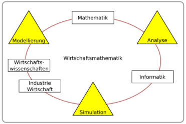
Time-periodic problems with time-dependent parameter functions

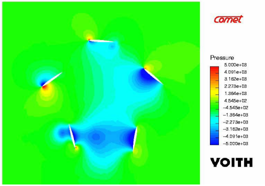
Workshop on Reduced Basis Methods



Research Training Group 1100

Kristina Steih
08. December 2010

Motivation



Questions

- Time-periodic problem
- Parameter-dependent domain
- Time-dependent parameter function
- Nonlinearities
- ⋮

Model Problem: Moving Boundary

Time-periodic problems

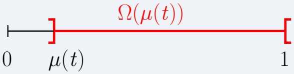
A posteriori error bounds

Offline Cost

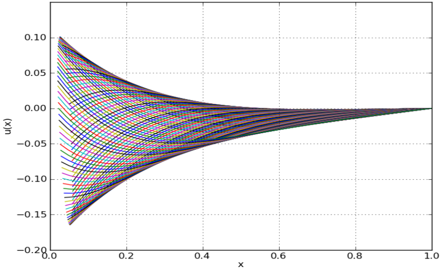
Model Problem: Moving Boundary

Problem

$$\mu : [0, T] \to [0, \mu_{max}]$$



$$
\begin{aligned}
u_t - \kappa u_{xx} &= 0 && \text{on } \Omega(\mu(t)), \\
u(t, 1) &= 0 && \forall t \in [0, T], \\
\frac{\partial}{\partial n} u(t, \mu(t)) &= -\dot{\mu}(t) && \forall t \in [0, T], \\
u(0, x) &= u(T, x) && \text{on } \Omega(\mu(t)).
\end{aligned}
$$



Model Problem: Moving Boundary

Mapping onto reference domain

- Stationary reference domain: $\hat{\Omega} := (0,1)$.
- Affine time-dependent mapping $S(t) : \hat{\Omega} \to \Omega(\mu(t))$

$V \hookrightarrow H \hookrightarrow V'$ Gelfand triple.

Variational Formulation

$$\int_{\hat{\Omega}} u_t(t,x)v(x)dx + \frac{\dot{\mu}(t)}{1-\mu(t)} \int_{\hat{\Omega}} (x-1)u_x(t,x)v(x)$$
$$+ \frac{\kappa}{(1-\mu(t))^2} \int_{\hat{\Omega}} u_x(t,x)v_x(x) = \frac{\kappa\dot{\mu}(t)}{(1-\mu(t))^2} v(0) \quad \forall v \in V, a.e.\ t \in [0,T].$$

Model Problem: Moving Boundary

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Variational Formulation

$$\int_{\hat{\Omega}} u_t(t,x)v(x)\,dx + \underbrace{\frac{\dot{\mu}(t)}{1-\mu(t)} \int_{\hat{\Omega}} (x-1)u_x(t,x)v(x)}_{=:b(t,u,v;\mu(t))}$$

$$+ \underbrace{\frac{\kappa}{(1-\mu(t))^2} \int_{\hat{\Omega}} u_x(t,x)v_x(x)}_{=:a(t,u,v;\mu(t))} = \underbrace{\frac{\kappa\dot{\mu}(t)}{(1-\mu(t))^2} v(0)}_{=:f(v;\mu(t))} \quad \forall v \in V, a.e.\ t \in [0,T].$$

Model Problem: Moving Boundary

Variational Formulation

$$\int_{\hat{\Omega}} u_t(t,x)v(x)dx + \Theta_a(\mu(t))a^1(u(t),v) + \Theta_b(\mu(t))b^1(u(t),v)$$
$$= \Theta_f(\mu(t))f^1(v), \qquad v \in V, t \in [0,T]$$

- Periodicity
  - solver?
  - error bounds?
  - basis: quality, computational effort?
- Parameter function:
  - (parameterized) function classes?
  - organization of training?
  - quality of basis for unknown functions?
- Time-dependent system
  - error bounds?
  - coercivity?
  - output: primal-dual approaches (in combination with periodicity)?

Model Problem: Moving Boundary

Time-periodic problems

A posteriori error bounds

Offline Cost

Time-Periodicity

Solver: Fixed Point Iterations

- Let $\mathcal{K} : V \to V,\ u_0 \mapsto u(T)$, where $u(T)$ solves the PDE for initial value $u(0) = u_0$.
- $u(0) = u(T) \iff \mathcal{K}(u(0)) = u(0)$, i.e. periodic solutions are fixed points of $\mathcal{K}$.
- Picard Iterations: sequence of initial value problems
  - Begin with an (arbitrary) $u_0^{(0)}$.
  - Solve PDE to obtain $u^{(0)}(T)$.
  - Set $u_0^{(1)} := u^{(0)}(T)$.
  - Repeat until $\|u_0^{(i)} - u_0^{(i-1)}\| < tol$.

