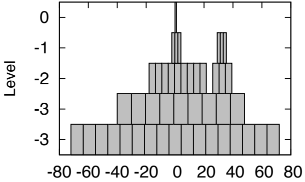

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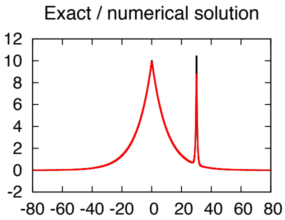

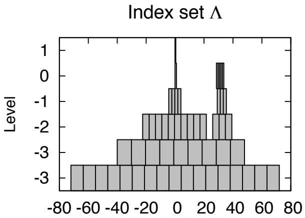

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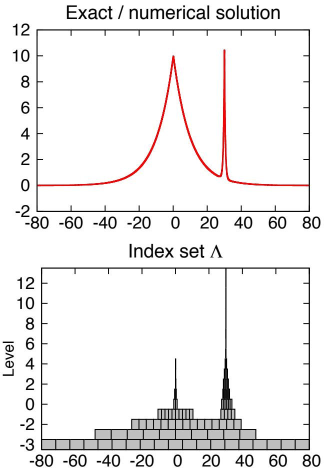

Adaptive wavelet algorithms for $\mathbf{A}\mathbf{u} = \mathbf{f}$: Implementation

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end for

Approximation of the infinite residual

Finitely supported approximation of the *infinite* matrix vector product $\mathbf{A}\mathbf{u}_{\mathbf{\Lambda}^{(k)}}$:

$$\rightsquigarrow \|\mathbf{APPLY}[\mathbf{u}_{\mathbf{\Lambda}^{(k)}}, \eta] - \mathbf{A}\mathbf{u}_{\mathbf{\Lambda}^{(k)}}\|_{\ell_2} \leq \eta,$$

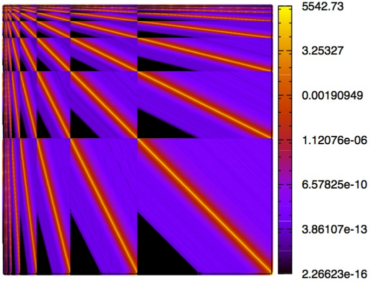
Finitely supported approximation of $\mathbf{f}$,

$$\rightsquigarrow \|\mathbf{RHS}[\eta] - \mathbf{f}\|_{\ell_2} \leq \eta.$$

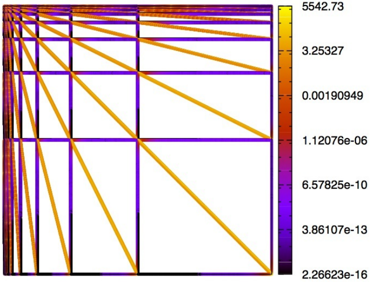
Replace $\mathbf{r}^{(k)}$ by

$$\tilde{\mathbf{r}}_\eta^{(k)} := \mathbf{APPLY}[\mathbf{u}_{\mathbf{\Lambda}^{(k)}}, \eta] - \mathbf{RHS}[\eta].$$

Adaptive wavelet algorithms for $\mathbf{A}u = f$: Implementation



$\mathbf{A}$



$\mathbf{A}_\eta$

- *Wavelet compression* (cf. [Ste04], [SS08]):

  $\|\mathbf{A} - \mathbf{A}_\eta\| \leq \eta$, $\mathbf{A}_\eta$ has only finitely many non-zeros per row and column.

- Use $\mathbf{A}_\eta$ for *approximate* matrix vector multiplication.

[Ste04] R. Stevenson. *On the compressibility of operators in wavelet coordinates*. SIAM Journal on Mathematical Analysis, 2004.

[SS08] C. Schwab and R.Stevenson. *Adaptive wavelet algorithms for elliptic PDEs on product domains*. Mathematics of Computation, 2008.

Adaptive wavelet algorithms for Au = f: Convergence

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end for

Best N-term approximation:

$$\|\mathbf{u} - \mathbf{u}_N\|_{\ell_2} \leq C N^{-s_{\max}}.$$

$\rightsquigarrow$ Besov regularity of u (e.g. [SU]).

