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Planar Analytic Dynamical Systems and their phase space structure

Master's Thesis

in Mathematics

by

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1 Introduction

The qualitative description of the phase space of a differential equation has been an important area of research in mathematics. In particular, the theory of Poincare and Bendixson provides a very good understanding of the dynamics of two-dimensional autonomous systems, i.e. dynamical systems in $\mathbb{R}^2 \cong \mathbb{C}$. When considering holomorphic (complex analytic) vector fields, even more structure of the phase space can be uncovered. More precisely, it is possible to characterize and describe the topological, geometrical and analytical properties of so-called *Planar Analytic Dynamical Systems* locally near equilibria as well as globally. This detailed analysis is the essential goal of this work. To achieve this objective, the thesis is mainly divided into two parts.

In the first part (Chapter 3), we explore the *quantitative* properties of analytic dynamical systems. Our investigation reveals that the existence and uniqueness of the solutions can be guaranteed, regardless of whether the time variable is real or complex.

The second objective of this study (Chapters 4 and 5) is to revise and augment the works of Broughan ([2] and [3]) by providing a detailed *qualitative* description and characterization of the topological and geometrical structure of the phase space. This endeavor involves two main steps: firstly, we aim to explore the local behavior near an equilibrium, and secondly, we will analyze the global properties of the so-called global neighborhood of an equilibrium. Our investigation reveals that the boundaries of such neighborhoods consist of, at most, countably many separatrix components.
Throughout this chapter, a variety of geometric structures will also be ruled out. For instance, there are no saddle points or limit cycles in analytical dynamical systems. This leads to a more accurate description of limit sets according to the Poincare-Bendixson theory. Additionally, examples and phase space plots will be used to illustrate the properties and characteristics of these geometric structures.

Throughout this work, all utilized sources and references will be appropriately cited. Any contributions made by me will be clearly stated. Sections that are authored entirely by me will be marked as such at the beginning of each section.

2 Some common notations

In the following some common notations and definitions are introduced, that are used throughout this thesis. Further definitions which require more context are introduced in the following chapters. The notations in the context of dynamical systems are introduced at the beginning of Chapter 4.

For a set O in a topological space X the boundary of O is denoted by ∂O, and the set of all inner points of O by $\mathring{O}$. For a complex number $z \in \mathbb{C}$ (or a vector $z \in \mathbb{R}^2 \cong \mathbb{C}$) we denote by $\Re(z)$ the real part, by $\Im(z)$ the imaginary part and by $\arg(z)$ the argument, i.e the angle with respect to the real axis, of z. If $z = 0$, the argument is not defined. Denote by $\mathcal{B}_r(x) \subset X$ the open ball with radius $r > 0$ and center $x \in X$. For a vector space V, the dual space of V is denoted by $V^\star$. Denote by $\langle x, y\rangle$ the Euclidean scalar product of two vectors $x, y \in \mathbb{C}^n$, $n \in \mathbb{N}$. For a matrix $A \in \mathbb{R}^n$, $n \in \mathbb{N}$, we denote by $\det(A)$ the determinant, by $\mathrm{tr}(A)$ the trace and by p_A the characteristic polonymial of A. The set $\mathrm{Eig}(A)$ contains all eigenvalues of A.

For an arbitrary measurable set $B \subset \mathbb{R}^n$, $n \in \mathbb{N}$, the Lebesgue measure of B is denoted by $\lambda(B)$.

For arbitrary metric spaces X and Y the space of continuous functions from X to Y is denoted by $\mathcal{C}^0(X;Y)$. The space $\mathcal{C}^k(X;Y)$, $k \in \mathbb{N}$, consists of all continuously differentiable functions from X to Y. The sets X and Y are often omitted in this notation if they result from the context. If $\Omega \subset \mathbb{R}^n$ is open and $f = (f_1, \ldots, f_m) \in \mathcal{C}^1(\Omega; \mathbb{R}^m)$, $n, m \in \mathbb{N}$, then the Jacobian matrix of f in $x_0 \in \Omega$ is denoted by $\mathcal{J}_f(x_0)$. The element of $\mathcal{J}_f(x_0)$ in the i^{th} row and j^{th} column is the partial derivative of f_j in the i^{th} Euclidean coordinate direction, i.e. $(\mathcal{J}_f(x_0))_{i,j} = \partial_i f_j(x_0)$.

For an open set $\Omega \subset \mathbb{C}^n$, $n \in \mathbb{N}$, we denote by $\mathcal{O}^m(\Omega) := \mathcal{O}(\Omega, \mathbb{C}^m)$, $m \in \mathbb{N}$, the complex linear space of vector-valued holomorphic functions $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$. Recall that such a function is holomorphic if and only if for all $\xi = (\xi_1, \ldots, \xi_n) \in \Omega$, all

$i \in \{1, \dots, n\}$ and all $j \in \{1, \dots, m\}$ the function $z \mapsto f_j(\xi_1, \dots, \xi_{i-1}, z, \xi_{i+1}, \xi_n)$ is holomorphic.[1] In the special case $m = 1$ we define $\mathcal{O}(\Omega) := \mathcal{O}^1(\Omega)$. For $f \in \mathcal{O}^n(\Omega)$ and $x_0 \in \Omega$ we set $\mathcal{J}_f(x_0) \in \mathbb{R}^{2n \times 2n}$ as the Jacobian matrix (see above) of f in x_0, understood as a function with $2n$ real input-variables and range $\mathbb{R}^{2n} \cong \mathbb{C}^n$. The residue of a holomorphic function $f \in \mathcal{O}(\mathcal{B}_r(x_0) \setminus \{x_0\})$, $r > 0$ sufficiently small, at an isolated singularity x_0 is denoted by $\operatorname{Res}(f, x_0)$.

Let $n \in \mathbb{N}$, $a_1 < b_1$ and $\gamma_1 : [a_1, b_1] \to \mathbb{R}^n$ be a path with image $\Gamma_1 := \gamma_1([a_1, b_1])$. Then we denote by $-\gamma_1$ or $-\Gamma_1$ (if there is a orientation of the parameterization given) the curve with the same image and opposite direction, i.e. the curve

$$-\gamma_1(t) := \gamma_1(a_1 + b_1 - t) \quad \forall\, t \in [a_1, b_1].$$

If $a_2 < b_2$ and $\gamma_2 : [a_2, b_2] \to \mathbb{R}^n$ is a second path with image $\Gamma_2 := \gamma_2([a_2, b_2])$ such that $\gamma_1(b_1) = \gamma_2(a_2)$, then the concatenation of γ_1 and γ_2 is denoted by $\gamma_1 + \gamma_2$. If the orientation of the curves is given, then we also denote the concatenation by $\Gamma_1 + \Gamma_2$. Correspondingly, we define $\gamma_1 - \gamma_2 := \gamma_1 + (-\gamma_2)$ and $\Gamma_1 - \Gamma_2 := \Gamma_1 + (-\Gamma_2)$, if $\gamma_1(b_1) = \gamma_2(b_2)$ is satisfied.

For a $c \in \mathbb{R}$ and a continuous function $f : [c, \infty) \to \mathbb{R}^m$ we define the positive limit set

$$w_+(f) := \{v \in \mathbb{R}^m : \exists\, (t_k)_{k \in \mathbb{N}} \subset [c, \infty) : t_k \to \infty \text{ and } f(t_k) \to v \text{ for } k \to \infty\}$$

and for $f : (-\infty, c] \to \mathbb{R}^m$ the negative limit set

$$w_-(f) := \{v \in \mathbb{R}^m : \exists\, (t_k)_{k \in \mathbb{N}} \subset (-\infty, c] : t_k \to -\infty \text{ and } f(t_k) \to v \text{ for } k \to \infty\}.$$

Important properties of limit sets can be found in [6, §4] and [7, Chapter 8.4]. In particular, if $w_+(f)$ (and $w_-(f)$, respectively) has only one element v_0, then

$$\lim_{t \to \infty} f(t) = v_0.$$

and the existential quantifier in the above sets can be replaced by a universal quantifier, cf. [7, Proposition 8.4.1].

[1] This result is known as *Hartogs' theorem on separate analyticity* and can be found in [4, Theorem 2.2.8]. A rigorous and detailed introduction to holomorphic functions in several variables can be found in [5, Chapter 1].

3 Solution Theory for Analytic Ordinary Differential Equations

In the first chapter, a quantitative theory for Analytic Ordinary Differential Equations will be explained. The existence and uniqueness of solutions with both, real-valued and complex-valued time will be proven. The underlying literature is [8, Chapter I.1], [9, Chapter 2.2] and [10, Chapter 2].

3.1 Solutions with real time

Definition 3.1
Let $n \in \mathbb{N}$ and $F = (F_1, \dots, F_n) \in \mathcal{O}^n(\Omega)$ be a holomorphic vector function defined on an open domain $\Omega \subset \mathbb{C}^n$. An *autonomous Analytic Ordinary Differential Equation with real time* by F on Ω is the system of n scalar function equations

$$x' = F(x). \tag{3.1}$$

A solution of the system (3.1) is a continuously differentiable curve $x = (x_1, \dots, x_n) : V \to \mathbb{C}^n$, defined on an open domain $V \subset \mathbb{R}$, satisfying the following two properties:

(i) The *phase curve* $x(V)$ of x belongs to Ω, i.e. $x(V) \subset \Omega$.

(ii) The equation $x'(t) = F(x(t))$ holds for all $t \in V$.

For a fixed point $(t_0, x_0) \in \mathbb{R} \times \Omega$ the *initial value problem* (IVP) is defined by (3.1) with an initial condition $x(t_0) = x_0$. Its solution is a solution of (3.1), whose phase curve passes the point x_0 at time t_0.

Theorem 3.2 (Existence and Uniqueness for real IVPs)
Let $\Omega \subset \mathbb{C}^n$ be an open domain and $F \in \mathcal{O}^n(\Omega)$. Then for every $(t_0, x_0) \in \mathbb{R} \times \Omega$ there exists a constant $\tau > 0$ such that the IVP

$$\begin{aligned} x' &= F(x) \\ x(t_0) &= x_0 \end{aligned} \tag{3.2}$$

has an unique solution $x : (t_0 - \tau, t_0 + \tau) \to \mathbb{C}^n$.

Proof
Let $t_0 \in \mathbb{R}$ and $x_0 = (x_{0,1}, \ldots, x_{0,n}) \in \Omega$ be arbitrarily fixed. Then there exists an $r > 0$ such that $\mathcal{B}_r(x_0) \subset \Omega$ with respect to the Euclidean norm on $\mathbb{C}^n$. The aim is to lead the problem back to a higher dimensional problem with real-valued right-hand side.
Set $y_0 := (\Re(x_{0,1}), \Im(x_{0,1}), \ldots, \Re(x_{0,n}), \Im(x_{0,n})) \in \mathbb{R}^{2n}$ and $\tilde{\Omega} := \mathcal{B}_r(y_0) \subset \mathbb{R}^{2n}$ with respect to the Euclidean norm on $\mathbb{R}^{2n}$. Write $F = (F_1, \ldots, F_n)$ and define $\tilde{F} : \tilde{\Omega} \to \mathbb{R}^{2n}$ by

$$\tilde{F}(y) := \sum_{j=1}^{n} \Re(F_j(y_1 + \mathrm{i}y_2, \ldots, y_{n-1} + \mathrm{i}y_n))e_{2j-1} + \Im(F_j(y_1 + \mathrm{i}y_2, \ldots, y_{n-1} + \mathrm{i}y_n))e_{2j}$$

where $y = (y_1, \ldots, y_{2n}) \in \tilde{\Omega}$ and e_k is the k^{th} canonical basis vector in $\mathbb{R}^{2n}$. Define a second IVP of dimension $2n$ by

$$\begin{aligned} y' &= \tilde{F}(y) \\ y(t_0) &= y_0 \end{aligned} \tag{3.3}$$

on $\tilde{\Omega}$. Now we have the following one-to-one correspondence: A function y is a solution of the IVP (3.3) if and only if x with $x_j := y_{2j-1} + \mathrm{i}y_{2j}$, $j \in \{1, \ldots, n\}$, is a solution of the IVP (3.2). On the other hand, a solution x of (3.2) leads to the solution

$$y := \sum_{j=1}^{n} \Re(x_j)e_{2j-1} + \Im(x_j)e_{2_j}$$

of (3.3). Here, the n and $2n$ input arguments for x and y, respectively, are transformed into each other in the same fashion as above.
Furthermore, since F is holomorphic, $\tilde{F} \in \mathcal{C}^1(\tilde{\Omega})$ and therefore locally Lipschitz-continuous. The Picard-Lindelöf theorem now ensures existence and uniqueness of the solution of (3.3). Because of the one-to-one correspondence between the two IVPs, the

existence and uniqueness is also proven for the system (3.2). More precisely, two different solutions of (3.2) lead to two different solutions of (3.3), which is impossible by the uniqueness of solutions.

□

Remark 3.3
Since we can transform each IVP of the form (3.2) into a real-valued IVP of the form (3.3), we can use the well-known Theory of Ordinary Differential Equations. Hence we get both, the unique maximum interval of existence for solutions and the continuous dependence of the initial data. Furthermore, there are only three cases to consider: the solution either exists globally, or reaches the boundary of Ω, or explodes in finite time („blow-up“). For more details see [9, Chapter 2.3 - 2.5]. The holomorphic dependence on initial values will be proven in Proposition 4.31.

3.2 Solutions with complex time

The explanations in this chapter correspond to the ideas in [8, Chapter I.1.] and [10, Chapter 2].

Definition 3.4
Let $n \in \mathbb{N}$ and $F = (F_1, \dots, F_n) \in \mathcal{O}^n(\Omega)$ be a holomorphic vector function defined on an open domain $\Omega \subset \mathbb{C}^{n+1}$. An *Analytic Ordinary Differential Equation with complex time* by F on Ω is the system of n scalar function equations

$$x' = F(t, x). \tag{3.4}$$

A solution of the system (3.4) is a holomorphic curve $x = (x_1, \dots, x_n) : V \to \mathbb{C}^n$, defined on an open domain $V \subset \mathbb{C}$, satisfying the following two properties:

(i) The *integral curve* $\{(t, x(t)) \in \mathbb{C}^{n+1} : t \in V\}$ of x belongs to Ω.

(ii) The equation $x'(t) = F(t, x(t))$ holds for all $t \in V$.

The system (3.4) is called autonomous, if F is independent of t. In this case the integral curve is called *phase curve* and is sometimes also denoted as $x(V) \subset \mathbb{C}^n$.

