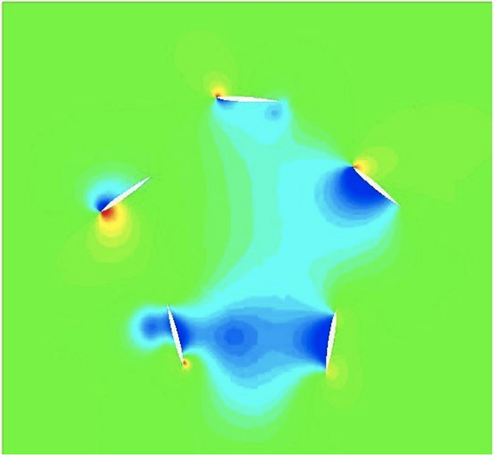

Adaptive Space-Time Methods in Reduced Basis Methods

Time-Periodic Problems

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Joint work with:

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Institute for Numerical Mathematics
Ulm University

August 20th, 2013 | RB Summer School 2013, Kopp

Introduction

Space-Time Adaptive Truth Computation

Adaptive RB Methods

So what's the problem?

Parametrized parabolic PDE with periodic BC in time:

$$u_t + \mathcal{A}(t;\mu)u = g(t;\mu) \qquad \text{on } \Omega \subset \mathbb{R}^d,$$

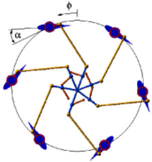
$$u(0) = u(T) \qquad \text{in } H.$$

Here:

- Parameter $\mu \in \mathcal{D} \subset \mathbb{R}^p$
- $V \hookrightarrow H \hookrightarrow V'$ a Gelfand triple
- $\mathcal{A}(t;\mu) : V \to V'$ uniformly continuous and coercive
- Dirichlet/Neumann BCs in space

Goal: **Construction of a Reduced Basis**

- good error estimators
- fast online evaluations
- feasible offline training phase



Doing it in space-time!

Well-posedness cf. [Schwab/Stevenson 2009]

Let $\mathcal{A}(t;\mu)$ be uniformly bounded and coercive. Then the problem

$$\text{Find } u(\mu) \in \mathcal{X}^{\text{per}} : \underbrace{\int_0^T \langle u_t, v\rangle\, dt + \int_0^T a(t,u,v;\mu)\, dt}_{=:b(u,v;\mu)} = \underbrace{\int_0^T g(t,v;\mu)\, dt}_{:=f(v;\mu)} \quad \forall\, v \in \mathcal{Y}.$$

is well-posed.

Bochner spaces:

$$\begin{aligned}
\mathcal{Y} &:= L_2(0,T;V), \\
\mathcal{X}^{\text{per}} &:= L_2(0,T;V) \cap H^1_{\text{per}}(0,T;V') \\
&= \{u \in L_2(0,T;V) :\ u_t \in L_2(0,T;V'), \\
&\qquad u(0) = u(T) \text{ in } H\}
\end{aligned}$$

C. Schwab, R. Stevenson. *Space-time adaptive wavelet methods for parabolic evolution problems*. Math. Comp., 2009.

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What do we gain?

- ✓ Periodic basis functions
- ✗ Additional dimension

Alternative:

- ▶ Time-stepping (fixed-point iterations)

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What do we gain for RB?

Basis construction

▶ Space-time basis functions	vs.	▶ Spatial basis functions (POD-Greedy)
▶ Space-time offline quantities		▶ Spatial offline quantities

- ✓ Full space-time information
- ✓ Time-dependent operators
- ✗ Large offline systems
- ✗ High memory requirement

What do we gain for RB?

Basis construction

- Space-time basis functions
- Space-time offline quantities

vs.

- Spatial basis functions (POD-Greedy)
- Spatial offline quantities

✓ Full space-time information

✓ Time-dependent operators

✗ Large offline systems

✗ High memory requirement

Online computations:

- 1 reduced system ($N_{\text{ST}} \times N_{\text{ST}}$)

vs.