[GHS07, Theorem 2.7]

Given *suitable* routines **APPLY** and **RHS**:

- $\|\mathbf{u} - \mathbf{u}_{\mathbf{\Lambda}^{(k)}}\|_{\ell_2} \lesssim N^{-s_{\max}},\ N := \operatorname{supp} \mathbf{u}_{\mathbf{\Lambda}^{(k)}}.$
- Linear complexity.

$\rightsquigarrow$ **Optimal scheme for bounded and <u>unbounded</u> domains**

[SU09] W. Sickel and T. Ullrich, *Tensor products of Sobolev-Besov spaces and applications to approximation form the hyperbolic cross*, Journal of Approximation Theory, 2009

Adaptive wavelet algorithms: From bounded to unbounded domains

Bounded domain: $\Omega = (a, b)$

$\nabla^{\Omega} := \{(j, k) : j \geq j_0, k \in \mathcal{I}_j\}, j_0 > 0$.

- ✓ Fixed minimal level j_0.
- ✓ Realization of **RHS** (e.g. [GHS07]):
- ✓ Realization of **APPLY** (e.g. [Urb09], [Ste04]).
- ✓ Quantitative analysis (e.g. [DHS07]).

Unbounded domain: $\mathbb{R}$ (cf. [KU11])

$\nabla^{\mathbb{R}} := \{(j, k) : j \geq j_0, k \in \mathbb{Z}\}, j_0 \in \mathbb{Z}$.

- ✓ *Good* choice of j_0:
  $\rightsquigarrow$ Diameter initial domain: $\sim 2^{-j_0}$.
- ✓ Construct *finite* $\nabla_\eta \subset \nabla^{\mathbb{R}}$ with

$$\|\mathbf{f} - \mathbf{f}_{\nabla_\eta}\|_{\ell_2(\nabla^{\mathbb{R}})} \leq \eta.$$

  $\rightsquigarrow$ Bound for translation indices.
- ✓ Special treatment of negative levels:

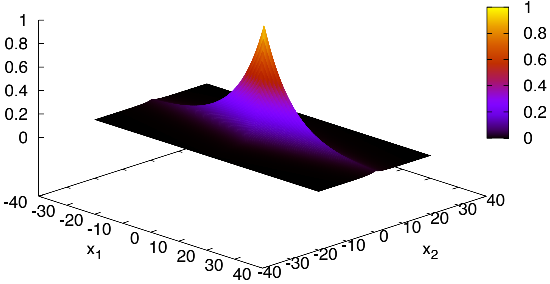
$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{+-} & \mathbf{A}_{++} \\ \mathbf{A}_{--} & \mathbf{A}_{-+} \end{pmatrix}.$$

- ✓ Adaptive domain truncation, local refinement, convergence.

[DHS07] W. Dahmen, H. Harbrecht and R. Schneider. *Adaptive methods for boundary integral equations: complexity and convergence estimates.* Mathematics of Computation, 2007.

[Urb09] K. Urban. *Wavelet methods for elliptic partial differential equations.* Oxford University Press, 2009.

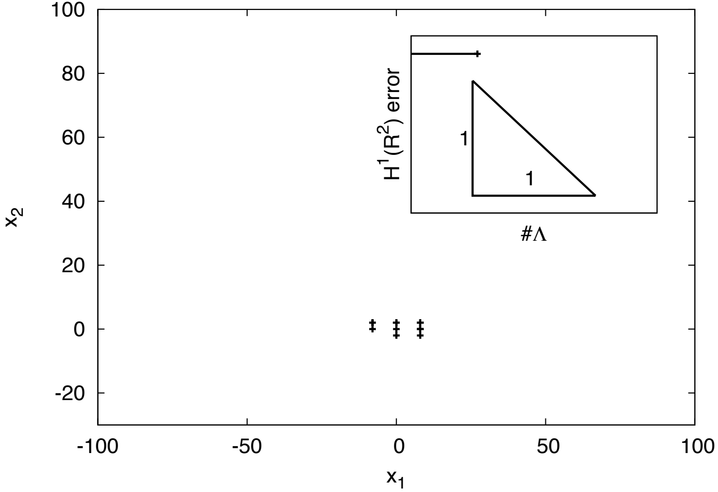
A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



- Conditions for optimal convergence are satisfied (cf. [KU11]).
- $s_{\max} = d - 1 - \varepsilon$ (independent of dimension n, cf. [SU09]).
- Tensor *multiwavelet* basis $d = 2$ (cf. [DGH96]).