For a fixed point $(t_0, x_0) \in \Omega$ the *initial value problem* (IVP) is defined by (3.4) with an initial condition $x(t_0) = x_0$. Its solution is a solution of (3.4), whose integral curve passes the point (t_0, x_0).

Remark 3.5
Let $x : V \to \mathbb{C}^n$ be a solution of the problem (3.4). From the real point of view, the integral curve C of x is a 2-dimensional smooth manifold in $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. For an arbitrary complex time $t = \Re(t) + \mathrm{i}\Im(t) = t_1 + \mathrm{i}t_2 \in V$ the curve C is parameterized by the two real variables t_1 and t_2. Its tangent space at the point $(s, y) \in C$ is spanned by the two real vectors $\Re(F(s, y))$ and $\Im(F(s, y))$.

Definition 3.6
Let $(t_0, x_0) \in \mathbb{C}^{n+1}$ be a fixed point and $\varepsilon > 0$. The *polydisk* $\mathcal{D}_\varepsilon$ centered in (t_0, x_0) is defined by

$$\mathcal{D}_\varepsilon := \left\{(t, x) \in \mathbb{C}^{n+1} : |t - t_0| < \varepsilon,\, |x - x_0| < \varepsilon\right\}.$$

Proposition 3.7
Let $\varepsilon, K > 0$ be two arbitrary constants and $(t_0, x_0) \in \mathbb{C}^{n+1}$ be a fixed point such that the polydisk $\mathcal{D}_\varepsilon \subset \mathbb{C}^{n+1}$ is centered in that point. Then the space

$$\mathcal{A}_K(\mathcal{D}_\varepsilon) := \left\{f \in \mathcal{O}^n(\mathcal{D}_\varepsilon) \cap \mathcal{C}^0(\overline{\mathcal{D}_\varepsilon}) : |f(t, z) - z| \leq K|t - t_0| \;\; \forall\, (t, z) \in \mathcal{D}_\varepsilon\right\}$$

equipped with the supremum-norm $\|f\|_\infty := \sup\limits_{z \in \mathcal{D}_\varepsilon} |f(z)|$ is a nonempty Banach space.

Proof
Obviously, the function $g : \overline{\mathcal{D}_\varepsilon} \to \mathbb{C}^n$, $g(t, z) := z$ is an element of $\mathcal{A}_K(\mathcal{D}_\varepsilon)$. Hence the set is nonempty.
Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{A}_K(\mathcal{D}_\varepsilon) \subset \mathcal{C}^0(\overline{\mathcal{D}_\varepsilon})$ be a Cauchy sequence. It is known that the space $(\mathcal{C}^0(\overline{\mathcal{D}_\varepsilon}), \|\cdot\|_\infty)$ is complete. Hence there exists a $f \in \mathcal{C}^0(\overline{\mathcal{D}_\varepsilon})$ such that $f_k \to f$ uniformly in $\overline{\mathcal{D}_\varepsilon}$ (and in any closed subset) for $k \to \infty$, i.e. the sequence converges compactly in $\mathcal{D}_\varepsilon$. By applying the Convergence Theorem of Weierstrass, [5, Theorem 1.4.20], we conclude that $\mathcal{O}^n(\mathcal{D}_\varepsilon)$ is a closed subspace of $\mathcal{C}^0(\mathcal{D}_\varepsilon)$ with respect to the topology induced by the compact convergence, i.e. $f \in \mathcal{O}^n(\mathcal{D}_\varepsilon)$.

In addition, by continuity of the norm $|\cdot|$ in $\mathbb{C}^n$, we have for arbitrary $(t,z) \in \mathcal{D}_\varepsilon$

$$|f(t,z) - z| = \lim_{k\to\infty} |f_k(t,z) - z| \leq \limsup_{k\to\infty} K|t-t_0| = K|t-t_0|,$$

since uniform convergence implies pointwise convergence. Hence, $f \in \mathcal{A}_K(\mathcal{D}_\varepsilon)$.

$\square$

Theorem 3.8 (Existence and Uniqueness for complex IVPs)
Let $\Omega \subset \mathbb{C}^{n+1}$ be an open domain and $F \in \mathcal{O}^n(\Omega)$. Then for every $(t_0, x_0) \in \Omega$ there exists a constant $\varepsilon > 0$ such that the IVP

$$\begin{aligned} x' &= F(t,x) \\ x(t_0) &= x_0 \end{aligned} \tag{3.5}$$

has a unique solution $x : \mathcal{B}_\varepsilon(t_0) \to \mathbb{C}^n$.

Proof
Let $(t_0, x_0) \in \Omega$ be arbitrarily fixed. Since F is holomorphic, the function $h_t : \mathcal{B}_\varepsilon(x_0) \to \mathbb{C}^n$, $h_t(z) := F(t,z)$ is Lipschitz continuous with Lipschitz constant $L < \infty$. By openness of Ω, there exists an $r > 0$ such that the polydisk $\mathcal{D}_r$ centered in (t_0, x_0) is a subset of Ω. Set

$$K := \max\left\{1, \max_{\zeta \in \overline{\mathcal{D}_r}} |F(\zeta)|\right\} < \infty$$

and $\varepsilon := \min\left\{\frac{r}{2K}, \frac{1}{2L}\right\}$. Define for $t \in \mathbb{C}$ the parameterized curve $\gamma_t : [0,1] \to \mathbb{C}$ as the convex combination of t_0 to t, i.e. $\gamma_t(s) := st + (1-s)t_0$ and the *Picard operator* $\mathcal{P} : \mathcal{A}_K(\mathcal{D}_\varepsilon) \to \mathcal{A}_K(\mathcal{D}_\varepsilon)$ by

$$\mathcal{P}(f)(t,z) := z + \int\limits_{\gamma_t} F(s, f(s,z))\,\mathrm{d}s = z + (t-t_0)\int\limits_0^1 F(\gamma_t(s), f(\gamma_t(s), z))\,\mathrm{d}s.$$

This integral is to be understood component-wise. The aim is to show that the operator has exactly one fixed point that leads to a solution of the IVP (3.5).

But first, we show that the operator $\mathcal{P}$ is well-defined. By definition of $\mathcal{A}_K(\mathcal{D}_\varepsilon)$ and the choice of ε, the estimation

$$\begin{aligned}|f(\gamma_t(s),z)-x_0| &\leq |f(\gamma_t(s),z)-z|+|z-x_0| \leq K|\gamma_t(s)-t_0|+|z-x_0| \\ &< K\varepsilon+\varepsilon \leq \frac{rK}{2K}+\frac{r}{2K} \leq \frac{r}{2}+\frac{r}{2}=r\end{aligned}$$

holds and thus $(\gamma_t(s), f(\gamma_t(s),z)) \in \mathcal{D}_r \subset \Omega$ for all $s \in [0,1]$ and $(t,z) \in \mathcal{D}_\varepsilon$. Hence the function F can be evaluated under the integral.
In addition, $\overline{\mathcal{D}_r}$ is convex and therefore the image of γ_t lies completely in $\overline{\mathcal{D}_r}$. Hence

$$|\mathcal{P}(f)(t,z)-z| \leq |t-t_0| \int_0^1 |F(\underbrace{\gamma_t(s), f(\gamma_t(s),z)}_{\in\overline{\mathcal{D}_r}})|\,\mathrm{d}s \leq |t-t_0| \int_0^1 K\,\mathrm{d}s = K|t-t_0|$$

holds for all $f \in \mathcal{A}_K(\mathcal{D}_\varepsilon)$ and $(t,z) \in \mathcal{D}_\varepsilon$, which implies $\mathcal{P}(\mathcal{A}_K(\mathcal{D}_\varepsilon)) \subset \mathcal{A}_K(\mathcal{D}_\varepsilon)$.
The second objective is to show that $\mathcal{P}$ is contracting on $\mathcal{A}_K(\mathcal{D}_\varepsilon)$. For all $f,g \in \mathcal{A}_K(\mathcal{D}_\varepsilon)$

$$\begin{aligned}\|\mathcal{P}f-\mathcal{P}g\|_\infty &\leq \sup_{(t,z)\in\mathcal{D}_\varepsilon} |t-t_0| \int_0^1 |F(\gamma_t(s), f(\gamma_t(s),z)) - F(\gamma_t(s), g(\gamma_t(s),z))|\,\mathrm{d}s \\ &\leq \sup_{(t,z)\in\mathcal{D}_\varepsilon} \varepsilon \int_0^1 \left|h_{\gamma_t(s)}(f(\gamma_t(s),z)) - h_{\gamma_t(s)}(g(\gamma_t(s),z))\right|\,\mathrm{d}s \\ &\leq \sup_{(t,z)\in\mathcal{D}_\varepsilon} \frac{1}{2L} \int_0^1 L\,\underbrace{|f(\gamma_t(s),z)-g(\gamma_t(s),z)|}_{\leq\|f-g\|_\infty}\,\mathrm{d}s \\ &\leq \frac{1}{2}\|f-g\|_\infty .\end{aligned}$$

holds true and so the Picard operator is contracting with Lipschitz constant $L_{\mathcal{P}} := \frac{1}{2} < 1$. Hence, by Proposition 3.7 and the Banach Fixed Point Theorem, there exists exactly one fixed point $f_0 \in \mathcal{A}_K(\mathcal{D}_\varepsilon)$ of $\mathcal{P}$, i.e.

$$f_0(t,z) = z + \int_{\gamma_t} F(s, f_0(s,z))\,\mathrm{d}s \quad \forall\,(t,z) \in \mathcal{D}_\varepsilon.$$

The third step is the definition of the explicit solution. Define $x : \mathcal{B}_\varepsilon(t_0) \to \mathbb{C}^n$ by $x(t) := f_0(t,x_0)$. Then clearly x ist holomorphic and satisfies the initial condition. Since

$\mathcal{D}_\varepsilon$ is simply connected (even convex), F has a primitive function on $\mathcal{D}_\varepsilon$ and so

$$x'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\gamma_t} F(s, x(s))\,\mathrm{d}s = F(t, x(t)) \quad \forall\, t \in \mathcal{B}_\varepsilon(t_0).$$

In addition, the integral curve of x belongs completely to Ω. To see this, we can estimate

$$|x(t) - x_0| = |f(t, x_0) - x_0| \leq K|t - t_0| < K\varepsilon \leq r \quad \forall\, t \in \mathcal{B}_\varepsilon(t_0).$$

Hence, $x(\mathcal{B}_\varepsilon(t_0)) \subset \mathcal{B}_r(x_0)$ with respect to the norm on $\mathbb{C}^n$. Finally, by Definition 3.4, the function x is indeed a solution of our IVP (3.5). This proves the existence. It remains to show the uniqueness of the solution. Although f_0 is the unique fixed point, it is hard to conclude from this that x is also unique. That is because we do not know yet that the solution depends continuously on the initial data. In [8, p. 5] the author does not prove the uniqueness. The following arguments are my own work.

Let $x^{(1)} : V_1 \to \mathbb{C}^n$ and $x^{(2)} : V_2 \to \mathbb{C}^n$ be two arbitrary solutions of the IVP (3.5). Then $V_1 \cap V_2$ is an open neighbourhood of t_0, thus there exists $\rho > 0$ such that $\mathcal{B}_\rho(t_0) \subset V_1 \cap V_2$. Set $y := x^{(1)} - x^{(2)} : \mathcal{B}_\rho(t_0) \to \mathbb{C}^n$ and fix a $t \in \mathcal{B}_\rho(t_0)$. Set $d := |t - t_0|$ and $\theta := \arg(t - t_0)$ as the radius and angle of t inside the ball $\mathcal{B}_\rho(t_0)$, i.e. $t = t_0 + de^{\mathrm{i}\theta}$. Define for an arbitrary $\xi \in [0, d]$ the holomorphic curve $\varphi_\xi : [0, \xi] \to \mathcal{B}_\rho(t_0)$, $\varphi_\xi(s) := t_0 + se^{\mathrm{i}\theta}$. Now the estimation

$$\begin{aligned}\left|y\left(t_0 + \xi e^{\mathrm{i}\theta}\right)\right| &= \left|\int\limits_{\varphi_\xi} x^{(1)\prime}(s) - x^{(2)\prime}(s)\,\mathrm{d}s\right| = \left|\int\limits_{\varphi_\xi} F(s, x^{(1)}(s)) - F(s, x^{(2)}(s))\,\mathrm{d}s\right| \\ &\leq K \int\limits_0^\xi \left|x^{(1)}(\varphi_\xi(s)) - x^{(2)}(\varphi_\xi(s))\right| \underbrace{|\varphi_\xi'(s)|}_{=1}\,\mathrm{d}s = K \int\limits_0^\xi \left|y\left(t_0 + se^{\mathrm{i}\theta}\right)\right|\,\mathrm{d}s\end{aligned}$$

holds for all $\xi \in [0, d]$. Define $u : [0, d] \to \mathbb{R}$ by $u(\xi) := \left|y\left(t_0 + \xi e^{\mathrm{i}\theta}\right)\right|$. Then u is continuous, non-negative and the estimation

$$u(\xi) \leq K \int\limits_0^\xi u(s)\,\mathrm{d}s$$

holds for all $\xi \in [0, d]$. Now we can apply Gronwall's lemma, [9, p. 79], and conclude $u(d) = 0$, i.e. $|y(t)| = 0$. Since $t \in \mathcal{B}_\rho(t_0)$ is arbitrary, we have $y \equiv 0$ and $x^{(1)} \equiv x^{(2)}$. Finally, this proves the uniqueness of the solution.

$\square$

Remark 3.9

The solution in Theorem 3.8 depends holomorphically on the initial data $x_0 \in \Omega$. This is clear from the fact that the unique fixed point f_0 of the Picard operator in the proof of Theorem 3.8 depends holomorphically on the variable z.