- Fixed-point iterations (in each timestep $N_{\text{FP}} \times N_{\text{FP}}$)

✓ Fast online systems

Space-Time Error Bounds

Rigorous and Effective Error Bounds

$$\|e_N(\mu)\|_{\mathcal{Y}} \leq \frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\alpha(\mu)} =: \Delta_N^{\mathsf{ST},\mathcal{Y}}(\mu),$$
$$\|e_N(\mu)\|_{\mathcal{X}} \leq \frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\beta(\mu)} =: \Delta_N^{\mathsf{ST},\mathcal{X}}(\mu),$$

where

- $r_N(\mu) : \mathcal{Y} \to \mathbb{R}$ is the space-time residual,
- $\beta(\mu) = \inf_{0 \neq u \in \mathcal{X}^{\mathrm{per}}} \sup_{0 \neq v \in \mathcal{Y}} \frac{|b(u,v;\mu)|}{\|u\|_{\mathcal{X}^{\mathrm{per}}} \|v\|_{\mathcal{Y}}}$ is the inf-sup constant of $b(\cdot,\cdot;\mu)$.

K. Steih, K. Urban. *Space-Time RBM for Time-Periodic Partial Differential Equations*. Proc. MATHMOD, 2012.

Space-Time Error Bounds

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Advantages:

- Bounds in the “correct” norm
- Space-time $\beta(\mu)$ reflects true system behaviour
- Sharp bounds, independent of timestep size

K. Steih, K. Urban. *Space-Time RBM for Time-Periodic Partial Differential Equations*. Proc. MATHMOD, 2012.

K. Urban, A.T. Patera *An improved error bound for RB approx. of linear parabolic systems*. Math. Comp., 2012 (*accepted*).

So what about the extra dimension?

Initial Value Problems:

Choose a (tensorized) space-time basis with

- piecewise linear trial functions in time
- piecewise constant test functions in time

$\Rightarrow$ **Crank-Nicolson Scheme**

M. Yano, A.T. Patera, K. Urban *A Space-Time Certified RBM for Burgers' Equation*. M3AS, 2012 (*submitted*).

M. Yano. *A Space-Time Petrov-Galerkin Certified RBM – Appl. to the Boussinesq Eq.*. SIAM J. Sc. Comp., 2012 (*submitted*).

So what about the extra dimension?

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Choose a (tensorized) space-time basis with

- piecewise linear trial functions in time
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⇒ Crank-Nicolson Scheme

Periodic Problems:

- Crank-Nicolson scheme has to be combined with fixed-point iterations.
- Calculations do not decouple.

⇒ Space-Time Adaptivity

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Doing it with wavelets!

Find $u(\mu) \in \mathcal{X}$: $\qquad b(u, v; \mu) = f(v; \mu) \qquad \forall\, v \in \mathcal{Y}$

$$\mathcal{Y} = L_2(0, T) \otimes V$$

$$\mathcal{X}^{\text{per}} = L_2(0, T) \otimes V \cap H^1_{\text{per}}(0, T) \otimes V'$$

A) Tensorized Riesz Wavelet Bases

Collection $\Upsilon := \{\gamma_i : i \in \mathbb{N}\} \subset \mathcal{H}$ (separable Hilbert): For $v = \sum_{i=1}^{\infty} v_i \gamma_i$

$$\exists\, \mathrm{c}, \mathrm{C} > 0 : \quad \mathrm{c}\|\mathbf{v}\|^2_{\ell_2(\mathbb{N})} \leq \|v\|^2_{\mathcal{H}} \leq \mathrm{C}\|\mathbf{v}\|^2_{\ell_2(\mathbb{N})} \qquad \forall\, \mathbf{v} = (v_i)_{i\in\mathbb{N}} \in \ell_2(\mathbb{N}).$$

- Collections for ranges of Sobolev spaces (after re-normalization) in 1D

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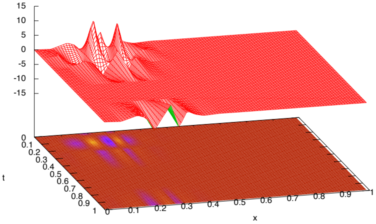
- Collections for ranges of Sobolev spaces (after re-normalization) in 1D
- Tensor product bases in higher dimensions:

$$\mathbf{\Psi}^{\mathcal{X}} = \mathbf{D}^{\mathcal{X}} (\Theta^{\text{per}} \otimes \mathbf{\Sigma}) = \{\psi^{\mathcal{X}}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathcal{J}^{\mathcal{X}}\}$$
$$\mathbf{\Psi}^{\mathcal{Y}} = \mathbf{D}^{\mathcal{Y}} (\Theta \quad \otimes \mathbf{\Sigma}) = \{\psi^{\mathcal{Y}}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathcal{J}^{\mathcal{Y}}\}$$



Doing it with wavelets!

