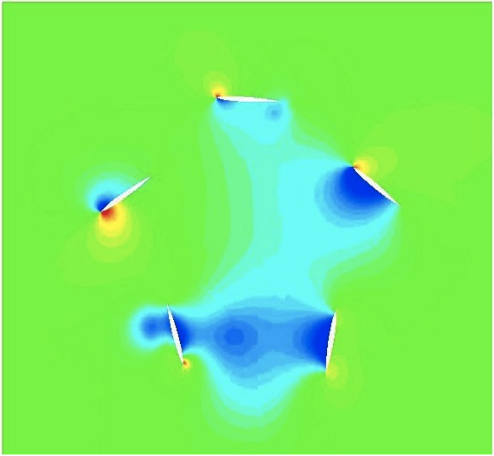

On Space-Time Approaches in Reduced Basis Methods

Time-Periodic Problems

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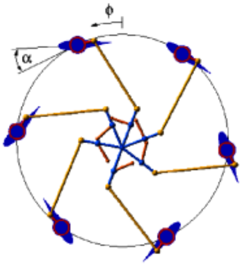
Motivation

Optimization of a ship propeller



- parameter studies
- long simulation times

⇒ Model reduction in parameter space

Challenges for Reduced Basis Methods

- **Time-periodic problem**
- Parameter-dependent domain [CTU]
- Time-dependent parameter function
- Nonlinearities [CTU]
- ⋮

C. Canuto, T. Tonn and K. Urban. *A-posteriori error analysis of the RBM for non-affine parameterized nonlinear PDE's*. SIAM J. Numer. Anal. 2009.

Mathematical Problem Formulation

Parameter-dependent time-periodic problem

Let $V \hookrightarrow H \hookrightarrow V'$ a Gelfand triple, $\mu \in \mathcal{D} \subset \mathbb{R}^p$, $\mathcal{A}(t;\mu) : V \to V'$.

$$J(u(\mu)) = \int_0^T \ell(u(\mu);\mu)\, dt,$$

$$u_t + \mathcal{A}(t;\mu)u = g(t;\mu) \qquad \text{on } \Omega,$$

$$u(0) = u(T) \qquad \text{in } H,$$

$$u(t,x;\mu) = h_D(x;\mu) \text{ on } \Gamma_D, \qquad \frac{\partial}{\partial n} u(t,x;\mu) = h_N(x;\mu) \text{ on } \Gamma_N.$$

Example:

$$\mathbf{V} = H^1_{(0)}(\Omega) \quad \hookrightarrow \quad \mathbf{H} := L_2(\Omega) \quad \hookrightarrow \quad \mathbf{V}' := H^{-1}(\Omega).$$

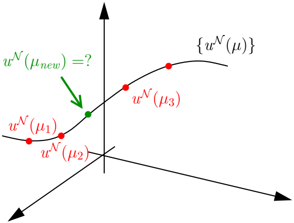
Introduction to RBM

Space-Time in RBM

Why Space-Time ?

Adaptive Space-Time Methods

The idea behind RBM



Assumption:

- High-dimensional discretization space $\mathcal{X}^{\mathcal{N}}$ (FEM, FV, wavelets,...)
- $u^{\mathcal{N}}(\mu)$ smooth in μ
- Low-dimensional solution space $\mathcal{M}^{\mathcal{N}} = \{u^{\mathcal{N}}(\mu) : \mu \in \mathcal{D}\}$

Procedure: Approximation of $\mathcal{M}^{\mathcal{N}}$

1. *Basis construction:* N solutions $u^{\mathcal{N}}(\mu_1), \dots, u^{\mathcal{N}}(\mu_N)$, $N \ll \mathcal{N}$.
2. *Projection:* Compute $u(\mu_{new})$ on $\mathcal{X}^N := \mathrm{span}\{u^{\mathcal{N}}(\mu_1), \dots, u^{\mathcal{N}}(\mu_N)\}$.

Application:

- Many-query context (e.g. optimization)
- Real-time applications

Reliable and efficient error bounds

$u(\mu) \in \mathcal{X}$ Function Space

Discretization

$u^{\mathcal{N}}(\mu) \in \mathcal{X}^{\mathcal{N}}$ Truth Approximation

Reduction

$u^{N}(\mu) \in \mathcal{X}^{N}$ RB Approximation

$e_N(\mu) = u^{\mathcal{N}}(\mu) - u_N(\mu)$

Error Bound

$$\|u^{\mathcal{N}}(\mu) - u_N(\mu)\| \leq \Delta_N(\mu)$$

- **rigorous**: strict upper bound
- **efficient**: $\mathcal{N}$-independent
- **sharp**: $\eta(\mu) := \frac{\Delta_N(\mu)}{\|e_N(\mu)\|} \approx 1$

- *Offline Phase:* Parameter choice for basis construction
- *Online Phase:* Verification of reduced solutions

Introduction to RBM

Space-Time in RBM

Why Space-Time ?