4 Equilibria and their local topological characteristics

In the following chapters we consider autonomous Analytic Ordinary Differential Equations with real time on the plane, i.e. systems of the form

$$x' = F(x) \tag{4.1}$$

with $F \in \mathcal{O}(\Omega)$, $\Omega \subset \mathbb{C}$ an open domain, an initial condition $x(0) = x_0 \in \mathbb{C}$ and $t_0 = 0$. We can rewrite this system to

$$\begin{aligned} x_1' &= F_1(x_1, x_2) \\ x_2' &= F_2(x_1, x_2) \end{aligned}$$

with the real right-hand side $F_1 := \Re(F) \in \mathcal{C}^\infty$ and $F_2 := \Im(F) \in \mathcal{C}^\infty$. Here we identify the complex initial point x_0 with the real point $(\Re(x_0), \Im(x_0)) \in \mathbb{R}^2$.
The *trajectory* or *orbit* through x_0 corresponding to (4.1) is the maximum phase curve and denoted by the set $\Gamma(x_0) := x(I)$, where x is the unique solution of (4.1) through x_0 and $I = I(x_0)$ the maximum interval of existence with respect to x_0. In this notation, set $\Gamma_+(x_0) := x(I \cap [0, \infty))$ and $\Gamma_-(x_0) := x(I \cap (-\infty, 0])$. Each orbit can be evaluated at a time $t \in I$ by $\Phi(t, x_0) := x(t)$, with $x(0) = x_0$. Using this notation, we have $\Gamma(x_0) = \Phi(I, x_0)$. The union of all trajectories is the *phase space*.
A well known result, cf. [7, Satz 4.4.2], is the following: If for any initial value the solution exists globally (to the right), then the map $(t, y) \mapsto \Phi(t, y)$ is a dynamical (semi-dynamical) system or flow (semi-flow). This result will be used throughout in the following chapters. Furthermore, by the Identity Theorem, we can always assume that the zeros of F in (4.1) do not have an accumulation point, i.e. on bounded sets there are at least finitely many zeros (Bolzano-Weierstrass theorem). The case $F \equiv 0$ is not interesting for us.
Moreover, in any considerations, the Jordan curve theorem, cf. [11, Theorem 63.4], is to

be presumed. i.e the complement of every closed Jordan curve (a simple closed piecewise continuously differentiable path) $\Gamma \subset \mathbb{C}$ consists of exactly two connected components. One of these components is bounded (the interior $\mathrm{Int}(\Gamma)$) and the other is unbounded (the exterior $\mathrm{Ext}(\Gamma)$). The curve Γ is the boundary of each component. Note that the Jordan curve theorem holds in both spaces, $\mathbb{C}$ and S^2 as its one-point-compactification, cf. [11, Lemma 61.1] and [11, p. 185, Example 4]. Additionally, if Γ lies in a simply connected domain $\Omega \subset \mathbb{C}$, then of course it holds that $\mathrm{Int}(\Gamma) \subset \Omega$.
Moreover, $\mathrm{Int}(\Gamma)$ is homeomorphic to $\mathcal{B}_1(0)$ and is always simply connected. This stronger version of the Jordan curve theorem is the Theorem of Jordan-Schoenflies. For more details see also the remarks made in [12, p. 169] and [11, pp. 376-377].[2]

4.1 Geometry of equilibria with vanishing derivative

In the following the geometric properties of the flow corresponding to the system (4.1) will be described. Especially the neighbourhood of equilibria in the phase space will be characterized. The definitions and assumptions can be found in [9, Chapter 1-2], and [7, Chapter I.3].

Definition 4.1
Let $\Omega \subset \mathbb{C}$ be an open domain and $F \in \mathcal{O}(\Omega)$. A point $a \in \Omega$ is an *equilibrium* of (4.1) if $F(a) = 0$. In particular, an equilibrium is called

(i) a *center* if there exists a $\delta > 0$ such that for all $y \in \mathcal{B}_\delta(a) \setminus \{a\} \subset \Omega$ the orbit $\Gamma(y)$ is a closed curve with $a \in \mathrm{Int}(\Gamma(y))$.

(ii) a (an) *stable (unstable) focus* if there exists a $\delta > 0$ such that for all $y \in \mathcal{B}_\delta(a) \setminus \{a\} \subset \Omega$ the solution through y exists globally to the right (left) and satisfies $|\Phi(t, y)| \to |a|$ and $|\arg(\Phi(t, y) - a)| \to \infty$ for $t \to \infty$ $(t \to -\infty)$, in particular $w_+(\Gamma(y)) = \{a\}$ $(w_-(\Gamma(y)) = \{a\})$. In this case, we say that $\Gamma(y)$ is a spiral.

(iii) a *stable (unstable) node* if there exist $\delta > 0$ such that for all $y \in \mathcal{B}_\delta(a) \setminus \{a\} \subset \Omega$ the solution through y exists globally to the right (left) and satisfies $|\Phi(t, y)| \to |a|$ and $\arg(\Phi(t, y) - a) \to \theta_0$ for $t \to \infty$ $(t \to -\infty)$ with a $\theta_0 \in [0, 2\pi)$, in particular $w_+(\Gamma(y)) = \{a\}$ $(w_-(\Gamma(y)) = \{a\})$. In this case, we say that $\Gamma(y)$ tends to *a in the definite direction θ_0*.

[2] The original proof by Jordan can be found in [13, pp. 587-594].

(iv) a *saddle* if there exist four trajectories $\Gamma_1, \ldots, \Gamma_4$ with $w_+(\Gamma_1) = w_+(\Gamma_2) = \{a\}$ and $w_-(\Gamma_3) = w_-(\Gamma_4) = \{a\}$ and a $\delta > 0$ such that for all $y \in \mathcal{B}_\delta(a) \setminus \{a\} \subset \Omega$ there exists a $\tau > 0$ with $\Phi(t, y) \notin \mathcal{B}_\delta(a)$ for all $|t| > \tau$.

Remark 4.2

a) If an orbit Γ tends to an equilibrium a in the definite direction $\theta_0 \in [0, 2\pi)$, then the pinned tangent vector of Γ tends to the vector $(\cos(\theta_0), \sin(\theta_0)) - a$. This tangent vector approaches the ray with angle θ_0 centered in a. In fact, the number $\arg(\Phi(t, y) - a)$ is the argument of the flow through y at time t with respect to the circle (or coordinate system) with center a.

b) A geometric visualization of the equilibria defined in Definition 4.1 can be looked up in [7, Chapter 5.2] and [9, Chapter 1.5]. A more detailed version of the different types of nodes and foci can be found in [7].

Theorem 4.3
Let $\Omega \subset \mathbb{C}$ be an open domain and $F \in \mathcal{O}(\Omega)$. Let $a \in \Omega$ be an equilibrium of (4.1) with $F'(a) = \alpha + \mathrm{i}\beta \neq 0$. Then:

(i) $\operatorname{Eig}(J_f(a)) = \left\{F'(a), \overline{F'(a)}\right\}$.

(ii) If $\alpha \neq 0$ and $\beta = 0$, a is a node. If $\alpha < 0$ ($\alpha > 0$), the node is asymptotically stable (unstable).

(iii) If $\alpha \neq 0$ and $\beta \neq 0$, a is a focus. If $\alpha < 0$ ($\alpha > 0$), the focus is asymptotically stable (unstable).

(iv) If $\alpha = 0$ and $\beta \neq 0$, a is a center or a focus.

In particular, under the above assumptions the equilibrium is not a saddle.

Proof
By the Cauchy-Riemann equations, we have

$$J := \mathcal{J}_F(a) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

and $\mathrm{p}_J(\lambda) = (\alpha - \lambda)^2 + \beta^2$. As zeros we get $\lambda_\pm := \alpha \pm \mathrm{i}\beta$. Hence, if $\beta \neq 0$, the two eigenvalues are the complex conjugated pair $\{\lambda_-, \lambda_+\}$. This proves (i).

Additionally, we have $\mathrm{tr}(J) = 2\alpha$, $\det(J) = \alpha^2 + \beta^2$ and $\mathrm{tr}(J)^2 - 4\det(J) = 4\alpha^2 - 4(\alpha^2 + \beta^2) = -4\beta^2$. Consider the linearized system $x' = Jx$ on $\mathbb{R}^2$. For this system we can use the already known results about equilibria of linear systems, [9, Chapter 1.5]. If $\alpha \neq 0$ and $\beta = 0$, there is just one real eigenvalue and a is a node for the linearized system. If $\alpha \neq 0$ and $\beta \neq 0$, a is a focus for the linearized system. And if $\alpha = 0$ and $\beta \neq 0$, a is a center for the linearized system.

With the help of the Hartman-Grobman Theorem, [9, Chapter 2.8], we can use [9, Chapter 2.10, Theorem 4] to conclude the assertions (ii) and (iii). The claim about the stability follows from the principle of linearized stability, [7, Satz 5.4.1]. The function F is real analytic in a and thus (iv) follows from the Corollary to [9, Chapter 2.10, Theorem 5].

□

4.2 Index Theory

The so-called Index Theory is a powerful tool for studying the phase space of (4.1). It provides us with an understanding of how the geometry, topology, and analysis of planar dynamical systems are interrelated. This chapter is based on [7, Chapter 9.6] and [6, §10]. The statements and proofs are suitably modified for holomorphic vector fields. Some proofs are entirely reformulated. The underlying area is mostly assumed to be a simply connected domain. For our later studies in this chapter, this condition is suitable, since we always want to ensure that the interior of closed Jordan curves lies completely in the domain.

Definition 4.4

Let $\Omega \subset \mathbb{R}^2$ be an open domain and $F = (F_1, F_2)^T \in \mathcal{C}^1(\Omega; \mathbb{R}^2)$ be vector field. Let $\gamma : [0,1] \to \Omega$ be a closed Jordan curve with image $\Gamma := \gamma([0,1])$ such that $\Gamma \cap F^{-1}(\{0\}) = \emptyset$. Then the *index* of F with respect to Γ is the net change in the argument of F over one trip around Γ, i.e. the integer

$$\mathrm{ind}(F, \Gamma) := \frac{1}{2\pi} \int_\Gamma \mathrm{d}(\arg(F)) = \frac{1}{2\pi} \int_\Gamma \nabla \arg(F) \,\mathrm{d}s.$$

The latter integral is understood as the line integral of 2^{nd} kind along γ.

Lemma 4.5
Let $\Omega \subset \mathbb{R}^2$ be an open domain and $F = (F_1, F_2)^T \in \mathcal{C}^1(\Omega; \mathbb{R}^2)$. Let $\gamma : [0,1] \to \Omega$ be a closed Jordan curve with image $\Gamma := \gamma([0,1])$ such that $\Gamma \cap F^{-1}(\{0\}) = \emptyset$. Then

$$\operatorname{ind}(F, \Gamma) = \frac{1}{2\pi} \int\limits_{\Gamma} \frac{F_1 \nabla F_2 - F_2 \nabla F_1}{|F|^2} \, \mathrm{d}s$$

and the index only depends on the orientation of the parameterization γ. In particular, only the sign of the index depends on the orientation. Furthermore, if we identify $\mathbb{R}^2$ with $\mathbb{C}$ and assume $F \in \mathcal{O}(\Omega)$, then the index of F is the winding number of the complex closed curve $F(\Gamma)$ around 0, i.e.

$$\operatorname{ind}(F, \Gamma) = \frac{1}{2\pi \mathrm{i}} \int\limits_{F(\Gamma)} \frac{1}{z} \, \mathrm{d}z.$$

In addition, the index is in fact an integer.

Proof
In the absence of any proof for this lemma in [7, Chapter 9.6], I will prove it myself.
We calculate

$$\nabla \arg(F) = \nabla \arctan\left(\frac{F_2}{F_1}\right) = \frac{1}{1 + \frac{F_2^2}{F_1^2}} \frac{F_1 \nabla F_2 - F_2 \nabla F_1}{F_1^2} = \frac{F_1 \nabla F_2 - F_2 \nabla F_1}{F_1^2 + F_2^2}.$$

This proves the first formula. Furthermore (F has no zeros on Γ)

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{F(\Gamma)} \frac{1}{z} \, \mathrm{d}z = \frac{1}{2\pi} \int\limits_{F(\Gamma)} -\frac{\mathrm{i}\overline{z}}{z\overline{z}} \, \mathrm{d}z = \frac{1}{2\pi} \int\limits_0^1 \frac{-F_2(\gamma(s)) - \mathrm{i}F_1(\gamma(s))}{|F(\gamma(s))|^2} (F \circ \gamma)'(s) \, \mathrm{d}s.$$

By applying the Cauchy-Riemann equations, we have

$$\begin{aligned} -(F_2 + \mathrm{i}F_1)F' &= -F_2 \partial_1 F_1 - \mathrm{i}F_2 \partial_1 F_2 - \mathrm{i}F_1 \partial_1 F_1 + F_1 \partial_1 F_2 \\ &= -F_2 \partial_1 F_1 + \mathrm{i}F_2 \partial_2 F_1 - \mathrm{i}F_1 \partial_2 F_2 + F_1 \partial_1 F_2 \\ &= F_1 \partial_1 F_2 - F_2 \partial_1 F_1 + \mathrm{i}(F_2 \partial_2 F_1 - F_1 \partial_2 F_2) \\ &= \left\langle F_1 \nabla F_2 - F_2 \nabla F_1, (1, -\mathrm{i})^T \right\rangle \end{aligned}$$

and thus for all $s \in [0,1]$

$$\begin{aligned}
-(F_2 + \mathrm{i}F_1)(\gamma(s))F'(\gamma(s))\gamma'(s) = &\Big\langle (F_1\nabla F_2 - F_2\nabla F_1)(\gamma(s)), (\gamma_1'(s), \gamma_2'(s))^T \Big\rangle \\
&+ \mathrm{i}\gamma_2'(s)(F_1\partial_1 F_2 - F_2\partial_1 F_1)(\gamma(s)) \\
&+ \mathrm{i}\gamma_1'(s)(F_2\partial_2 F_1 - F_1\partial_2 F_2)(\gamma(s)).
\end{aligned}$$

Here, the derivative of γ is to be understood as a number in $\mathbb{C}$. Now define $h : \Gamma \to \mathbb{R}^2$ by

$$h := \frac{1}{F_1^2 + F_2^2} \begin{pmatrix} F_2\partial_2 F_1 - F_1\partial_2 F_2 \\ F_1\partial_1 F_2 - F_2\partial_1 F_1 \end{pmatrix}.$$

With this definition we conclude

$$\begin{aligned}
\frac{1}{2\pi\mathrm{i}} \int\limits_{F(\Gamma)} \frac{1}{z}\,\mathrm{d}z &= \frac{1}{2\pi} \int\limits_0^1 \frac{\langle (F_1\nabla F_2 - F_2\nabla F_1)(\gamma(s)), \gamma'(s)\rangle}{|F(\gamma(s))|^2}\,\mathrm{d}s + \frac{\mathrm{i}}{2\pi} \int\limits_0^1 \langle h(\gamma(s)), \gamma'(s)\rangle\,\mathrm{d}s \\
&= \frac{1}{2\pi} \int\limits_\Gamma \frac{F_1\nabla F_2 - F_2\nabla F_1}{|F|^2}\,\mathrm{d}s + \frac{\mathrm{i}}{2\pi} \int\limits_\Gamma h\,\mathrm{d}s.
\end{aligned}$$

Here, the derivative of γ is to be understood as a vector in $\mathbb{R}^2$. Since the winding number is real, [14, Theorem 10.10], the second term (the imaginary part of the integral) is equal to 0. Additionally, line integrals are parameterization-independent. Hence, the index, or more precisely, the sign of the index, only depends on the orientation of the parameterization γ. Since the winding number is an integer, so is the index.