Problems

- Computationally intensive, especially for non-linear PDEs.
- Good choice of $u_0^{(0)}$?

Alternatives / Improvements: Multi-Grid, adaptive space-time solver.

Model Problem: Moving Boundary

Time-periodic problems

A posteriori error bounds

Offline Cost

Error Bounds

Backward Euler, time steps $k = 1, \ldots, K$.

$$m(u^k, v) + \Delta t\big(a(u^k, v; \mu^k) + b(u^k, v; \mu^k)\big) = m(u^{k-1}, v) + \Delta t \, f(v, \mu^k).$$

Error Bounds

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Error: The error $e^k = e^k(\mu) = u^k_{\mathcal{N}}(\mu) - u^k_N$ satisfies

$$m(e^k, e^k) - \underbrace{m(e^0, e^0)}_{\text{not computable}} + \Delta t \sum_{l=1}^{k} \big(a(e^l, e^l; \mu^l) + b(e^l, e^l; \mu^l)\big) \leq \Delta t \sum_{l=1}^{k} \frac{\|\hat{\varepsilon}(\mu^l)\|_V^2}{\alpha(\mu^l)}$$

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$$\Delta t \sum_{k=1}^{K} \underbrace{\big(a(e^k, e^k; \mu^k) + b(e^k, e^k; \mu^k)\big)}_{\geq \alpha(\mu^k)\|e^k\|_V^2} \leq \Delta t \sum_{k=1}^{K} \frac{\|\hat{\varepsilon}(\mu^k)\|_V^2}{\alpha(\mu^k)}$$

Error Bounds

Backward Euler, time steps $k = 1, \dots, K$.

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Error bound: $L_2(0, T; V)$ - Norm

$$\left(\Delta t \sum_{k=1}^{K} \|e^k\|_V^2\right)^{\frac{1}{2}} \leq \Delta_N^{per}(\mu) := \left(\frac{\Delta t}{\alpha_{min}(\mu)} \sum_{k=1}^{K} \frac{\|\hat{\varepsilon}(\mu^k)\|_V^2}{\alpha(\mu^k)}\right)^{\frac{1}{2}}$$

Error Bounds

$$s(u(\mu)) := \int_0^T \ell(t, u(\mu(t)); \mu(t))\, dt, \qquad \ell : V \to \mathbb{R} \text{ linear.}$$

$$|s(u(\mu))| \leq \left(\int_0^T \|\ell(\mu(t))\|_{V'}^2\, dt \right)^{\frac{1}{2}} \left(\int_0^T \|u(\mu(t))\|_V^2\, dt \right)^{\frac{1}{2}}$$

Error bound: Output

$$|s(u_{\mathcal{N}}(\mu) - u_N(\mu))| \leq \left(\sum_{k=0}^{K} w_k \|\hat{\ell}(\mu^k)\|_V^2 \right)^{\frac{1}{2}} \Delta_N^{per}(\mu)$$

Error Bounds

Difficulties for periodic problems

- Initial error unknown
- Error bounds only for complete time period
- Error bounds for single time points?
- L_2-error bounds?
- Approaches for non-coercive problems?
  - stability constant: $\|e^k\|_{L_2}^2$ needed
  - better: space-time inf-sup constant?

Model Problem: Moving Boundary

Time-periodic problems

A posteriori error bounds

Offline Cost

Offline Cost

POD-Greedy

Given N_0, basis Φ_{N_0}.
for $N = N_0 + 1$ **to** N_{max} **do**
Compute new parameter $\mu_N^* = \text{argmax}_{\mu \in \Xi_{train}} \Delta_{N-1}^{per}(\mu)$.
if $\Delta_{N-1}^{per}(\mu_N^*) < \epsilon$ **exit.**
Compute snapshot $u_{\mathcal{N}}(\mu_N^*)$.
Compute basis function $\varphi_N = \text{POD}\big(e_{N-1,proj}^k(\mu_N^*),\ k = 1, \dots, K\big)$.
$\Phi_N = \Phi_{N-1} \cup \varphi_N$.
end for

Offline Cost

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- n_{train} RB fixed point solutions

Offline Cost

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- n_{train} RB fixed point solutions
- 1 truth fixed point solution

Offline Cost

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end for

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- 1 truth fixed point solution

- Initial values $u_{\mathcal{N}}^{(0)}(0,x)$, $u_N^{(0)}(0,x)$ for fixed point solver?
- Periodicity / Number of fixed point iterations?