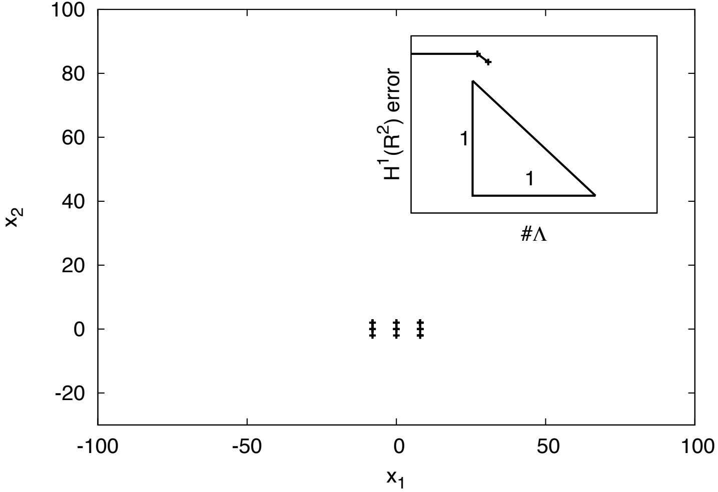
[DGH96] G. Donovan, J. Geronimo and D. Hardin. *Intertwining multiresolution analyses and the construction of piecewise-polynomial wavelets*. SIAM Journal on Mathematical Analysis, 1996.