□

Proposition 4.6
Let $\Omega \subset \mathbb{C}$ be an open domain, $F = F_1 + \mathrm{i}F_2 \in \mathcal{O}(\Omega)$, $F \not\equiv 0$, and $\Gamma \subset \Omega$ a closed Jordan curve (passed counterclockwise) with $\overline{\mathrm{Int}(\Gamma)} \cap F^{-1}(\{0\}) = \emptyset$. Then $\mathrm{ind}(F, \Gamma) = 0$.

Proof
Assume w.l.o.g. that Γ is parameterized by γ with $|\gamma'| \equiv 1$, i.e. by its arc length. This is possible, since Γ is piecewise differentiable and $\mathrm{ind}(F,\Gamma)$ does not depend on the parameterization γ. Define the vector field $g : \Omega \setminus F^{-1}(\{0\}) \to \mathbb{R}^2$ by

$$g := \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} := \frac{1}{|F|^2} \begin{pmatrix} F_1\partial_2 F_2 - F_2\partial_2 F_1 \\ -F_1\partial_1 F_2 + F_2\partial_1 F_1 \end{pmatrix}$$

and $\tilde{g}_1 := F_1\partial_2F_2 - F_2\partial_2F_1$, $\tilde{g}_2 := -F_1\partial_1F_2 + F_2\partial_1F_1$. Let $L(\Gamma)$ be the length of Γ. Using this definition yields to

$$\operatorname{ind}(F,\Gamma) = (2\pi)^{-1}\int_0^{L(\Gamma)} \left\langle g(\gamma(s)), \begin{pmatrix} \gamma_2'(s) \\ -\gamma_1'(s) \end{pmatrix} \right\rangle \underbrace{|\gamma'(s)|}_{=1}\,\mathrm{d}s = (2\pi)^{-1}\int_\Gamma \langle g(x), \nu(x)\rangle\,\mathrm{dS}(x).$$

The latter integral is to be understood as a surface integral over the 1-dimensional submanifold $\Gamma \subset \mathbb{R}^2$ with the outer unit normal $\nu := (-\gamma_2', \gamma_1')$. Choose $R > 0$ big enough such that $\Gamma \subset \mathcal{B}_R(0)$. Since there are only finitely many zeros of F in $\Omega \cap \mathcal{B}_R(0)$, there exists an open domain $V \subset \Omega \cap \mathcal{B}_R(0)$ such that $g \in \mathcal{C}^1(V;\mathbb{R}^2)$ and $\overline{\operatorname{Int}(\Gamma)} \subset V$. More precisely, finitely many points cannot be arbitrarily close to the compact set $\overline{\operatorname{Int}(\Gamma)}$. In the following calculations the Cauchy-Riemann equations and the symmetry of second derivatives of F (Schwarz's theorem) are used throughout. We have

$$\partial_j g_j = -|F|^{-4}\partial_j|F|^2\tilde{g}_j + |F|^{-2}\partial_j\tilde{g}_j = |F|^{-2}\partial_j\tilde{g}_j - 2|F|^{-4}\tilde{g}_j\,\underbrace{(F_1\partial_jF_1 + F_2\partial_jF_2)}_{=\tilde{g}_j}$$

with $j \in \{1,2\}$. Furthermore,

$$\begin{aligned}\partial_1\tilde{g}_1 + \partial_2\tilde{g}_2 =& \partial_1F_1\partial_2F_2 + F_1\partial_1\partial_2F_2 - \partial_1F_2\partial_2F_1 - F_2\partial1\partial_2F_1 \\ &- \partial_2F_1\partial_1F_2 - F_1\partial_2\partial_1F_2 + \partial_2F_2\partial_1F_1 + F_2\partial_2\partial_1F_1 \\ =& 2\partial_1F_1\partial_2F_2 - 2\partial_2F_1\partial_1F_2 = 2|\nabla F_2|^2\end{aligned}$$

and

$$\begin{aligned}\tilde{g}_1^2 + \tilde{g}_2^2 =& F_1^2(\partial_2F_2)^2 + F_2^2(\partial_2F_1)^2 - 2F_1F_2\partial_2F_2\partial_2F_1 \\ &+ F_1^2(\partial_1F_2)^2 + F_2^2(\partial_1F_1)^2 - 2F_1F_2\partial_1F_2\partial_1F_1 \\ =& F_1^2\left((\partial_2F_2)^2 + (\partial_1F_2)^2\right) + F_2^2\left((\partial_2F_1)^2 + (\partial_1F_1)^2\right) \\ =& F_1^2\left((\partial_2F_2)^2 + (\partial_1F_2)^2\right) + F_2^2\left((-\partial_1F_2)^2 + (\partial_2F_2)^2\right) = |\nabla F_2|^2\left(F_1^2 + F_2^2\right).\end{aligned}$$

Hence, we conclude that the divergence of g is given by

$$\begin{aligned}\operatorname{div}(g) = \partial_1g_1 + \partial_2g_2 &= |F|^{-2}\left(\partial_1\tilde{g}_1 + \partial_2\tilde{g}_2\right) - 2|F|^{-4}\left(\tilde{g}_1^2 + \tilde{g}_2^2\right) \\ &= |F|^{-2}\left(2|\nabla F_2|^2 - 2|F|^{-2}|\nabla F_2|^2\left(F_1^2 + F_2^2\right)\right) \\ &= 2|F|^{-2}\left(|\nabla F_2|^2 - |\nabla F_2|^2\right) = 0.\end{aligned}$$

Therefore, the divergence theorem implies

$$\operatorname{ind}(F,\Gamma) = (2\pi)^{-1} \int\limits_{\operatorname{Int}(\Gamma)} \operatorname{div}(g)(x)\,\mathrm{d}x = 0.$$

$\square$

Lemma 4.7
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F = F_1 + \mathrm{i}F_2 \in \mathcal{O}(\Omega)$ and $a \in \Omega$ fixed. For $r > 0$ set $\Gamma_r := \partial\mathcal{B}_r(a) \subset \Omega$, Assume that Γ_r is passed counterclockwise. Let $0 < r_1 < r_2$ be two radii such that $\Gamma_{r_2} \subset \Omega$. Assume that there are no zeros in the circle ring with the radii r_1 and r_2, i.e. $\left(\overline{\mathcal{B}_{r_2}(a)} \setminus \mathcal{B}_{r_1}(a)\right) \cap F^{-1}(\{0\}) = \emptyset$. Then it holds that

$$\operatorname{ind}(F,\Gamma_{r_1}) = \operatorname{ind}(F,\Gamma_{r_2}).$$

Proof
In [7, Chapter 9.6] the index of equilibria is constructed in a slightly different way. The following arguments are therefore my own.
Parameterize (counterclockwise) the curve Γ_r by $t \mapsto a + re^{2\pi \mathrm{i}t}$, $t \in [0,1]$ and define $h : [0,1] \times [r_1,r_2] \to \mathbb{C}$ by

$$h(t,r) := r\frac{(F_1\nabla F_2 - F_2\nabla F_1)(a + re^{2\pi \mathrm{i}t})}{|F(a + re^{2\pi \mathrm{i}t})|^2}.$$

Here we identify $h(t,r) \in \mathbb{R}^2$ as an element of $\mathbb{C}$. Since $|F|^2$ has no zeros in $\overline{\mathcal{B}_{r_2}(a)} \setminus \mathcal{B}_{r_1}(a)$, h is well-defined and continuous on the compact set $[0,1] \times [r_1,r_2]$. So the Heine-Cantor theorem implies the uniform continuity of h. Now, let $\varepsilon > 0$ be arbitrary. Then there exists a $\delta > 0$ such that for all $\rho_1, \rho_2 \in [r_1,r_2]$ with $|\rho_1 - \rho_2| < \delta$ we have

$$|\operatorname{ind}(F,\Gamma_{\rho_1}) - \operatorname{ind}(F,\Gamma_{\rho_2})| \leq \frac{1}{2\pi}\int\limits_0^1 \underbrace{|h(t,\rho_1) - h(t,\rho_2)|}_{<\varepsilon}\underbrace{\left|2\pi\mathrm{i}e^{2\pi \mathrm{i}t}\right|}_{=2\pi}\,\mathrm{d}t < \varepsilon.$$

Hence the map $r \mapsto \operatorname{ind}(F,r)$ is also uniformly continuous on $[r_1,r_2]$. On the other hand, this continuous map takes values in the discrete set $\mathbb{Z}$, cf. Lemma 4.5. So it must be constant. The assertion follows.

$\square$

Definition 4.8

Let $\Omega \subset \mathbb{C}$ be an open domain, $F \in \mathcal{O}(\Omega)$ and a an equilibrium of (4.1). Assume $F \not\equiv 0$. Then there exists an $r > 0$ such that $\overline{\mathcal{B}_r(a)} \cap F^{-1}(\{0\}) = \{a\}$. The *index* of a with respect to F is defined as the integer

$$\operatorname{ind}(F, a) := \operatorname{ind}(F, \partial \mathcal{B}_r(a)).$$

The curve $\partial \mathcal{B}_r(a)$ is parameterized counterclockwise, e.g. by $t \mapsto a + re^{2\pi i t}$, $t \in [0, 1]$.

Remark 4.9

The number $\operatorname{ind}(F, a)$ in Definition 4.8 does not depend on the choice of the radius r. This follows directly by applying Lemma 4.7.

Theorem 4.10

Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F = F_1 + \mathrm{i}F_2 \in \mathcal{O}(\Omega)$, $F \not\equiv 0$, and $\Gamma \subset \Omega$ a closed Jordan curve with $\Gamma \cap F^{-1}(\{0\}) = \emptyset$. Assume that Γ is passed counterclockwise. Then it holds that

$$\operatorname{ind}(F, \Gamma) = \sum_{\substack{a \in \operatorname{Int}(\Gamma) \\ F(a)=0}} \operatorname{ind}(F, a).$$

Proof

This proof is based on the ideas given in [6, §11.1] and [9, p. 302].

By the Identity Theorem, there are in fact only finitely many equilibria in $\operatorname{Int}(\Gamma)$, so the sum is well-defined. If there are no zeros in $\operatorname{Int}(\Gamma)$, nothing is to show, cf. Lemma 4.5. Let $\{a_1, \dots, a_\ell\}$, $\ell \in \mathbb{N}$, be the zeros of F in $\operatorname{Int}(\Gamma)$. The following explains the geometrical idea of the proof.

For $j \in \{1, \dots, \ell\}$ choose a radius $r_j > 0$ and a curve $\Lambda_j := \partial \mathcal{B}_{r_j}(a_j)$ such that $\Lambda_j \subset \operatorname{Int}(\Gamma)$. The radii can be assumed small enough to ensure $\overline{\operatorname{Int}(\Lambda_{j_2})} \cap \overline{\operatorname{Int}(\Lambda_{j_1})} = \emptyset$, if $j_1 \neq j_2$. Additionally, choose a $\mathcal{C}^1$-curve Ξ_j to connect Λ_j with Γ such that these connecting curves have empty pairwise intersections. Since Γ is continuously differentiable and there are only finitely many equilibria in $\operatorname{Int}(\Gamma)$, all these objects exist. By this construction, we have

$$\sum_{\substack{a \in \operatorname{Int}(\Gamma) \\ F(a)=0}} \operatorname{ind}(F, a) = \sum_{j=1}^{\ell} \operatorname{ind}(F, \Lambda_j).$$

Now parameterize for $j \in \{1, \dots, \ell\}$ the curve Ξ_j by $\xi_j : [0,1] \to \mathbb{C}$ such that $\xi_j(0) \in \Lambda_j$ and $\xi_j(1) \in \Gamma$. Set $p_j := \xi_j(1)$ and suppose that the points $p_1, \dots, p_\ell$ are arranged counter-clockwise with $p_{\ell+1} := p_1$ on Γ. Let Γ_j be the part of Γ connecting the two points p_j and p_{j+1} and parameterized by $\gamma_j : [0,1] \to \mathbb{C}$. Using this notation, we have $\gamma_j(0) = p_j$ and $\gamma_j(1) = \gamma_{j+1}(0) = p_{j+1}$. Parameterize Λ_j by $\lambda_j : [0,1] \to \mathbb{C}$ such that $\lambda_j(0) = \lambda_j(1) = \xi(1)$. This construction can be illustrated by the following figure.

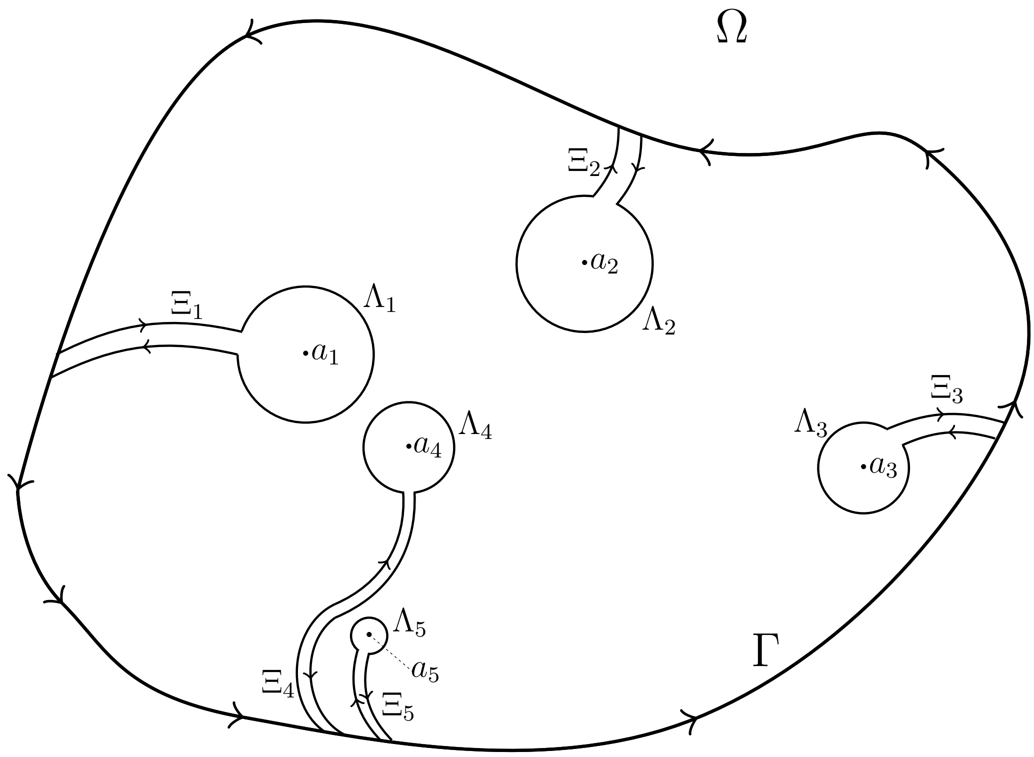

Figure 4.1: Geometrical visualization of the proof of Theorem 4.10 with $\ell = 5$

Using the definition of concatenation of paths, leads to the piecewise continuously differentiable path $\Pi : [0,1] \to \mathbb{C}$ defined by

$$\Pi := \sum_{j=1}^{\ell} \left(p_j - \xi_j - \lambda_j + \xi_j + \gamma_j\right).$$

Since $p_1 = \gamma_\ell(1)$, this path is closed.