Reducing Offline Cost I: Choice of initial values

	Zero Initialization			
N	Truth	RB (avg)	Truth	RB (avg)
1	29	2.87		
2	30	11.87		
3	30	24.60		
4	30	24.67		
5	30	24.67		
6	25	24.67		
7	30	24.67		
8	30	24.67		
Total	234	$159.35 \cdot n_{train}$		

Figure: Number of iterations needed to reach given fixed point tolerance $\|u(0) - u(T)\|_2 \leq 10^{-8}$

Reducing Offline Cost I: Choice of initial values

1. Truth initialization: $u_{\mathcal{N}}^{(0)}(0, x; \mu_N^*) = u_{N-1}^{per}(T, x; \mu_N^*)$.

	Zero Initialization		Truth Init	
N	Truth	RB (avg)	Truth	RB (avg)
1	29	2.87	29	
2	30	11.87	30	
3	30	24.60	27	
4	30	24.67	11	
5	30	24.67	4	
6	25	24.67	2	
7	30	24.67	2	
8	30	24.67	2	
Total	234	$159.35 \cdot n_{train}$	107	

Figure: Number of iterations needed to reach given fixed point tolerance $\|u(0) - u(T)\|_2 \leq 10^{-8}$

Reducing Offline Cost I: Choice of initial values

1. Truth initialization: $u_{\mathcal{N}}^{(0)}(0, x; \mu_N^*) = u_{N-1}^{per}(T, x; \mu_N^*)$.
2. RB initialization: $u_N^{(0)}(0, x; \mu^i) = u_{N-1}^{per}(T, x; \mu^i)$, $i = 1, \ldots, n_{train}$.

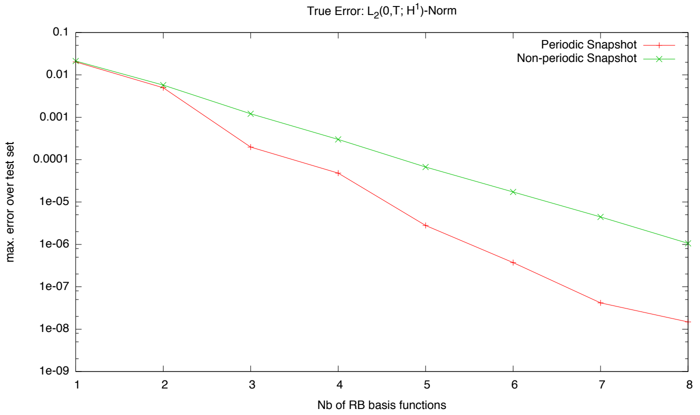
	Zero Initialization		Truth + RB Init.	
N	Truth	RB (avg)	Truth	RB (avg)
1	29	2.87	29	2.86
2	30	11.87	30	11.53
3	30	24.60	27	19.47
4	30	24.67	11	5.00
5	30	24.67	3	2.67
6	25	24.67	2	1.93
7	30	24.67	2	1.93
8	30	24.67	1	1.93
Total	234	$159.35 \cdot n_{train}$	105	$46.65 \cdot n_{train}$

Figure: Number of iterations needed to reach given fixed point tolerance $\|u(0) - u(T)\|_2 \leq 10^{-8}$

Reducing Offline Cost II: Periodicity of snapshots

Comparison:

- Periodic snapshots: $\|u_{\mathcal{N}}(0) - u_{\mathcal{N}}(T)\|_2 \leq 10^{-8}$
- Non-periodic snapshots: allow only 1 fixed point iteration in truth solve



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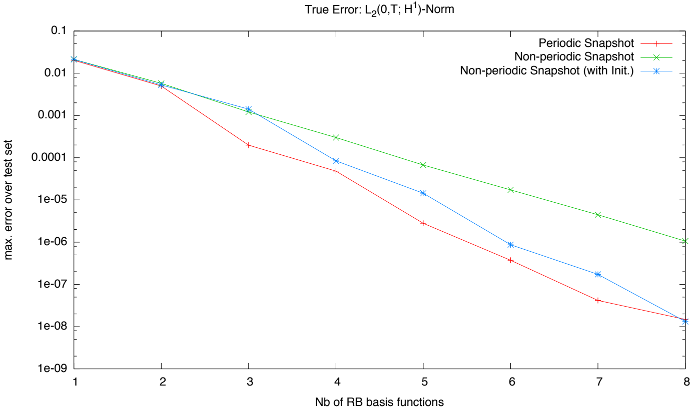

Summary

- LTV model problem:
  - error bounds
- Periodic problems:
  - fixed point solver
  - (time-dependent) error bounds for coercive case
  - identification of problems / differences to initial value problems
  - POD-Greedy: reduction of offline cost
- Parameter Function
  - parameterized function class