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



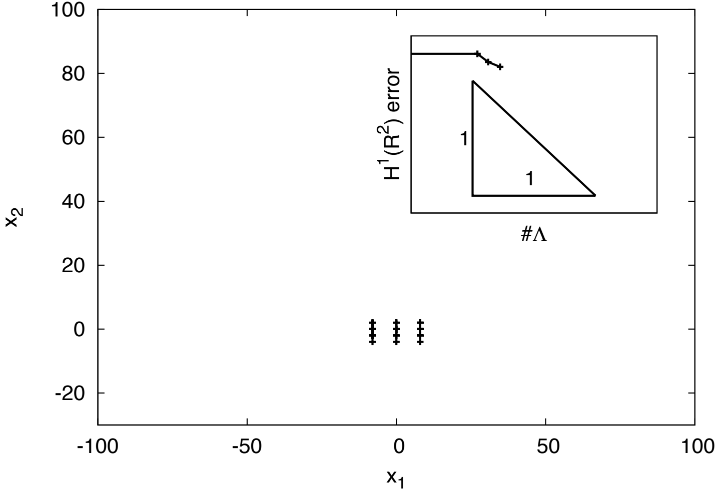
Iteration 1

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



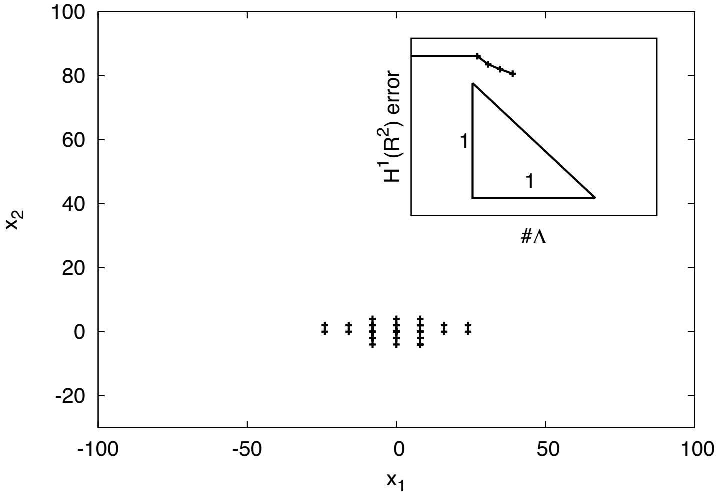
Iteration 2

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



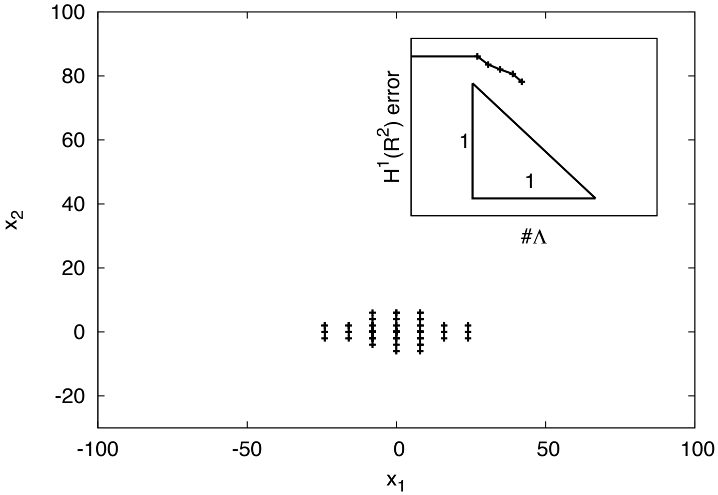
Iteration 3

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



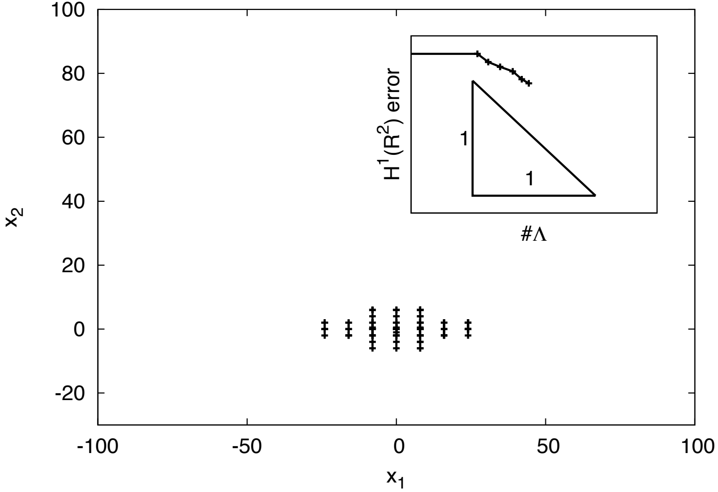
Iteration 4

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



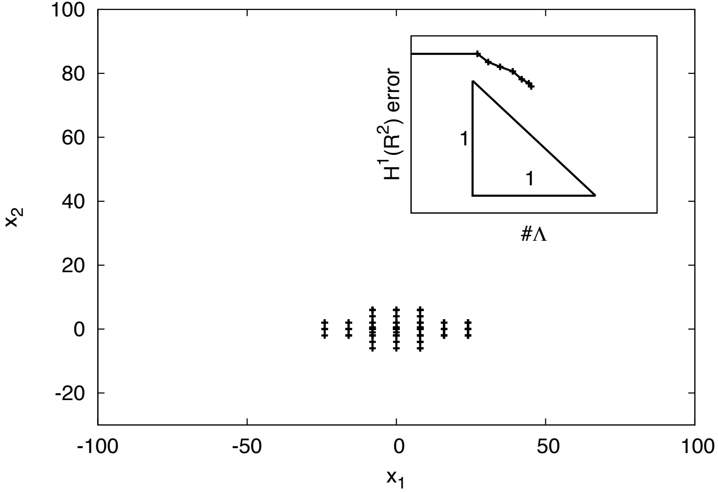
Iteration 5

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



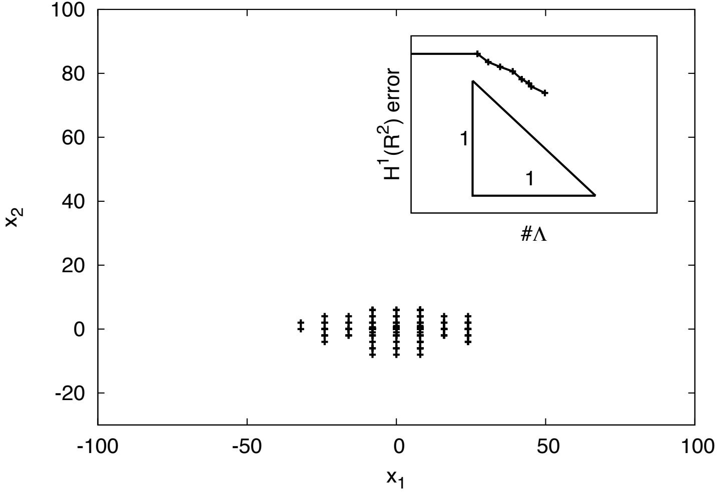
Iteration 6