By construction and Proposition 4.6, we conclude

$$\sum_{\substack{a\in\mathrm{Int}(\Gamma)\\ F(a)=0}} \mathrm{ind}(F,a) - \sum_{j=1}^{\ell} \mathrm{ind}(F,\Lambda_j) = \frac{1}{2\pi}\int_{\Pi} \frac{F_1\nabla F_2 - F_2\nabla F_1}{|F|^2}\,\mathrm{d}s = \mathrm{ind}(F,\Pi) = 0.$$

In this equation we used the fact that the line integral is linear with respect to the paths and vanishes along the connecting paths Ξ_j, $j \in \{1,\dots,\ell\}$. In addition we assumed the following: For each $j \in \{1,\dots,\ell\}$ the path Ξ_j can be approximated by two parallel paths such that Π is indeed a closed Jordan curve without any zeros in its interior, cf. Figure 4.1. This can be done by approximation of Π and the continuous dependence of the index on the chosen Jordan curve, cf. [7, p. 244].

□

Theorem 4.11
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F \in \mathcal{O}(\Omega)$ and $\Gamma \subset \Omega$ a periodic orbit of (4.1) with $\mathrm{Int}(\Gamma) \neq \emptyset$. Assume that Γ is parameterized as solution of (4.1). Then:

(i) $\mathrm{ind}(F,\Gamma) \in \{1,-1\}$.

(ii) $\mathrm{Int}(\Gamma) \cap F^{-1}(\{0\}) \neq \emptyset$.

(iii) $\sum_{\substack{a\in\mathrm{Int}(\Gamma)\\ F(a)=0}} \mathrm{ind}(F,a) = 1$.

In particular, there exists at least one equilibrium of (4.1) in $\mathrm{Int}(\Gamma)$.

Proof
First, Γ cannot be an equilibrium, since the interior of Γ is nonempty. Let $T > 0$ be the period of Γ and parameterize Γ by $\gamma : [0,T] \to \mathbb{R}^2$ where γ is a $\mathcal{C}^1$-solution of (4.1), i.e. $\gamma'(t) = F(\gamma(t))$ for all $t \in [0,T]$. By [9, Chapter 2.3, Remark 1], this solution is even a $\mathcal{C}^2$-curve.
The proof of this theorem is essentially based on Hopf's Umlaufsatz, a result of differential geometry, cf. [15, p. 28, Theorem 2.28]. By applying [15, p. 26, Definition 2.25] and Lemma 4.5, we get

$$\mathrm{ind}(F,\Gamma) = \frac{1}{2\pi\mathrm{i}}\int_{F(\Gamma)} \frac{1}{z}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int_{\gamma'} \frac{1}{z}\,\mathrm{d}z \in \{1,-1\}.$$

This proves (i). Suppose there is no equilibrium in $\mathrm{Int}(\Gamma)$. Then, by Proposition 4.6, we conclude the contradiction $\mathrm{ind}(F,\Gamma) = 0 \notin \{1,-1\}$. Thus (ii) holds. The sign of $\mathrm{ind}(F,\Gamma)$ is positive if and only if Γ is parameterized counterclockwise, cf. the proof of [15, p. 28, Theorem 2.28]. If this is the case, then (iii) is implied by (i) and Theorem 4.10. If $\mathrm{ind}(F,\Gamma) = -1$, then Γ is parameterized clockwise and $\tilde{\gamma} := -\gamma$ is a counterclockwise parameterization. Hence, again by Theorem 4.10, we have

$$\sum_{\substack{a\in\mathrm{Int}(\Gamma)\\ F(a)=0}} \mathrm{ind}(F,a) = \mathrm{ind}(F,\tilde{\gamma}([0,T])) = -\mathrm{ind}(F,\gamma([0,T])) = -\underbrace{\mathrm{ind}(F,\Gamma)}_{=-1} = 1$$

Here we used the fact that only the sign of $\mathrm{ind}(F,\Gamma)$ depends on the orientation of Γ. This proves (iii) also for this case.

$\square$

Theorem 4.12
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain and $F \in \mathcal{O}(\Omega)$ with $F \not\equiv 0$. Let $a \in \Omega$ be an equilibrium of (4.1) with $F'(a) \neq 0$. Then $\mathrm{ind}(F,a) = 1$.

Proof
This proof is based on [7, p. 247] with minor changes. The omitted steps of the proof are performed by me, in particular the equations (4.2) and (4.3).
Assume w.l.o.g. that $a = 0$, which is possible since the phase space structure of the analytic vector filed $x \mapsto F(x+a)$ is just shifted and has the same topological properties. This simplifies the notation in the following arguments.
The proof is based on the idea that the index is invariant under linearization of the vector field. Set $\alpha := \Re(F'(a))$, $\beta := \Im(F'(a))$,

$$J := \mathcal{J}_F(a) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

and $B := (J^{-1})^T$. Note that $\det(J) > 0$. Define $H, \tilde{H} : \mathbb{R}^2 \to \mathbb{R}^2$, $H(x) := Jx = F'(a)x$, $\tilde{H}(x) := Bx$ and $g : [0,1] \to \mathcal{O}(\Omega)$ as the convex combination of F and H, i.e. $g(\tau) := \tau H + (1-\tau)F$, $\tau \in [0,1]$. Since H is entire, i.e. holomorphic on $\mathbb{C}$, we indeed have $g(\tau) \in \mathcal{O}(\Omega)$ for all $\tau \in [0,1]$.
Denote by $\arg(x,y)$ the angle between the two vectors $x, y \in \mathbb{R}^2$. At first, consider the

following estimation for all $r > 0$ and $x \in \partial\mathcal{B}_r(0)$

$$
\begin{aligned}
|H(x)| \left\|\tilde{H}\right\| &= |Jx| \sup_{|y|=1} |By| = \frac{1}{r} \sup_{|y|=r} |Jx||By| \geq \frac{1}{r} \sup_{|y|=r} |\underbrace{\langle By, Jx\rangle}_{=\langle x,y\rangle}| \\
&= \frac{1}{r} \sup_{|y|=r} |\cos(\arg(x,y))| \underbrace{|x||y|}_{=r^2} = r \underbrace{\max_{|y|=r} |\cos(\arg(x,y))|}_{=1 \ \text{(with } y=x\text{)}} = r.
\end{aligned}
\tag{4.2}
$$

Here we used the Cauchy-Schwarz inequality and the operator norm for $\tilde{H} \in (\mathbb{R}^2)^\star$. Choose $\varepsilon := \frac{1}{2}\left\|\tilde{H}\right\|^{-1} > 0$. Then, by differentiability of F in a, there exists a $\delta > 0$ such that

$$
|F(x) - H(x)| = |F(x) - F(a) - J(x-a)| \leq \varepsilon |x| \quad \forall\, x \in \mathcal{B}_\delta(a). \tag{4.3}
$$

Choose $r \in (0, \frac{\delta}{2})$ sufficiently small such that $\overline{\mathcal{B}_r(a)} \subset \Omega$ and $\overline{\mathcal{B}_r(a)} \cap F^{-1}(\{0\}) = \{a\}$. By applying the reverse triangle inequality and the estimations (4.2) and (4.3), this leads to

$$
\begin{aligned}
|g(\tau)(x)| &= |\tau H(x) + (1-\tau)F(x)| = |H(x) - \underbrace{(1-\tau)}_{\leq 1}(H(x) - F(x))| \\
&\geq |H(x)| - |F(x) - H(x)| \geq \left\|\tilde{H}\right\|^{-1} r - \varepsilon|x| = \frac{r}{2\left\|\tilde{H}\right\|} > 0
\end{aligned}
$$

for all $\tau \in [0,1]$ and $x \in \partial\mathcal{B}_r(a) \subset \mathcal{B}_\delta(a)$. Parameterize $\partial\mathcal{B}_r(a)$ by $\gamma(s) := re^{2\pi \mathrm{i} s}$, $s \in [0,1]$. Then $\Psi(\tau, s) := g(\tau)(\gamma(s))$, $(\tau, s) \in [0,1]^2$, satisfies $\Psi \in \mathcal{C}^0([0,1]^2; \mathbb{C} \setminus \{0\})$. Hence, it is a path homotopy between $\Psi(0,\cdot)$ and $\Psi(1,\cdot)$ on $\mathbb{C}\setminus\{0\}$. Actually, since $\Psi([0,1]^2)$ is compact in $\mathbb{C}\setminus\{0\}$, we can even find a $\rho > 0$ such that $\Psi([0,1]^2) \subset \mathbb{C} \setminus \mathcal{B}_\rho(0)$, i.e. Ψ has away from zero. By applying the homotopy version of Cauchy's Integral Theorem, we conclude

$$
\int\limits_{F(\partial\mathcal{B}_r(a))} z^{-1}\,\mathrm{d}z = \int\limits_{\Psi(0,[0,1])} z^{-1}\,\mathrm{d}z = \int\limits_{\Psi(1,[0,1])} z^{-1}\,\mathrm{d}z = \int\limits_{H(\partial\mathcal{B}_r(a))} z^{-1}\,\mathrm{d}z.
$$

Applying Lemma 4.5 implies $\operatorname{ind}(F,a) = \operatorname{ind}(H,a)$, i.e. the index is invariant under linearization. Furthermore, we have $(H \circ \gamma)'(s) = F'(a)2\pi \mathrm{i} re^{2\pi \mathrm{i} s} = 2\pi\mathrm{i} H(\gamma(s))$, $s \in [0,1]$. Hence, we conclude

$$
\operatorname{ind}(F,a) = \operatorname{ind}(H,a) = \frac{1}{2\pi\mathrm{i}} \int_0^1 \frac{(H\circ\gamma)'(s)}{(H\circ\gamma)(s)}\,\mathrm{d}s = \frac{1}{2\pi\mathrm{i}} \int_0^1 \frac{2\pi\mathrm{i} H(\gamma(s))}{H(\gamma(s))}\,\mathrm{d}s = 1.
$$

□

4.3 Sectorial decomposition of equilibria

The aim of this section is to characterize the local structure of an equilibrium with non-vanishing derivative, i.e. the order of the zero is at least 2. The definitions are based on [1, Chapter 1.5] and the results in [6, §20]. The main theorem in this chapter is our first fundamental result: There are a certain number of so-called elliptic sectors in an equilibrium and locally no other sectors can occur.

Definition 4.13
Let $\Omega \subset \mathbb{C}$ be an open domain, $F \in \mathcal{O}(\Omega)$, $F \not\equiv 0$, $a \in \Omega$ an equilibrium of (4.1) and $r > 0$ sufficiently small. Set $\Gamma := \partial\mathcal{B}_r(a)$. Assume that $\overline{\mathrm{Int}(\Gamma)} \subset \Omega$ and $\overline{\mathrm{Int}(\Gamma)} \cap F^{-1}(\{0\}) = \{a\}$.

a) For $p, q \in \Gamma$, $p \neq q$, we denote by $\Gamma(p, q)$ the closed (i.e. including p and q) curve section of Γ from p to q in counterclockwise direction.

b) A *sector* $S \subset \mathrm{Int}(\Gamma)$ of (4.1) in a with respect to Γ is a compact set such that there exist two characteristic orbits Γ_1, Γ_2 of (4.1) and two intersection points $p_1, p_2 \in \Gamma$ with the following properties:

   (i) For $i \in \{1, 2\}$ it holds that $\Gamma_i \cap \Gamma = \{p_i\}$ and furthermore this cutting is transverse, i.e. the embedded tangent spaces of the two 1-dimensional $\mathcal{C}^1$-surfaces Γ and Γ_i at p_i are not equal.

   (ii) The boundary of S is given by

$$\partial S = \mathrm{Int}(\Gamma) \cap (\{a\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma(p_1, p_2)) \, .$$

Definition 4.14
Let $\Omega \subset \mathbb{C}$ be an open domain, $F \in \mathcal{O}(\Omega)$, $F \not\equiv 0$, $a \in \Omega$ an equilibrium of (4.1) and $r > 0$ sufficiently small. Set $\Gamma := \partial\mathcal{B}_r(a)$. Assume that $\overline{\mathrm{Int}(\Gamma)} \subset \Omega$ and $\overline{\mathrm{Int}(\Gamma)} \cap F^{-1}(\{0\}) = \{a\}$. Let $\nu : \Gamma \to S^1$ be the outer unit normal of Γ. Let S be a sector of (4.1) in a with respect to Γ with characteristic orbits Γ_1, Γ_2, intersection points $p_1, p_2 \in \Gamma$ and the curve piece $\Lambda := \Gamma(p_1, p_2)$. Then S is called

a) an *attracting (repelling) parabolic sector* if for all $y \in \Lambda$ and $z \in S \setminus \{a\}$ the following properties are satisfied:

   (i) $\langle F(y), \nu(y) \rangle < (>) \; 0$.

(ii) $w_+(\Gamma(z)) = \{a\}$ $(w_-(\Gamma(z)) = \{a\})$.

(iii) $\Gamma_-(z) \cap \Lambda \neq \emptyset$ $(\Gamma_+(z) \cap \Lambda \neq \emptyset)$.

In particular, S is positive (negative) invariant.

b) an *elliptic sector with clockwise (counterclockwise) direction* if there exists a point $E \in \Lambda \setminus \{p_1, p_2\}$ such that the following properties are satisfied:

(i) $\Gamma(E) \subset \mathrm{Int}(\Gamma) \cup \{E\}$.

(ii) $w_+(\Gamma(E)) = w_-(\Gamma(E)) = \{a\}$.