B) Equivalent Bi-infinite Matrix-Vector Problem

$$\text{Find } \mathbf{u} \in \ell_2(\mathcal{J}^{\mathcal{X}}) : \qquad \mathbf{B}\mathbf{u} = \quad \mathbf{f}, \qquad \mathbf{f} \in \ell_2(\mathcal{J}^{\mathcal{Y}}).$$

with $\mathbf{B} = \left[b(\psi^{\mathcal{X}}_{\boldsymbol{\mu}}, \psi^{\mathcal{Y}}_{\boldsymbol{\lambda}}) \right]_{\boldsymbol{\lambda} \in \mathcal{J}^{\mathcal{X}}, \boldsymbol{\mu} \in \mathcal{J}^{\mathcal{Y}}}, \quad \mathbf{f} = \left[\langle f, \psi^{\mathcal{Y}}_{\boldsymbol{\lambda}} \rangle \right]_{\boldsymbol{\lambda} \in \mathcal{J}^{\mathcal{Y}}}.$

C)

A. Cohen, W. Dahmen, R. DeVore. *Adaptive wavelet methods. II. Beyond the elliptic case*. Found. Comput. Math., 2002.

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B) Equivalent Bi-infinite Matrix-Vector Problem – Normal Equations

Find $\mathbf{u} \in \ell_2(\boldsymbol{\mathcal{J}}^{\mathcal{X}}) :$ $$\mathbf{B}^\top \mathbf{B} \mathbf{u} = \mathbf{B}^\top \mathbf{f}, \qquad \mathbf{f} \in \ell_2(\boldsymbol{\mathcal{J}}^{\mathcal{Y}}).$$

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- Normal Equations: Galerkin problem, $\mathbf{B}^\top \mathbf{B}$ symmetric positive definite

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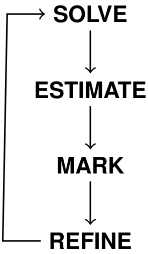
C) Adaptive Wavelet Galerkin Methods


SOLVE

ESTIMATE

MARK

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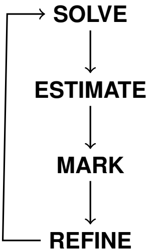
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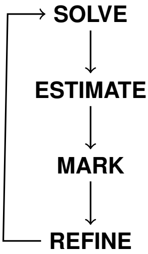
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Find $\mathbf{r} \in \ell_2(\boldsymbol{\mathcal{J}}^{\mathcal{X}})$: $\quad \mathbf{r} = \mathbf{B}^\top \mathbf{B} \mathbf{u}_{\boldsymbol{\Lambda}_k^{\mathcal{X}}} - \mathbf{B}^\top \mathbf{f}$

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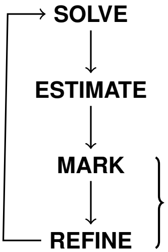

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Bulk Chasing: Find $\boldsymbol{\Lambda}^{\mathcal{X}}_{k+1}$ s.t. $\|\mathbf{r}_{\boldsymbol{\Lambda}^{\mathcal{X}}_{k+1}}\|_{\ell_2(\boldsymbol{\Lambda}^{\mathcal{X}}_{k+1})} \geq \delta \|\mathbf{r}\|_{\ell_2(\boldsymbol{\mathcal{J}})}$

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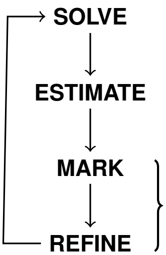

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Doing it with wavelets!