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



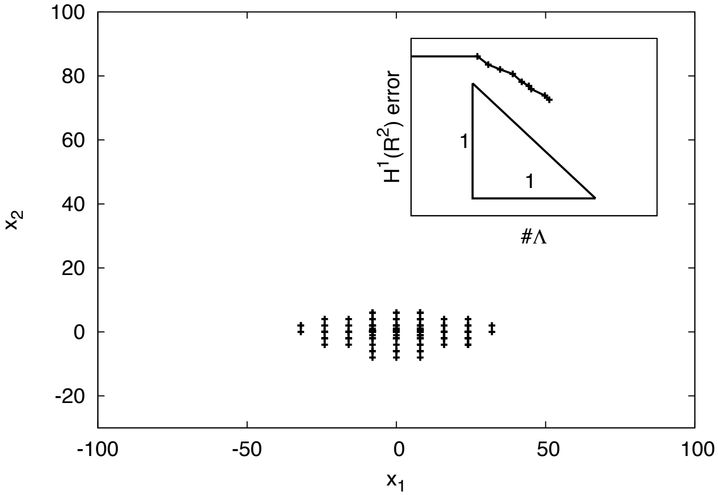
Iteration 7

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



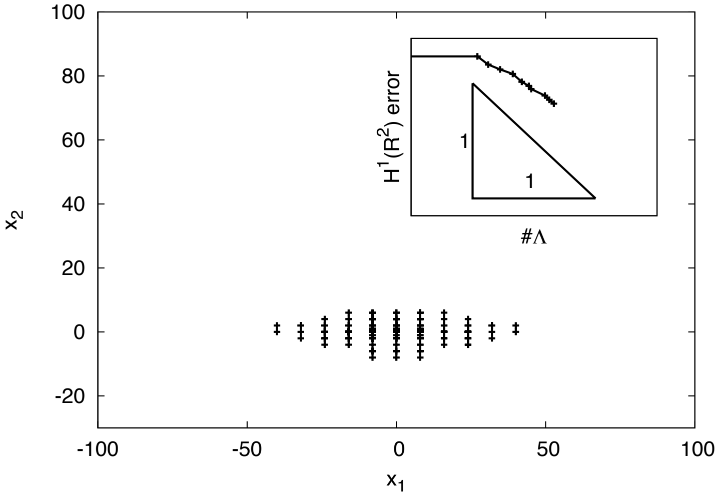
Iteration 8

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



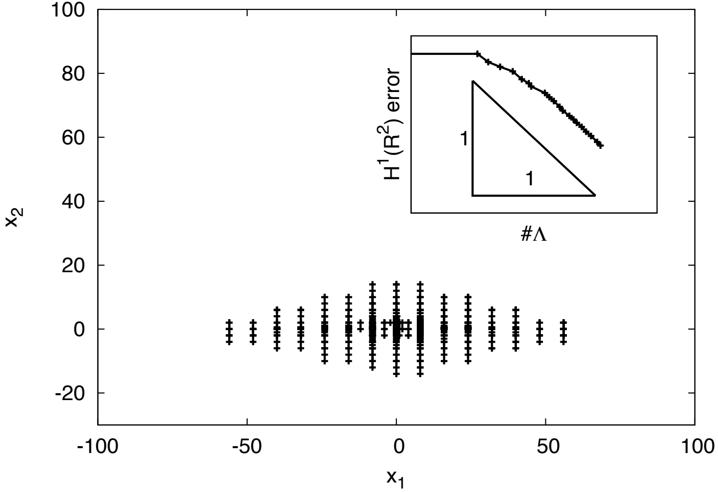
Iteration 9

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



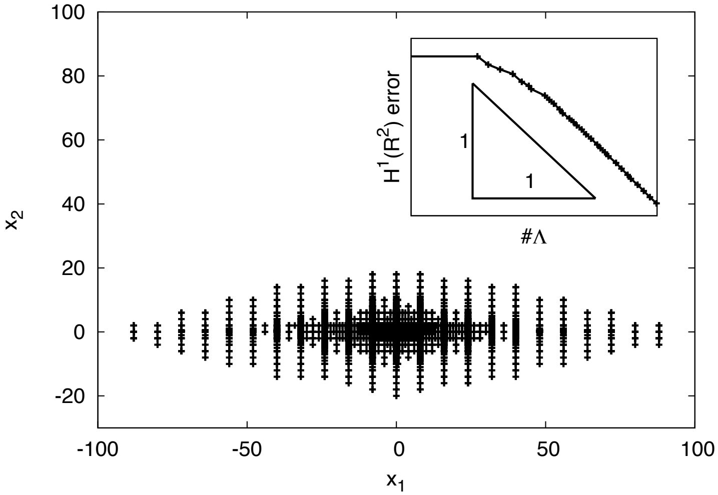
Iteration 10

A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



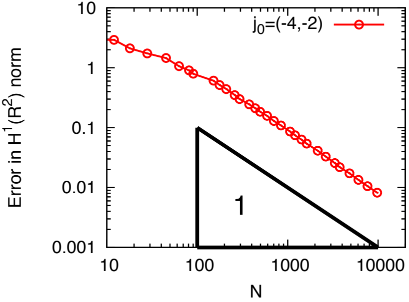
Iteration 20

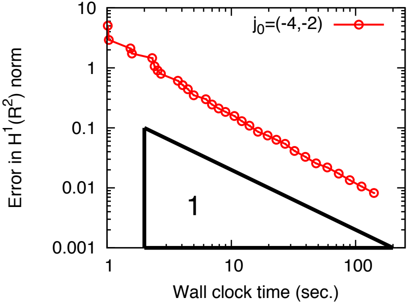
A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.