(iii) For all $y_1 \in \Lambda_1 := \Gamma(p_1, E) \setminus \{E\}$ and $y_2 \in \Lambda_2 := \Gamma(E, p_2) \setminus \{E\}$ it holds:

- $\langle F(y_1), \nu(y_1)\rangle < (>)\ 0$.
- $\langle F(y_2), \nu(y_2)\rangle > (<)\ 0$.

(iv) For all $y_1 \in \Lambda_1$ and $y_2 \in \Lambda_2$ it holds:

- $\Gamma_+(y_1) \cup \Gamma_-(y_2) \subset \overline{\mathrm{Int}(\Gamma)}$ $\big(\Gamma_-(y_1) \cup \Gamma_+(y_2) \subset \overline{\mathrm{Int}(\Gamma)}\big)$.
- $w_+(\Gamma(y_1)) = w_-(\Gamma(y_2)) = \{a\}$ $(w_-(\Gamma(y_1)) = w_+(\Gamma(y_2)) = \{a\})$.

(v) $\Gamma(z) \subset \overline{\mathrm{Int}(\Gamma)}$ and $w_+(\Gamma(z)) = w_-(\Gamma(z)) = \{a\}$ for all $z \in S \setminus S_E$, where

$$S_E := \{p\} \cup \Gamma(E) \cup \bigcup_{y_1 \in \Lambda_1} \Gamma_+(y_1) \cup \bigcup_{y_2 \in \Lambda_2} \Gamma_-(y_2)$$
$$\left(S_E := \{p\} \cup \Gamma(E) \cup \bigcup_{y_1 \in \Lambda_1} \Gamma_-(y_1) \cup \bigcup_{y_2 \in \Lambda_2} \Gamma_+(y_2)\right).$$

In particular, $S \setminus S_E$ is an invariant attractor for a.

c) a *hyperbolic or saddle sector with clockwise (counterclockwise) direction* if there exists a point $E \in \Lambda \setminus \{p_1, p_2\}$ such that the following properties are satisfied:

(i) The cutting of $\Gamma(E)$ with Γ is tangent and satisfies $\Gamma(E) \cap \overline{\mathrm{Int}(\Gamma)} = \{E\}$, i.e. the embedded tangent spaces of the two 1-dimensional $\mathcal{C}^1$-surfaces Γ and $\Gamma(E)$ at E are equal and the orbit through E remains outside of $\overline{\mathrm{Int}(\Gamma)}$.

(ii) For all $y_1 \in \Gamma(p_1, E) \setminus \{E\}$ and $y_2 \in \Gamma(E, p_2) \setminus \{E\}$ it holds:

- $\langle F(y_1), \nu(y_1)\rangle > (<)\ 0$.
- $\langle F(y_2), \nu(y_2)\rangle < (>)\ 0$.

(iii) $\Gamma_-(z) \cap \Gamma \neq \emptyset$ and $\Gamma_+(z) \cap \Gamma \neq \emptyset$ for all $z \in S \setminus \overline{\{E, a\} \cup \Gamma_1 \cup \Gamma_2}$.

In particular, there is no invariant subset of S. Every orbit starting in $\mathring{S}$ exits S in positive and negative finite time.

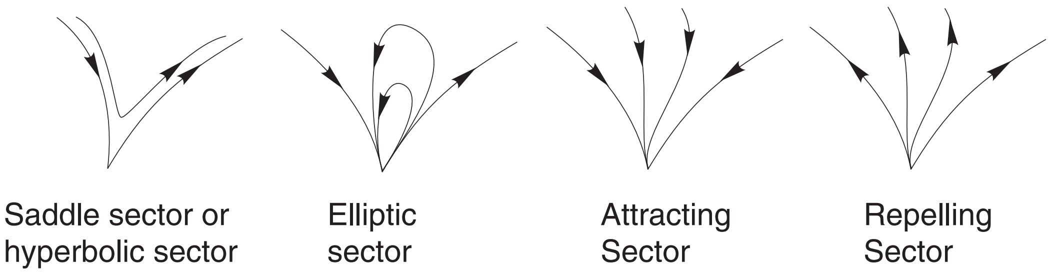

Figure 4.2: Geometrical visualization of a hyperbolic, elliptic, attracting parabolic and repelling parabolic sector ([1, Fig. 1.8])

Definition 4.15

Let $\Omega \subset \mathbb{C}$ be an open domain, $F \in \mathcal{O}(\Omega)$, $F \not\equiv 0$, $a \in \Omega$ an equilibrium of (4.1) and $r > 0$ sufficiently small. Set $\Gamma := \partial \mathcal{B}_r(a)$. Assume that $\overline{\mathrm{Int}(\Gamma)} \subset \Omega$ and $\overline{\mathrm{Int}(\Gamma)} \cap F^{-1}(\{0\}) = \{a\}$.

a) The system (4.1) has a *finite sectorial decomposition* (FSD) of order $d \in \mathbb{N} \setminus \{1\}$ in a, if the equilibrium is neither a center nor a focus and if there are d characteristic orbits $\Gamma_1, \ldots, \Gamma_d$ with corresponding points $p_1 \ldots, p_d \in \Gamma$ such that the following properties are satisfied for all $i \in \{1, \ldots, d\}$:

(i) The set
$$\tilde{S} := \mathrm{Int}(\Gamma) \cap (\{a\} \cup \Gamma_i \cup \Gamma_{i+1} \cup \Gamma(p_i, p_{i+1}))$$
is a simple closed piecewise continuously differentiable path.

(ii) The set $S := \overline{\mathrm{Int}(\tilde{S})}$ is a parabolic, elliptic or hyperbolic sector of (4.1) in a with respect to Γ whose characteristic orbits are Γ_i, Γ_{i+1} and whose intersection points are $p_i, p_{i+1} \in \Gamma$.

(iii) The characteristic orbits with the corresponding points are ordered cyclic and counterclockwise with respect to a.

Using this notation, we set $\Gamma_{d+1} := \Gamma_1$ and $p_{d+1} := p_1$.

b) A FSD of order d is *minimal* if there is no possibility to construct a FSD of order $\tilde{d} \in \{2, \ldots, d-1\}$. If $d = 2$, the FSD is always already minimal.

Remark 4.16
Since the constructions in the definitions above are not straightforward, the following remarks provide a more elaborate view.

a) A more detailed geometrical description of the scheme of equilibria and the structure of sectors can be found in [6, Chapter VIII]. Here the author constructs the different types of sectors step by step, by means of arcs and subarcs without contact to the circle Γ, cf. [6, §3]. In particular, the constructions starts with semipaths (parts of orbits) that tend to the equilibrium and form so-called separatrices. This semipaths correspond to our characteristic orbits $\Gamma_1, \dots, \Gamma_d$. However, in our case this property is a priori not given and will be proven later in this section.

b) Whether a sector is elliptic, hyperbolic or parabolic, does not depend on the choice of the radius r, i.e. the topological structure of a finite sectorial decomposition is a local property of the equilibrium. To ensure this, the radius must be chosen small enough. Recall that there are only finitely many zeros of F in any bounded set. Also the cyclic order of the characteristic orbits and intersection points does not depend on the radius. This can be proven with lengthy geometric contradiction arguments, cf. [6, §17, Lemma 1].

Remark 4.17
The order of a given FSD can be reduced by joining two adjacent parabolic sectors, in other words, not accepting two adjacent parabolic sectors, and by adding a parabolic sector to an elliptic one if it is adjacent to it. Hence in a minimal FSD the parabolic sectors can only be the ones lying between two hyperbolic sectors.

Example 4.18
The simplest already known example of an equilibrium with a minimal FSD is a saddle, cf. Definition 4.1. The characteristic orbits are $\Gamma_1, \dots, \Gamma_4$ and the order is 4. All sectors are hyperbolic. Additionally, there are no other FSDs and there is no way to construct one with either smaller order, or with an elliptic sector.
Geometrically, a node is an equilibrium with arbitrary many parabolic sectors. But there is no way to construct a minimal FSD for nodes, since it would have only one sector, i.e. the order of the minimal FSD would be $1 < 2$.

Proposition 4.19
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F \in \mathcal{O}(\Omega)$, $F \not\equiv 0$ and $a \in \Omega$ an equilibrium of (4.1) with order $m \in \mathbb{N}$. Then $\operatorname{ind}(F, a) = m$.

Proof
If $m = 1$, the assertion follows directly from Theorem 4.12. Assume $m \geq 2$ and let $r > 0$ be small enough such that $\operatorname{ind}(F, a) = \operatorname{ind}(F, \partial\mathcal{B}_r(a))$. Set $\Gamma := \partial\mathcal{B}_r(a)$. By Lemma 4.5,

$$\operatorname{ind}(F, a) = \frac{1}{2\pi \mathrm{i}} \int\limits_{F(\Gamma)} z^{-1}\, \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma} \frac{F'(z)}{F(z)}\, \mathrm{d}z = \operatorname{Res}\left(\frac{F'}{F}, a\right).$$

Since the order of the zero is m, the residue of $\frac{F'}{F}$ in a is also m. Thus $\operatorname{ind}(F, a) = m$.

□

Proposition 4.20
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $a := 0 \in \Omega$, $F = (F_1, F_2)^T \in \mathcal{C}^\infty(\Omega; \mathbb{R}^2)$ real analytic with at most finitely many zeros in every bounded set and $F(a) = 0$. Assume that the order of a is $m \in \mathbb{N} \setminus \{1\}$. Let $F_1^{[k]}$ and $F_2^{[k]}$ be the sum of all terms of the Taylor series at a of F_1 and F_2 with degree $k \in \mathbb{N}$ and set

$$H(x_1, x_2) := x_1 F_2^{[m]}(x_1, x_2) - x_2 F_1^{[m]}(x_1, x_2)$$

with $(x_1, x_2)^T \in \mathbb{R}^2$. Assume that H is not identically zero. Then:

a) Any orbit Γ of (4.1) tending to a for $t \to \infty$ or $t \to -\infty$ is either a spiral, or tends to a in a definite direction[3] $\theta_0 \in [0, 2\pi)$.

b) If Γ is an orbit of (4.1) tending to a in a definite direction $\theta_0 \in [0, 2\pi)$, then θ_0 satisfies the equation $H(\cos(\theta_0), \sin(\theta_0)) = 0$.

c) If there exists a spiral Γ of (4.1) tending to a for $t \to \infty$ ($t \to -\infty$), then a is a stable (unstable) focus.

Proof
The formulated Proposition coincides with [6, §20, Theorem 64]. In the following some technical details of the proof are left out. A more detailed version can be found in [6, §20].

[3] We defined this expressions in Definition 4.1.

At first we apply polar coordinates to deduce

$$\frac{\mathrm{d}|x|}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{x_1^2 + x_2^2} = \frac{2x_1x_1' + 2x_2x_2'}{2\sqrt{x_1^2 + x_2^2}} = \frac{x_1F_1 + x_2F_2}{|x|}$$

and

$$\frac{\mathrm{d}\arg(x)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\arctan\left(\frac{x_2}{x_1}\right) = \frac{1}{1 + \frac{x_2^2}{x_1^2}}\frac{x_1x_2' - x_2x_1'}{x_1^2} = \frac{x_1F_2 - x_2F_1}{|x|^2}$$

for an arbitrary solution $x = (x_1, x_2)$. This leads to the system

$$\begin{aligned} \rho'(t) &= F_1(\rho\cos(\theta), \rho\sin(\theta))\cos(\theta) + F_2(\rho\cos(\theta), \rho\sin(\theta))\sin(\theta) \\ \theta'(t) &= \frac{1}{\rho}\Big(F_2(\rho\cos(\theta), \rho\sin(\theta))\cos(\theta) - F_1(\rho\cos(\theta), \rho\sin(\theta))\sin(\theta)\Big) \end{aligned} \tag{4.4}$$

on the simply connected open domain $\Omega_1 := \mathbb{R} \times (0, \rho^\star)$ with a small $\rho^\star > 0$. The relation between the orbits of (4.1) and (4.4) is determined in [6, §8.3]. For $j \in \{1, 2\}$, $k \in \mathbb{N}$ and $\theta \in \mathbb{R}$ set $\tilde{F}_j^{[k]}(\theta) := F_j^{[k]}(\cos(\theta), \sin(\theta))$. With this notation we have

$$F_j^{[k]}(\rho\cos(\theta), \rho\sin(\theta)) = \sum_{\substack{0 \le k_1 \le k_2 \\ k_1 + k_2 = k}} \frac{\partial^{(k_1,k_2)}F_j(0)}{k_1!k_2!}\rho^{k_1}\cos^{k_1}(\theta)\rho^{k_2}\sin^{k_2}(\theta) = \rho^k\tilde{F}_j^{[k]}(\theta).$$

By applying the time transformation $\mathrm{d}\tau = \rho^{m-1}\mathrm{d}t$ and the real analyticity of F_1 and F_2 in a, there exist two analytic functions Ψ_1 and Ψ_2 such that the orbits of (4.4) coincide with those of the system

$$\begin{aligned} \rho'(\tau) &= \rho\tilde{F}_1^{[m]}(\theta)\cos(\theta) + \rho\tilde{F}_2^{[m]}(\theta)\sin(\theta) + \rho^2\Psi_1(\rho, \theta) \\ \theta'(\tau) &= \tilde{F}_2^{[m]}(\theta)\cos(\theta) - \tilde{F}_1^{[m]}(\theta)\sin(\theta) + \rho\Psi_2(\rho, \theta) \end{aligned} \tag{4.5}$$

on the domain Ω_1. More precisely, Ψ_1 and Ψ_2 are the Taylor series of F_1 and F_2 without the terms of degree less than or equal to m and multiplied by ρ^{1-m}. This system can also be considered on $\Omega_2 := \mathbb{R} \times (-\rho^\star, \rho^\star)$. But only on Ω_1 we can relate the orbits of (4.1) with those of (4.5), cf. [6, §8.3].
Now, let Γ be an arbitrary orbit of (4.1) tending to a for $t \to \infty$ with the corresponding orbits Γ_1 of (4.4) and Γ_2 of (4.5). The argument for $t \to -\infty$ is analogous. Fix a $z \in \Gamma \cap \mathcal{B}_{\rho^\star}(0)$. It is clear that $|\Phi(t, z)| \to 0$ for $t \to \infty$. Thus $\rho(t) \to 0$ for $t \to \infty$ on Γ_1. By a short contradiction argument it can be shown that $\tau \to \infty$ if $t \to \infty$ and the other

way around. Note that the used time transformation is monotone increasing. Hence it follows $\rho(\tau) \to 0$ for $\tau \to \infty$ on Γ_2. Moreover, this shows that the orbit Γ_2 exists globally to the right on Ω_2. In order to ascertain whether Γ tends to a in a definite direction, we have to analyze the function $\tau \mapsto \theta(\tau)$ corresponding to the orbit Γ_2. A priori we have three possibilities:

(i) $|\theta(\tau)| \to \infty$ for $\tau \to \infty$.