B) **Equivalent Bi-infinite Matrix-Vector Problem – Normal Equations**

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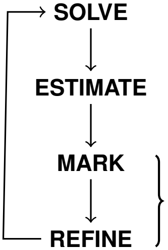
- Normal Equations: Galerkin problem, $\mathbf{B}^\top\mathbf{B}$ symmetric positive definite

C) **Multitree-based Least-Squares Adaptive Wavelet Galerkin Methods**



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$$\text{Find } \mathbf{r} \in \ell_2(\boldsymbol{\Xi}^{\mathcal{X}}_k) : \quad \mathbf{r}_{\boldsymbol{\Xi}^{\mathcal{X}}_k} = {}_{\boldsymbol{\Xi}^{\mathcal{X}}_k}\mathbf{B}^\top_{\boldsymbol{\Xi}^{\mathcal{Y}}_k \boldsymbol{\Xi}^{\mathcal{Y}}_k}\mathbf{B}_{\boldsymbol{\Lambda}^{\mathcal{X}}_k}\mathbf{u}_{\boldsymbol{\Lambda}^{\mathcal{X}}_k} - {}_{\boldsymbol{\Xi}^{\mathcal{X}}_k}\mathbf{B}^\top_{\boldsymbol{\Xi}^{\mathcal{Y}}_k}\mathbf{f}_{\boldsymbol{\Xi}^{\mathcal{Y}}_k}$$

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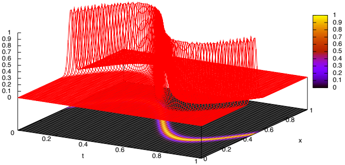
S. Kestler, K. Steih, K. Urban. *An efficient space-time AWGM for time-periodic PDEs.* 2013 (submitted).

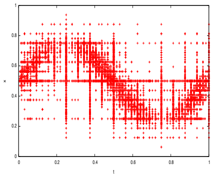
MT–LS–AWGM

$$\begin{cases} u_t - u_{xx} + u_x + u = f(t,x) & \text{on } \Omega = (0,1), \\ u(t,0) = u(t,1) & \text{for all } t \in [0,\, T], \\ u(0,x) = u(T,x) = 0 & \text{on } \overline{\Omega}, \end{cases}$$

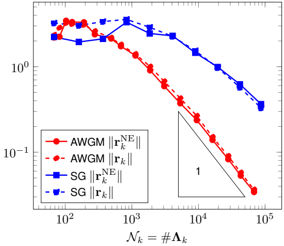


(a) Exact Solution:

$$u(t,x) = e^{-1000\left(x-\left(\frac{1}{2}+\frac{1}{4}\sin(2\pi t)\right)\right)^2}$$



(b) Support centers



(c) Residual behaviour

Introduction

Space-Time Adaptive Truth Computation

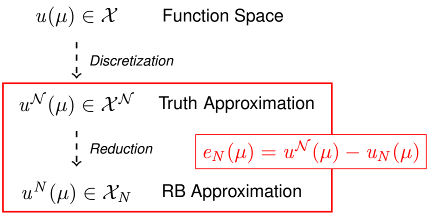
Adaptive RB Methods

There's no truth in the world..

Find $u(\mu) \in \mathcal{X}$: $\quad b(u,v;\mu) = f(v;\mu) \quad \forall\, v \in \mathcal{Y}$

Usual RB Point of View:

Assumption: Truth spaces $\mathcal{X}^{\mathcal{N}}$, $\mathcal{Y}^{\mathcal{N}}$ are "good enough" for all $\mu \in \mathcal{D}$.