Adaptive Space-Time Methods

Standard Approaches

Initial Value Problems

- Method of lines + time-stepping schemes (implicit Euler/Crank-Nicolson)

For $\{t_k\}_{k=1,\ldots,K}$, find $u^k(\mu) = u(t_k;\mu) \in V$ with $u^0(\mu) = u_0(\mu)$ and

$$\langle u^k, v\rangle + \Delta t\, a(t_k, u^k, v;\mu) = \langle u^{k-1}, v\rangle + \Delta t\, g(t_k, v;\mu) \qquad \forall\, v \in V.$$

Note:
A Crank-Nicolson time-stepping scheme can be formulated as a space-time problem for the special choice by choosing a finite tensorized basis with

- piecewise linear trial functions in time,
- piecewise constant test functions in time.

M. Grepl and A.T. Patera. *A Posteriori Error Bounds for Reduced-Basis Approximations of Parametrized Parabolic Partial Differential Equations*. M2AN, 2005.

B. Haasdonk and M. Ohlberger. *Reduced Basis Method for Finite Volume Approximations of Parameterized Linear Evolution Equations*. M2AN, 2008.

Standard Approaches

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Periodic Problems

- Method of lines + time-stepping + **fixed-point iterations**: $u^0(\mu) = u^K(\mu)$

Note:
Consequently, such a solution of a periodic problem requires

- an a-priori unknown number of repetitions of an initial value problem.

Usually, these approaches only yield a spatial reduced basis $V^N \subset V^{\mathcal{N}_x}$ and thus also require time-stepping methods in the online phase.

RB Error Bounds in time-stepping approaches

Time-discrete error bound for periodic problems

The reduction error for a reduced basis based on an implicit Euler scheme can be bounded as follows:

$$\|e_N(\mu)\|_{\mathcal{Y}} \approx \left(\Delta t \sum_{k=1}^{K} \|e_N^k(\mu)\|_V^2\right)^{\frac{1}{2}} \leq \left(\frac{\Delta t}{\alpha^2(\mu)} \sum_{k=1}^{K} \|r_N^k(\cdot;\mu)\|_{V'}^2\right)^{\frac{1}{2}} =: \Delta_N^{\mathsf{FP},\mathcal{Y}}(\mu),$$

where

- $r_N^k(\mu) : V \to \mathbb{R}$ is the residual at time step t_k
- $\alpha(\mu)$ is the coercivity constant of $a(t, \cdot, \cdot; \mu)$
- $\mathcal{Y} := L_2(0, T; V)$

Here:

- Calculation of $\|r_N^k(\cdot;\mu)\|_{V'}$
  - using Riesz representors $\hat{r}_N^k(\mu) \in V$
  - offline-online decomposition

K. Steih and K. Urban. *Space-Time Reduced Basis Methods for Time-Periodic Partial Differential Equations*. Proc. MATHMOD, 2012.

Space-Time Formulation

Notation:

$$\mathcal{Y} := L_2(0,T;V),$$

$$\mathcal{X}^{per} := L_2(0,T;V) \cap H^1_{per}(0,T;V') := \{u \in L_2(0,T;V) \colon u_t \in L_2(0,T;V'),\ u(0)=u(T) \text{ in } H\}$$

Well-posedness

Let $\mathcal{A}(t)$ be uniformly bounded and coercive. Then the problem

$$\text{Find } u(\mu) \in \mathcal{X}^{per} : \underbrace{\int_0^T \langle u_t, v\rangle\, dt + \int_0^T a(t,u,v;\mu)\, dt}_{=:b(u,v;\mu)} = \underbrace{\int_0^T g(t,v;\mu)\, dt}_{:=f(v;\mu)} \quad \forall\, v \in \mathcal{Y}.$$

is well-posed.

Proof: Babuška-Aziz, cf. [Schwab/Stevenson 2009]. □

$\Rightarrow$ **Additional dimension**

C. Schwab and R. Stevenson. *Space-time adaptive wavelet methods for parabolic evolution problems*. Mathematics of Computation, 2009.