(ii) There exists a $C > 0$ such that $|\theta| \leq C$ on $[t_0, \infty)$ with $t_0 \in \mathbb{R}$ sufficiently large.

(iii) θ is not bounded by a constant and does not tend to ∞ or $-\infty$ for $\tau \to \infty$.

By using $H \not\equiv 0$ it can be shown that possibility (iii) cannot occur. The possibility (i) fits to the case that Γ is a spiral. And (ii) describes a bounded orbit away from the boundary of Ω_2. Hence $w_+(\Gamma_2) \neq \emptyset$. But we also have $\rho(\tau) \to 0$ for $\tau \to \infty$. Thus, by [6, §4, Theorem 9], we get $w_+(\Gamma_2) \subset \{0\} \times \mathbb{R}$. This cannot be a limit cycle. So a Corollary to the Poincare-Bendixson theorem, cf. [7, Satz 9.2.4], implies that $w_+(\Gamma_2)$ consists of exactly one equilibrium $a_1 := (0, \theta_0) \in \Omega_2$. In summary, we get $\arg(\Phi(t, z)) \to \theta_0$ for $t \to \infty$, i.e. Γ tends to a in the definite direction θ_0. This proves a). Moreover, a_1 is a zero of the right-hand side of (4.5). This implies

$$H(\cos(\theta_0), \sin(\theta_0)) = \tilde{F}_2(\theta_0)\cos(\theta_0) - \tilde{F}_1(\theta_0)\sin(\theta_0) = 0.$$

This shows b). The assertion c) is proved in [6, §20.2, Remark 2]. Here one makes use of arcs without contact, cf. [6, §3, Lemma 14].

□

Remark 4.21

The singular case of Proposition 4.20, i.e. $H \equiv 0$, is described in [6, §20.3]. However, we do not need this result in our further considerations.

In the proof of [2, Theorem 2.5] the author uses the equation $H(\cos(\theta_0), \sin(\theta_0)) = 0$, cf. Proposition 4.20, not only as a necessary but also as a sufficient condition for definite directions. But why does this hold? The author does not provide an explanation. Hence we need the following proposition to ensure the correctness of [2, Theorem 2.5]. The proof of this proposition is my own work and uses the geometrical properties of the phase space belonging to system (4.5).

Definition and Proposition 4.22 (Sufficient condition for definite directions)
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $a := 0 \in \Omega$, $F = F_1 + \mathrm{i}F_2 \in \mathcal{O}(\Omega)$, $F \not\equiv 0$ and $F(a) = 0$. Assume that the order of a is $m \in \mathbb{N} \setminus \{1\}$. Let $F_1^{[k]}$ and $F_2^{[k]}$ be the sum of all terms of the Taylor series at a of F_1 and F_2 with degree $k \in \mathbb{N}$ and set

$$H(x_1, x_2) := x_1 F_2^{[m]}(x_1, x_2) - x_2 F_1^{[m]}(x_1, x_2)$$

with $(x_1, x_2) \in \mathbb{R}^2$. Then for all $\theta_0 \in [0, 2\pi)$ satisfying $H(\cos(\theta_0), \sin(\theta_0)) = 0$ there exist $r > 0$ and $\delta > 0$ such that for every $x_0 \in \{x \in \mathbb{R}^2 : |x - a| < r, |\arg(x - a) - \theta_0| < \delta\} \subset \Omega$ the orbit though x_0 tends to a in the definite direction θ_0. These orbits tend to a for $t \to \infty$ ($t \to -\infty$) if and only if $\cos(\arg(F^{(m)}(a)) + \theta_0(m-1)) < (>)\ 0$. Define the set

$$\mathcal{E}(F, m) := \left\{ \frac{\ell\pi - \arg(F^{(m)}(a))}{m-1} \mod 2\pi : \ell \in \mathbb{Z} \right\} \subset [0, 2\pi).$$

Then additionally there are exactly $2m - 2$ such definite directions for a and every orbit tending to a does so in a definite direction $\theta_0 \in \mathcal{E}(F, m)$.

Proof
The proof is divided into several steps. The first step is to constitute the real Taylor expansions of F_1 and F_2 in an appropriate manner. Set $c_k := \frac{1}{k!} F^{(k)}(0)$ for all $k \in \mathbb{N}$. Note that under the above assumptions we always have $c_m \neq 0$. This is used throughout this proof. The complex Taylor expansion of F in a is

$$F(z) = \sum_{k=1}^{\infty} c_k z^k \quad \forall\, z \in \mathcal{B}_{r_0}(0) \subset \Omega$$

with a sufficiently small radius of convergence $r_0 > 0$. In particular, reduce r_0 such that $\mathcal{B}_{r_0}(0) \cap F^{-1}(\{0\}) = \{0\}$. The functions $z \mapsto \Re(z)$ and $z \mapsto \Im(z)$ are continuous on $\mathbb{C}$. Hence it follows

$$F_1(z) = \Re(F(z)) = \Re\left(\sum_{k=1}^{\infty} c_k z^k\right) = \sum_{k=1}^{\infty} \Re\left(c_k z^k\right) \quad \forall\, z \in \mathcal{B}_{r_0}(0)$$

and analogous with $F_2 = \Im(F)$.

Fix $k \in \mathbb{N}$. By the Binomial theorem we get for the real part of the k^{th} term $\overline{F_1}^{[k]} : \mathbb{R}^2 \to \mathbb{R}$

$$\overline{F_1}^{[k]}(x,y) := \Re\left(c_k(x+\mathrm{i}y)^k\right) = \Re\left(c_k \sum_{j=0}^{k} \binom{k}{j} x^{k-j} (\mathrm{i}y)^j\right) = \sum_{j=0}^{k} \binom{k}{j} \Re\left(c_k \mathrm{i}^j\right) x^{k-j} y^j$$

for all $(x,y) \in \mathbb{R}^2$. Thus a term of the (real) Taylor expansion of F_1 has degree k if and only if it is a term of $\overline{F_1}^{[k]}$, i.e. $\overline{F_1}^{[k]}$ is exactly the sum of all addends with degree k in the (real) Taylor expansion. Note that the Taylor expansion is unique. So we have $F_1^{[k]} \equiv \overline{F_1}^{[k]}$. Of course, the same holds for $F_2^{[k]}(x,y) = \overline{F_2}^{[k]}(x,y) := \Im\left(c_k(x+\mathrm{i}y)^k\right)$ with $(x,y) \in \mathbb{R}^2$. As in the proof of Proposition 4.20, set $\tilde{F}_j^{[k]}(\theta) := F_j^{[k]}(\cos(\theta), \sin(\theta))$ with $j \in \{1,2\}$, $k \in \mathbb{N}$ and $\theta \in \mathbb{R}$.

The second step is to prove the following: For all $\theta \in \mathbb{R}$ it is impossible that both, $\tilde{F}_1^{[m]}(\theta) = 0$ and $\tilde{F}_2^{[m]}(\theta) = 0$. Suppose that such a θ exists. Then we have

$$0 = \tilde{F}_1^{[m]}(\theta) + \mathrm{i}\tilde{F}_2^{[m]}(\theta) = \Re\left(c_m \left(e^{\mathrm{i}\theta}\right)^m\right) + \mathrm{i}\Im\left(c_m \left(e^{\mathrm{i}\theta}\right)^m\right) = c_m e^{\mathrm{i}\theta m}.$$

Hence $c_m = 0$, which is impossible.

Define now $\tilde{H}(\theta) := H(\cos(\theta), \sin(\theta)$, $\theta \in \mathbb{R}$. The next claim is the following: There is no $\theta \in [0, 2\pi)$ satisfying the system of equations

$$\begin{cases} 0 = \tilde{H}(\theta) \\ 0 = \tilde{F}_1^{[m]}(\theta)\cos(\theta) + \tilde{F}_2^{[m]}(\theta)\sin(\theta) \end{cases}.$$

Suppose again that such a θ exists. Then θ satisfies

$$\begin{cases} \tilde{F}_1^{[m]}(\theta)\sin(\theta) = \tilde{F}_2^{[m]}(\theta)\cos(\theta) \\ \tilde{F}_1^{[m]}(\theta)\cos(\theta) = -\tilde{F}_2^{[m]}(\theta)\sin(\theta) \end{cases}.$$

We can exclude the following impossible cases: If $\tilde{F}_{j_1}^{[m]}(\theta) = 0$, then $\tilde{F}_{j_2}^{[m]}(\theta) = 0$ or $\cos(\theta) = \sin(\theta) = 0$ for $j_1, j_2 \in \{1,2\}$, $j_1 \neq j_2$. If $\cos(\theta) = 0$, then $\sin(\theta) \neq 0$ and so $\tilde{F}_1^{[m]}(\theta) = \tilde{F}_2^{[m]}(\theta) = 0$.

Hence we must have

$$-\frac{\tilde{F}_1^{[m]}(\theta)}{\tilde{F}_2^{[m]}(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\tilde{F}_2^{[m]}(\theta)}{\tilde{F}_1^{[m]}(\theta)}$$

and thus $\left(\tilde{F}_1^{[m]}(\theta)\right)^2 = -\left(\tilde{F}_2^{[m]}(\theta)\right)^2$. We conclude $\tilde{F}_1^{[m]}(\theta) = \tilde{F}_2^{[m]}(\theta) = 0$, which is impossible.

In the next step we prove $H \not\equiv 0$. Suppose this is the case. Then we have

$$x\Im\left(c_m(x+\mathrm{i}y)^m\right) = y\Re\left(c_m(x+\mathrm{i}y)^m\right)$$

for all $(x,y) \in \mathbb{R}^2$. We would like to derive a contradiction by choosing $x, y \in \mathbb{R}$ suitably. First, choose $x = 1$ and $y = 0$. This leads to $\Im(c_m) = 0$, i.e. $c_m \in \mathbb{R}$. Second, for arbitrary $\theta \in \mathbb{R}$ and $(x,y) = (\cos(\theta), \sin(\theta))$ we can conclude

$$\cos(\theta)\Im\left(c_m e^{\mathrm{i}\theta m}\right) - \sin(\theta)\Re\left(c_m e^{\mathrm{i}\theta m}\right) = 0.$$

Hence $c_m = 0$ (this is impossible) or $c_m\cos(\theta)\sin(m\theta) - c_m\sin(\theta)\cos(m\theta) = 0$. By using a trigonometric addition formula, we get $0 = \sin(m\theta - \theta) = \sin(\theta(m-1))$. So choosing $\theta := \frac{\pi}{2(m-1)}$ leads to the contradiction $0 = 1$. Note that $m \geq 2$, i.e. $m - 1 \neq 0$.

In the next step we calculate the zeros of $\tilde{H}$. Since $\tilde{F}_1^{[m]}$ and $\tilde{F}_2^{[m]}$ cannot be both zero (see above), we can characterize these zeros appropriately. Set $c := |c_m|$ and $\beta := \arg(c_m) = \arg(F^{(m)}(0))$. For $\theta \in [0, 2\pi)$ we have

$$\frac{\tilde{F}_2^{[m]}(\theta)}{\tilde{F}_1^{[m]}(\theta)} = \frac{\Im\left(ce^{\mathrm{i}\beta}e^{\mathrm{i}\theta m}\right)}{\Re\left(ce^{\mathrm{i}\beta}e^{\mathrm{i}\theta m}\right)} = \frac{\Im\left(e^{\mathrm{i}(\beta+\theta m)}\right)}{\Re\left(e^{\mathrm{i}(\beta+\theta m)}\right)} = \frac{\sin(\beta+\theta m)}{\cos(\beta+\theta m)} = \tan(\beta+\theta m).$$

Hence θ satisfies $\tilde{H}(\theta) = 0$ if and only if $\tan(\theta) = \tan(\beta + \theta m)$. And this is the case if and only if there exists a $\ell \in \mathbb{Z}$ such that $\beta + \theta m = \theta + \ell\pi$, which is equivalent to $\theta = \frac{\ell\pi - \beta}{m-1}$. Note that $x \mapsto \tan(x)$ is π-periodic. So we conclude $\tilde{H}^{-1}(\{0\}) = \mathcal{E}(F, m)$. Moreover,

$$\frac{(\ell + 2m - 2)\pi - \beta}{m-1} = \frac{\ell\pi - \beta}{m-1} + \frac{(2m-2)\pi}{m-1} = \frac{\ell\pi - \beta}{m-1} + 2\pi$$

for all $\ell \in \mathbb{Z}$. Hence $\mathcal{E}(F, m)$ has exactly $2m - 2$ elements. Now we have all auxiliary results to proof our proposition.