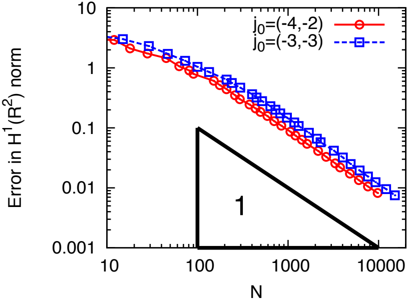
Iteration 30

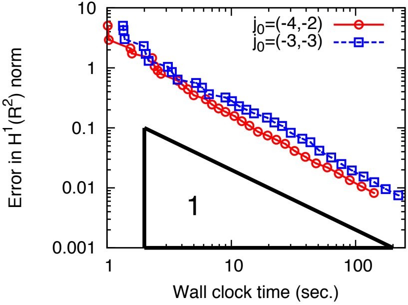
A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.





A first test: $-\Delta u + u = f$ in $H^{-1}(\mathbb{R}^2)$.

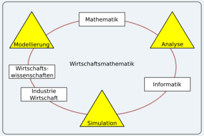




- Convergence for any minimal level.
- Best praxis: A priori estimate of largest index in $\mathbf{f}$.
- Realization: **L**ibrary for **A**daptive **W**avelet **A**ppliciations (**LAWA**).

[LAWA] `http://lawa.sourceforge.net/`

Contact



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http://www.uni-ulm.de/en/einrichtungen/research-training-group-1100.html

Thank you for your attention!