Space-Time RB Error Bound

Error bounds for a periodic space-time problem

The reduction error for a space-time reduced basis can be bounded as follows:

$$\|e_N(\mu)\|_{\mathcal{Y}} \leq \frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\alpha(\mu)} =: \Delta_N^{\mathsf{ST},\mathcal{Y}}(\mu),$$

$$\|e_N(\mu)\|_{\mathcal{X}} \leq \frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\beta(\mu)} =: \Delta_N^{\mathsf{ST},\mathcal{X}}(\mu),$$

where

- $r_N(\mu) : \mathcal{Y} \to \mathbb{R}$ is the space-time residual,
- $\beta(\mu) = \inf_{0\neq u\in\mathcal{X}^{per}} \sup_{0\neq v\in\mathcal{Y}} \frac{|b(u,v;\mu)|}{\|u\|_{\mathcal{X}^{per}}\|v\|_{\mathcal{Y}}}$ is the inf-sup constant of $b(\cdot,\cdot;\mu)$.

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Space-Time RB Error Bound

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- Calculation of $\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}$:
  - using Riesz representors $\hat{r}_N(\mu) \in \mathcal{Y}$
  - offline-online decomposition
- Both $\alpha(\mu)$ and $\beta(\mu)$ are usually replaced by efficient (i.e. $\mathcal{N}$-independent) lower bounds $\alpha_{\mathsf{LB}}(\mu)$, $\beta_{\mathsf{LB}}(\mu)$.

K. Steih and K. Urban. *Space-Time Reduced Basis Methods for Time-Periodic Partial Differential Equations*. Proc. MATHMOD, 2012.

Short comparison

Space-Time	**Fixed-Point Iterations**
Solution procedure offline: ▶ Periodic basis functions ▶ 1 large system ($\mathcal{O}(\mathcal{N}_t\mathcal{N}_x)$)	*Solution procedure offline:* ▶ Time-stepping with unknown number of iterations M ▶ Many small systems ($MK\mathcal{O}(\mathcal{N}_x)$)
Basis construction: ▶ Space-time basis functions ▶ Space-time offline quantities → high memory requirements ▶ Time-dependent operators: ✓	*Basis construction:* ▶ Spatial basis functions (POD) ▶ Spatial offline quantities → low memory requirements ▶ Time-dependent operators: ?
Solution procedure online: ▶ 1 reduced system ($\mathcal{O}(N_{\mathsf{ST}})$)	*Solution procedure online:* ▶ MK reduced systems (à $\mathcal{O}(N_{\mathsf{FP}})$)

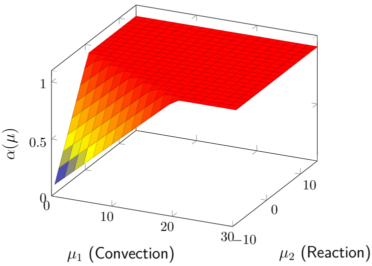
Model Problem

Convection-Diffusion-Reaction Example

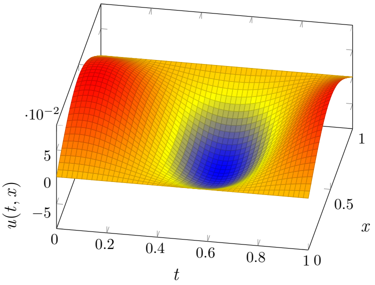
$$u_t - u_{xx} + \mu_1(\tfrac{1}{2} - x)u_x + \mu_2 u = \cos(2\pi t) \qquad \text{on } \Omega = (0,1),$$
$$u(t,0) = u(t,1) = 0,$$
$$u(0,x) = u(T,x), \qquad (T = 1).$$

Parameter Domain: $\mu \in \mathcal{D} := [0,30] \times [-9,15]$.

Coercivity constant



Sample Solution ($\mu = (10,0)^T$)



Why Space-Time? – Effective error bounds

Space-Time inf-sup constant $\beta(\mu)$

Consider **non-coercive initial-value** systems.

- Case 1: Diffusion-Convection $(\mu_2 = 0)$ – system asymptotically stable
- Case 2: Diffusion-Reaction $(\mu_1 = 0)$ – system asymptotically unstable

K. Urban and A.T. Patera *An improved error bound for RB approximation of linear parabolic systems. Submitted to* Math. Comp., 2012.

Why Space-Time? – Effective error bounds

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Standard Crank-Nicolson error bound contains

- stability constant $\rho_N(t^k, \mu)$ (energy estimates),
- with long-term behaviour $e^{-\mu_i T}$, $i = 1, 2$.

$\Rightarrow$ **exponentially growing** error bounds.

Space-time inf-sup constant:

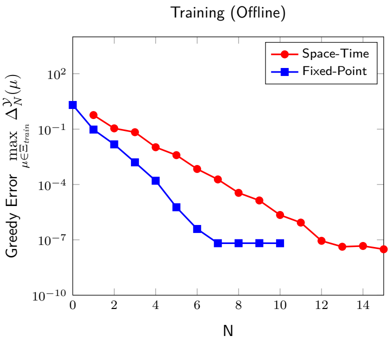
- Case 1: long-term behaviour $(\mu_1 T)^{-1}$
- Case 2: long-term behaviour $e^{-\mu_2 T}$

$\Rightarrow$ reflects **true system behaviour**.