$u(\mu) \in \mathcal{X}$ Function Space

Discretization

$u^{\mathcal{N}}(\mu) \in \mathcal{X}^{\mathcal{N}}$ Truth Approximation

Reduction

$e_N(\mu) = u^{\mathcal{N}}(\mu) - u_N(\mu)$

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[Offline-Online decomposition:

$$(\hat{f}^q, v)_{\mathcal{Y}} = f^q(v) \qquad \forall\, v \in \mathcal{Y}^{\mathcal{N}} \qquad , \forall\, q$$

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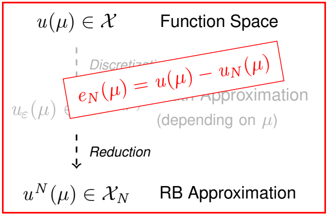

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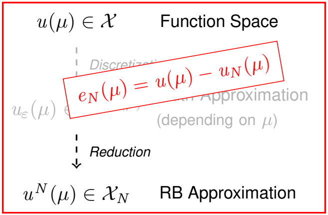

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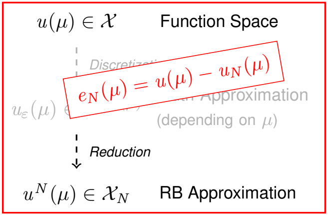

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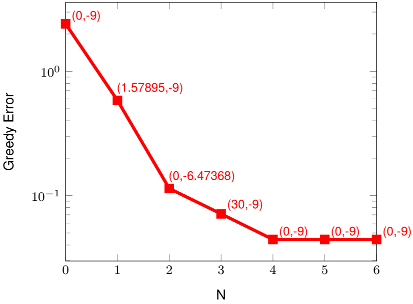
Periodic Convection-Diffusion-Reaction Problem

$$\begin{cases} u_t - u_{xx} + \mu_1 u_x + \mu_2 u = \cos(2\pi t) & \text{on } \Omega = (0,1), \\ u(t,0) = u(t,1) & \text{for all } t \in [0,T], \\ u(0,x) = u(T,x) = 0 & \text{on } \overline{\Omega}. \end{cases}$$

Coercive for parameter range $\mu \in [0,30] \times [-9,15]$.

Fixed adaptivity tolerances:

- Snapshots:
  $\varepsilon_u = 0.005$
- Riesz Representors:
  $\varepsilon_{\hat{f}} = \varepsilon_{\hat{b}} = 0.0005$



Snapshot Accuracy

Greedy Convergence [Binev et.al.]

Suppose that the *Kolmogorov n-width* for some compact set $\mathcal{F}$ fulfills $d_0(\mathcal{F}) \leq M$, $d_n(\mathcal{F}) \leq M n^{-\alpha}$ for some M, $\alpha > 0$.
For an approximation $\widehat{F}_n := \mathrm{span}\{\hat{f}_0, \dots, \hat{f}_{n-1}\}$ with $\|f_i - \hat{f}_i\| \leq \varepsilon$, the weak Greedy algorithm with parameter γ then has the convergence rate

$$\sup_{f \in \mathcal{F}} \mathrm{dist}(f, \widehat{F}_n) \leq C \max\{M n^{-\alpha}, \varepsilon\}, \qquad C = C(\alpha, \gamma).$$

P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, P. Wojtaszczyk. *Convergence Rates for Greedy Algorithms in RBM.* SIAM J. Math. Anal., 2010.

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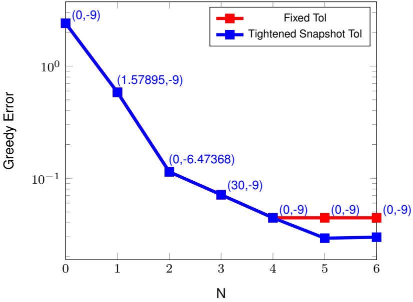
Idea:

- Tighten snapshot accuracy "when necessary"
- Here: at repeated selection of same parameter μ
- Reduction factor: $\varepsilon_u^{\text{new}} = 0.1 \cdot \varepsilon_u$



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Let $S_n := \{\mu_0, \ldots, \mu_{n-1}\}$ the selected parameters and $\varepsilon_n^* := \max_{0 \leq i < n} \min_{0 \leq j < n, \mu_j = \mu_i} \varepsilon_i$, $\|f_i - \hat{f}_i\| \leq \varepsilon_i$. Then the Greedy convergence is

$$\sup_{f \in \mathcal{F}} \operatorname{dist}(f, \widehat{F}_n) \leq C \max\{M|S_n|^{-\alpha}, \varepsilon_n^*\}, \qquad C = C(\alpha, \gamma).$$