Outlook

- Modified POD-Greedy for more realistic problems
- Training over different parameter function classes
- Space-time inf-sup constant
- Nonlinear problems

Literature

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A posteriori error bounds for reduced-basis approximations of parameterized parabolic partial differential equations.
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M. Steuerwalt.
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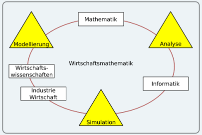
Contact information



Research Training Group 1100
Ulm University

Kristina Steih
kristina.steih@uni-ulm.de

Thank you for your attention

Model Problem: Moving Boundary

Coercivity Constraints

- Consider the parameter-independent constant

$$\rho := \inf_{v \in V} \frac{\int_{\hat{\Omega}} v_x(x) v_x(x)\, dx}{\int_{\hat{\Omega}} v(x) v(x)\, dx + v(0)^2}.$$

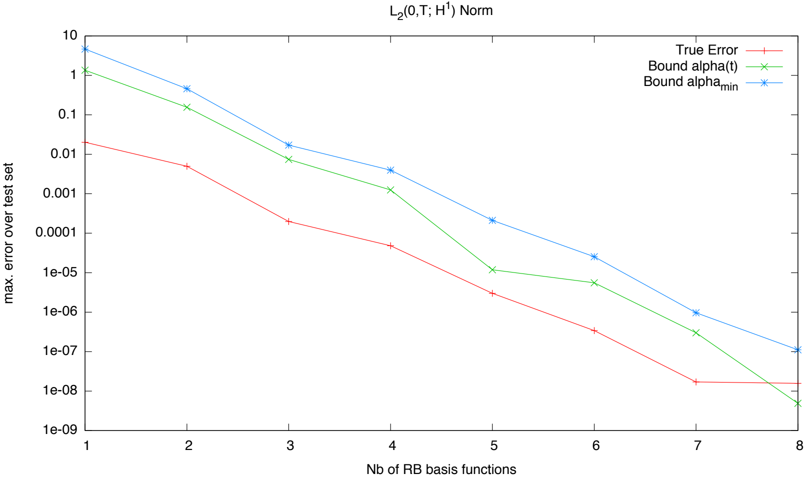
- Then

$$|\dot{\mu}(t)| < \frac{\kappa\rho}{1-\mu(t)} \implies a(t, v, v; \mu(t)) + b(t, v, v; \mu(t)) \geq \alpha(\mu(t)) \|v\|_V^2,$$

with

$$\alpha(\mu(t)) = \frac{1}{2(1-\mu(t))^2} \min\{1, \kappa\rho - \dot{\mu}(t)(1-\mu(t))\} > 0$$

Time-dependent coercivity constants



Periodic Problems: Non-Coercive Case

Difficulties for periodic problems

- Error bounds for single time points (initial error unknown)?
- L_2-error bounds?

Stabilization approach

Stability constant $\tau(\mu(t)) := \inf_{v \in L_2} \frac{\frac{1}{2} a(v, v, \mu(t)) + 2b(v, v; \mu(t))}{\|v\|_{L_2}^2}.$

$$\frac{1}{\Delta t}\left[m(e^k, e^k) - m(e^{k-1}, e^{k-1})\right] + a(e^k, e^k; \mu^k) + 2b(e^k, e^k, \mu^k) \leq \frac{\|\hat{\varepsilon}(\mu^k)\|_V^2}{\alpha_a(\mu^k)}$$

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$$(1 - \Delta t \tau(\mu^k)) m(e^k, e^k) - m(e^{k-1}, e^{k-1}) + \frac{\Delta t}{2} a(e^k, e^k) \leq \frac{\Delta t \|\hat{\varepsilon}(\mu^k)\|_V^2}{\alpha_a(\mu^k)}.$$

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$$\frac{\Delta t}{2} \sum_{l=1}^{k} a(e^l, e^l, \mu) \leq m(e^0, e^0) - m(e^k, e^k) - \sum_{l=1}^{k} \Delta t\, \tau(\mu^l) m(e^l, e^l) + \sum_{l=1}^{k} \frac{\Delta t \|\hat{\varepsilon}(\mu^l)\|_V^2}{\alpha_a(\mu^l)}.$$

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$$\frac{\Delta t}{2} \sum_{k=1}^{K} a(e^k, e^k, \mu) \leq - \sum_{k=1, \tau(\mu^k)<0}^{K} \Delta t \, \tau(\mu^k) + \sum_{k=1}^{K} \frac{\Delta t \|\hat{\varepsilon}(\mu^k)\|_V^2}{\alpha_a(\mu^k)}.$$

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$\Longrightarrow$ Better approach: space-time inf-sup constant?

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