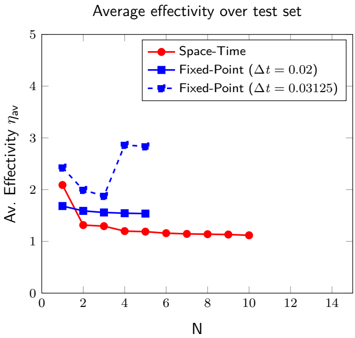
K. Urban and A.T. Patera *An improved error bound for RB approximation of linear parabolic systems. Submitted to* Math. Comp., 2012.

Why Space-Time? – Effective error bounds

$$\|r_N(\cdot;\mu)\|_{\mathcal{Y}'} \;\leftrightarrow\; \left(\sum_{k=1}^{K} \|r_N^k(\cdot;\mu)\|_{V'}^2\right)^{1/2}$$



(a) Training ($n_{\text{train}} = 400$)

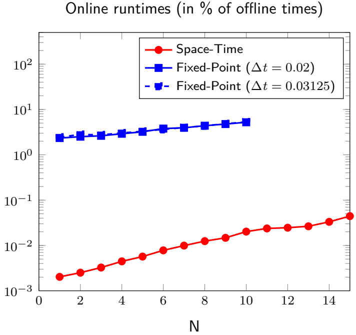


(b) Average Effectivity ($n_{\text{test}} = 225$)

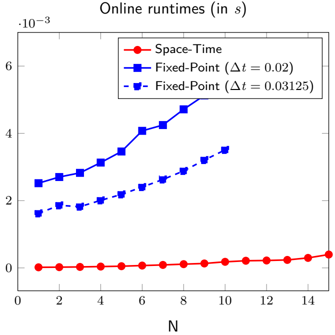
$$\eta_{\text{av}} := \frac{1}{n_{\text{test}}} \sum_{\mu\in\Xi_{\text{test}}} \frac{\Delta_N^{\mathcal{Y}}(\mu)}{\|e_N(\mu)\|}$$

Why Space-Time? – Online speed-up

$$\mathcal{O}(N_{\mathsf{ST}}) \leftrightarrow MK\,\mathcal{O}(N_{\mathsf{FP}})$$



(a) Runtime reduction



(b) Online runtimes

Implementations:

- Space-Time: `LAWA`
- Fixed-Point Iterations: `rbOOmit/libmesh`

	Time Truth (in s)
Space-Time	0.8965
Fixed-Point ($\Delta t = 0.02$)	0.1076
Fixed-Point ($\Delta t = 0.03125$)	0.0662

`LAWA`: Library for Adaptive Wavelet Algorithms, *lawa.sourceforge.net*, 2001
`libmesh`: Kirg, Stogner and Carey, *A C++ Library for Parallel Adaptive Mesh Refinement / Coarsening Simulations*, Eng. with Comp., 2006

Adaptive Space-Time Methods

Initial Value Problems:

- "Right" choice of space-time basis leads to Crank-Nicolson scheme.
- Calculations decouple.

Periodic Problems:

- Crank-Nicolson scheme has to be combined with fixed-point iterations.
- Calculations do not decouple.

Idea

Adaptive space-time algorithm.

Wavelets:

Variational formulation equivalent to bi-infinite matrix-vector problem:

$$b(u, v; \mu) = f(v; \mu), \quad u \in \mathcal{X}, v \in \mathcal{Y} \quad \Longleftrightarrow \quad \mathbf{Bu} = \mathbf{f}, \qquad u, f \in \ell_2$$

C. Schwab and R. Stevenson. *Space-time adaptive wavelet methods for parabolic evolution problems*. Mathematics of Computation, 2009.

Conclusion

Space-Time methods can lead to

- inf-sup constants that reflect the true behaviour of the underlying systems,
- more effective error bounds,
- faster reduced models.

The drawback of the additional dimension can be circumvented:

- Initial value problems: specific basis yields Crank-Nicolson scheme. [YPU]
- Periodic problems: adaptive space-time methods.

M. Yano, A.T. Patera and K. Urban *A Space-Time Certified RBM for Burgers' Equation*. *Submitted to* Math. Models and Methods in Appl. Sc. , 2012.

M. Yano. *A Space-Time Petrov-Galerkin Certified RBM – Application to the Boussinesq Equation*. *Submitted to* SIAM J. Sc. Comp. , 2012.

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Thank you for your attention