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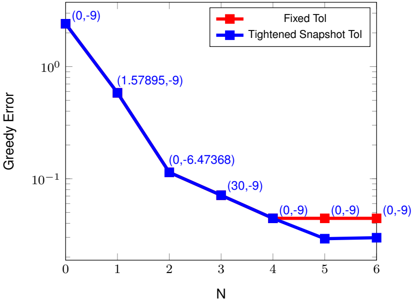

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Error Estimator Accuracy

Error bound w.r.t. exact solution

$$\|u(\mu) - u_N(\mu)\|_{\mathcal{X}} \leq \Delta_N(\mu) := \frac{\|\widehat{r}_N(\cdot;\mu)\|_{\mathcal{Y}}}{\beta(\mu)}, \qquad r_N(\cdot;\mu) : \mathcal{Y} \to \mathbb{R}$$

Problem:

- Computable: $\Delta_{N,\varepsilon}(\mu) := \frac{\|\widehat{r}_{N,\varepsilon}(\cdot;\mu)\|_{\mathcal{Y}}}{\beta(\mu)}$ with $\|\widehat{r}_N(\cdot;\mu) - \widehat{r}_{N,\varepsilon}(\cdot;\mu)\|_{\mathcal{Y}} \leq \varepsilon$.
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Idea:

- Tighten Riesz representor accuracy as well
  - $\hat{b}$: for all new snapshots
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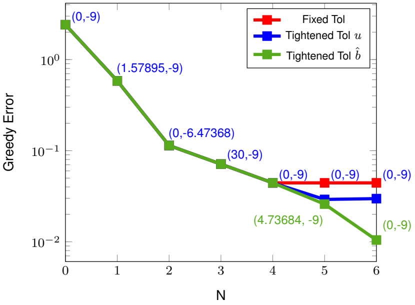

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  - $\hat{f}$: recalculate representors
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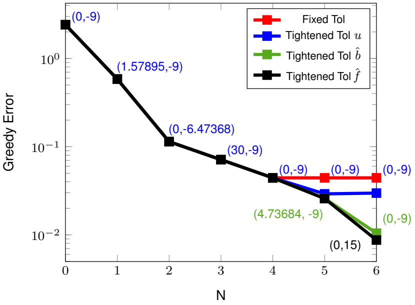

Equivalent Error Estimator

Recall:

- Error bound accuracy: $|\Delta_N(\mu) - \Delta_{N,\varepsilon}(\mu)| \leq \frac{\varepsilon}{\beta(\mu)}$
- Greedy training relies on *equivalent* error estimator
- Error bound accuracy can be chosen *independently* of snapshot accuracy

Question:

- For which tolerances ε is $\Delta_{N,\varepsilon}(\mu)$ still equivalent to $e_N(\mu)$?

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A-posteriori equivalence condition

Assume $\Delta_{N,\varepsilon}(\mu) \leq \Delta_N(\mu)$ and let

$$\varepsilon = \varepsilon(\mu) \leq \frac{1-c}{2-c}\beta(\mu)\Delta_{N,\varepsilon}(\mu), \qquad c \in (0,1).$$

Then: $$c\|e_N(\mu)\|_{\mathcal{X}} \;\leq\; \Delta_{N,\varepsilon}(\mu) \;\leq\; \frac{\gamma_b(\mu)}{\beta(\mu)}\|e_N(\mu)\|_{\mathcal{X}}.$$

- Assumption realistic, as $\mathcal{Y}_\varepsilon \subset \mathcal{Y}$.
- Offline-online decomposition: accuracy ε can be bounded a-posteriori

Conclusion

Adaptive calculations:

- Compute snapshots and Riesz representors *adaptively* up to a certain accuracy
- Consider error with respect to *exact* solution
- Update snapshots/representors when necessary
- Accuracy of basis functions and error estimator can be determined *separately*
- A-posteriori criteria to ensure *equivalence* of error estimator

Goal:

- Control *real* error
- *Minimize computational cost* for target RB tolerance

Outlook:

- Different update strategies
- Minimize error bound decomposition by using EIM ([Casenave])
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F. Casenave. *Accurate a posteriori error evaluation in the reduced basis method* . C. R. Math., 2012.

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