Let $\theta_0 \in [0, 2\pi)$ satisfy $\tilde{H}(\theta) = 0$. We have to find $r > 0$ and $\delta > 0$ such that all orbits Γ of (4.1) trough a point of the set $\{x \in \mathbb{R}^2 : |x - a| < r, |\arg(x - a) - \theta| < \delta\}$ tend to a in the definite direction θ_0. Consider the $\mathcal{C}^\infty$-system (4.5) on $\Omega_2 := \mathbb{R} \times (-\rho^\star, \rho^\star)$ with $\rho^\star > 0$ sufficiently small. Set $\Omega_1 := \mathbb{R} \times (0, \rho^\star)$. The point $a_1 := (0, \theta_0)$ is an equilibrium of

(4.5). Denote by $G : \Omega_2 \to \mathbb{R}^2$ the right-hand side of (4.5). We calculate the linearization

$$\mathcal{J}_G(a_1) = \begin{pmatrix} \tilde{F}_1^{[m]}(\theta_0)\cos(\theta_0) + \tilde{F}_2^{[m]}(\theta_0)\sin(\theta_0) & 0 \\ \Psi_2(0,\theta_0) & \tilde{H}'(\theta_0) \end{pmatrix}$$

and the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\tilde{F}_1^{[m]}(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\Re\left(c_m e^{\mathrm{i}\theta m}\right) = m\Re\left(ic_m e^{\mathrm{i}\theta m}\right) = -m\Im\left(c_m e^{\mathrm{i}\theta m}\right) = -m\tilde{F}_2^{[m]}(\theta)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\tilde{F}_2^{[m]}(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\Im\left(c_m e^{\mathrm{i}\theta m}\right) = m\Im\left(ic_m e^{\mathrm{i}\theta m}\right) = m\Re\left(c_m e^{\mathrm{i}\theta m}\right) = m\tilde{F}_1^{[m]}(\theta).$$

Here we used the formulas $\Re(iz) = -\Im(z)$ and $\Im(iz) = \Re(z)$, which are valid on $\mathbb{C}$. Hence

$$\begin{aligned}
\tilde{H}'(\theta_0) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\tilde{F}_2^{[m]}(\theta)\cos(\theta) - \tilde{F}_1^{[m]}(\theta)\sin(\theta)\right]_{\theta=\theta_0} \\
&= \left[m\tilde{F}_1^{[m]}(\theta)\cos(\theta) - \tilde{F}_2^{[m]}(\theta)\sin(\theta) + m\tilde{F}_2^{[m]}(\theta)\sin(\theta) - \tilde{F}_1^{[m]}(\theta)\cos(\theta)\right]_{\theta=\theta_0} \\
&= \tilde{F}_1^{[m]}(\theta_0)\cos(\theta_0)(m-1) + \tilde{F}_2^{[m]}(\theta_0)\sin(\theta_0)(m-1) \\
&= (m-1)\left(\tilde{F}_1^{[m]}(\theta_0)\cos(\theta_0) + \tilde{F}_2^{[m]}(\theta_0)\sin(\theta_0)\right).
\end{aligned}$$

We can identify the two real eigenvalues $\lambda_1 := \tilde{F}_1^{[m]}(\theta_0)\cos(\theta_0) + \tilde{F}_2^{[m]}(\theta_0)\sin(\theta_0)$ and $\lambda_2 := (m-1)\lambda_1$. λ_1 can be calculated explicitly by

$$\lambda_1 = \cos(\beta + \theta_0 m)\cos(\theta_0) + \sin(\beta + \theta_0 m)\sin(\theta_0) = \cos(\beta + \theta_0 m - \theta_0).$$

Here we used again a trigonometric addition formula. Since $\lambda_1 \neq 0$ (see above) and $m - 1 > 0$, the linearization $\mathcal{J}_G(a_1)$ has two real non-zero eigenvalues with the same sign. Thus a_1 is a stable or unstable node of (4.5), cf. [9, Chapter 2.10, Theorem 4]. More precisely, a_1 is stable if and only if $\lambda_1 < 0$ and unstable if and only if $\lambda_1 > 0$, cf. [9, Chapter 1.5]. Assume w.l.o.g $\lambda_1 < 0$. The other case is analogous. Additionally, it can be calculated that $\Psi_2(0,\theta_0) = \tilde{F}_2^{[m+1]}(\theta_0)$.

By using Definition 4.1, we find $r \in (0, \min\{r_0, \rho^\star\})$ and $\delta > 0$ such that for all $\xi \in (-r, r) \times (\theta_0 - \delta, \theta_0 + \delta) \cap \Omega_1$ we have $|\Phi(\tau,\xi)| \to \theta_0$ for $\tau \to \infty$. Fix such a ξ. We claim, that the orbit $\Gamma_+(\xi)$ stays in Ω_1. Suppose that this is not the case. Then we find a $\tau_0 \in \mathbb{R}$ such that $\Phi(\tau_0, \xi) \in \{0\} \times \mathbb{R}$, i.e. $\rho(\tau_0) = 0$ on $\Gamma_+(\xi)$. But due to (4.5) the orbit through a point in $\{0\} \times \mathbb{R}$ stays on $\{0\} \times \mathbb{R}$. But $\xi \notin \{0\} \times \mathbb{R}$. Thus $\Gamma_+(\xi) \subset \Omega_1$.

Additionally, on Ω_1 we can relate the orbits of (4.1) with those of (4.5), cf. the proof of Proposition 4.20. So for $x_0 \in \{x \in \mathbb{R}^2 : |x-a| < r, |\arg(x-a) - \theta_0| < \delta\}$ and the orbit Γ through x_0 we find $\xi \in (-r, r) \times (\theta_0 - \delta, \theta_0 + \delta) \cap \Omega_1$ and the orbit Γ_1 through ξ of (4.5) which corresponds to Γ. So we have $\Gamma_+(\xi) \subset \Omega_1$ and Γ_1 tends to a_1. Hence the corresponding orbit Γ tends to a in the definite direction θ_0. This relation also implies that Γ tends to a for $t \to \infty$ if and only if $\lambda_1 < 0$ and for $t \to -\infty$ if and only if $\lambda_1 > 0$. Finally it remains to show that we have exactly $2m-2$ such definite directions. We have already shown that every $\theta_0 \in \mathcal{E}(F, m)$ is a definite direction in the sense described above, i.e. there are at least $2m-2 > 0$ definite directions. Now let Γ be an arbitrary orbit tending to a. Since we have $H \not\equiv 0$ (see above), we can apply Proposition 4.20. So if Γ was a spiral, a would be a focus. But we have already found orbits tending to a in certain definite directions. This is a contradiction to Definition 4.1. Hence, by Proposition 4.20, Γ tends to a in a definite direction $\tilde{\theta_0}$ and satisfies $\tilde{H}(\tilde{\theta_0}) = 0$. Thus $\tilde{\theta_0} \in \mathcal{E}(F, m)$ and there are no more definite direction than these in $\mathcal{E}(F, m)$. We summarize that every orbit tending to a does so in a definite direction $\theta_0 \in \mathcal{E}(F, m)$.

$\square$

Lemma 4.23

Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F = F_1 + \mathrm{i}F_2 \in \mathcal{O}(\Omega)$, $F \not\equiv 0$ and $a \in \Omega$ an equilibrium of (4.1) with order $m \in \mathbb{N} \setminus \{1\}$. Let $r > 0$ be sufficiently small and S an elliptic sector of (4.1) in a with respect to $\Gamma := \partial\mathcal{B}_r(0)$ with two characteristic orbits Γ_1, Γ_2 and two intersection points $p_1, p_2 \in \Gamma$. Then there are two directions $\theta_1, \theta_2 \in \mathcal{E}(F, m)$ such that Γ_i tends to a in the definite direction θ_i for $i \in \{1, 2\}$ and $\theta_1 \neq \theta_2$.

Proof

Assume w.l.o.g. $a = 0$. By applying Definition 4.13 we have

$$\partial S = \mathrm{Int}(\Gamma) \cap (\{a\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma(p_1, p_2)) \,.$$

This implies that Γ_1 and Γ_2 connect a with Γ, i.e., by Proposition 4.22, these characteristic orbits tend to a in some definite directions $\theta_1, \theta_2 \in \mathcal{E}(F, m)$. For $i \in \{1, 2\}$ set $\lambda_i := \cos(\arg(F^{(m)}(a)) + \theta_i(m-1))$. Suppose $\theta_1 = \theta_2$. Then $\lambda_1 = \lambda_2$. By Definition 4.14, there exists a point $E \in \Gamma(p_1, p_2)$ such that $w_+(\Gamma(E)) = w_-(\Gamma(E)) = \{a\}$. But then Proposition 4.22 implies $\lambda_1 < 0$ as well as $\lambda_1 > 0$, which is impossible. Thus we have $\theta_1 \neq \theta_2$.

$\square$

Theorem 4.24 (Existence of elliptic decomposition)
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F = F_1 + \mathrm{i}F_2 \in \mathcal{O}(\Omega)$, $F \not\equiv 0$ and $a \in \Omega$ an equilibrium of (4.1) with order $m \in \mathbb{N} \setminus \{1\}$. Then a is not a node. Furthermore, the system (4.1) has a minimal finite sectorial decomposition of order $d := 2m - 2$ with only elliptic sectors and there are neither hyperbolic nor parabolic sectors in a. Additionally, the characteristic orbits of each sector in this minimal FSD tend to a in adjacent definite directions given by $\mathcal{E}(F, m)$, i.e. the sectors have pairwise empty intersection up to the characteristic orbits.

Proof
Assume w.l.o.g. $a = 0$. Since $F \not\equiv 0$, we can find an $r_0 > 0$ with $\mathcal{B}_{r_0}(0) \cap F^{-1}(\{0\}) = \{a\}$. By applying Proposition 4.22 the equilibrium has exactly d definite directions $\theta_1, \ldots, \theta_d$, given by $\mathcal{E}(F, m)$, and every orbit tending to a does so in a definite direction $\theta \in \mathcal{E}(F, m)$. Assume that the angles in $\mathcal{E}(F, m)$ are ordered cyclic and counterclockwise with respect to a. Additionally, for every $i \in \{1, \ldots, d\}$ we have $r_i > 0$ and $\delta_i > 0$ such that for every $x_0 \in \{x \in \mathbb{R}^2 : |x - a| < r_i, |\arg(x - a) - \theta_i| < \delta_i\} \subset \Omega$ the orbit though x_0 tends to a in the definite direction θ_i. It is geometrically clear that a cannot be a center or focus, cf. Definition 4.1. Fix an $i \in \{1, \ldots, d\}$. Set $\beta := \arg(F^{(m)}(0))$ and $\lambda_i := \cos(\beta + \theta_i m - \theta_i)$. Proposition 4.22 characterizes whether the orbits near the ray with angle $\theta_i \in \mathcal{E}(F, m)$ tend to a for positive or negative time. The equilibrium a is reached for $t \to \infty$ if and only if $\lambda_i < 0$ and for $t \to -\infty$ if and only if $\lambda_i > 0$. We calculate

$$\lambda_i = \cos(\beta + \theta_i(m-1)) = \cos\left(\beta + (m-1)\frac{i\pi - \beta}{m-1}\right) = \cos(i\pi) = \begin{cases} 1 & \text{if } i \text{ even} \\ -1 & \text{if } i \text{ odd} \end{cases}$$

and conclude that there is a pair of orbits, one reaching a with positive time and one with negative time. Note that $d \geq 2$, i.e. there are always at least two definite directions. Hence the equilibrium cannot be a node. In addition, the directions are alternating, i.e. every ray has no adjacent ray with the same direction.
Set $r := \min\{r_i : 0 \leq i \leq d\} > 0$ and $\Gamma := \partial\mathcal{B}_r(0)$. Let S be a hyperbolic sector of (4.1) in a with respect to Γ and with two characteristic orbits Γ_1, Γ_2 and two intersection points $p_1, p_2 \in \Gamma$. By Definition 4.13, we know that Γ_1 and Γ_2 connect a with Γ, i.e., by Proposition 4.22, these characteristic orbits tend to a in some definite directions given by $\mathcal{E}(F, m)$. In addition, there exists a point x_0 in S such that the orbit though x_0 tends to a. Note that $\delta > 0$ in Proposition 4.22. But this contradicts Definition 4.14, i.e. S cannot

be hyperbolic. It follows that there are neither hyperbolic nor parabolic sectors in every minimal FSD, cf. Remark 4.17.
For the next step of the proof let S be an elliptic sector of (4.1) in a with two characteristic orbits Γ_1, Γ_2 tending to a in the definite directions $\eta_1, \eta_2 \in \mathcal{E}(F,m)$, $\eta_1 \neq \eta_2$, and two intersection points $p_1, p_2 \in \Gamma$, cf. Lemma 4.23. Assume w.l.o.g. $\eta_1 < \eta_2$. We claim now the following: All other directions $\theta \in \mathcal{E}(F,m) \setminus \{\eta_1, \eta_2\}$ satisfy either $\theta < \eta_1$, or $\eta_2 < \theta$, i.e. there are no definite directions between η_1 and η_2 and these two directions are adjacent. Suppose, this is not the case and such a θ exists. Then there is an $i \in \{1, \dots, d\}$ such that $\theta = \theta_i$ with the corresponding „attracting region“ defined by $r_i > 0$ and $\delta_i > 0$. We have $\eta_1 < \theta_i < \eta_2$. Assume w.l.o.g. that S has clockwise direction. The other case can be treated equally. Then there exits an $E \in \Gamma(p_1, p_2)$ such that for all $y_1 \in \Lambda_1 := \Gamma(p_1, E) \setminus \{E\}$ and $y_2 \in \Lambda_2 := \Gamma(E, p_2) \setminus \{E\}$ it holds:

(i) $\langle F(y_1), \nu(y_1) \rangle < 0$, i.e. the vector field points inwards.

(ii) $\langle F(y_2), \nu(y_2) \rangle > 0$, i.e. the vector field points outwards.

Assume w.l.o.g that $E \in \Gamma(p_1, re^{\mathrm{i}\theta_i})$ and set $\Theta := \theta_i + \frac{\delta_i}{2}$. The case $E \in \Gamma(re^{\mathrm{i}\theta_i}, p_2)$ is analogous, here we would have $\Theta := \theta_i - \frac{\delta_i}{2}$. The vector field points outwards at $re^{\mathrm{i}\Theta} \in \Lambda_2$, i.e. $\lambda_i = 1$ and $w_-(re^{\mathrm{i}\Theta}) = a$. The orbit Γ_2 tends to a for negative time as well. Note that $p_2 \in \Lambda_2$. Now there are two possibilities: Either $\eta_2 = \theta_{i+1}$ or there exists at least one more definite direction between θ_i and η_2. In the first case we conclude directly $\lambda_i = \lambda_{i+1} = 1$, which is impossible. And in the second case there exists $re^{\mathrm{i}\theta_{i+1}} \in \Lambda_2 \setminus \{p_2\}$. At this point the vector field points outwards as well and we have again $\lambda_i = \lambda_{i+1} = 1$. This is a contradiction. Hence every elliptic sector consists of exactly two definite directions.
Denote by $E(F,a)$ and $H(F,a)$ the number of elliptic and hyperbolic sectors in a with respect to Γ. In the next step we want to apply the Poincare-Bendixson Index Theorem, cf. [6, Appendix, p. 511] and [16, Theorem 2.2][4],

$$\operatorname{ind}(F,a) = 1 + \frac{E(F,a) - H(F,a)}{2}.$$

By Proposition 4.19, we have $\operatorname{ind}(F,a) = \operatorname{ind}(F,\Gamma) = m$. Thus we get

$$E(F,a) = H(F,a) + 2m - 2 = H(F,a) + d.$$

[4]The original proof by Bendixson can be found in [17].

Since we do not have any hyperbolic sectors, we conclude $E(F, a) = d$. Overall, we can conclude that there must be d elliptic sectors, each between exactly two definite directions. But we also have only d definite directions. Hence the only areas in $\overline{\mathrm{Int}(\Gamma)}$ where other topological structures than elliptic sectors can occur are the „attracting regions" near the definite directions. If that were the case, we would have two elliptic sectors, each having an characteristic orbit with a common definite direction θ_i, but the two characteristic orbits near this definite direction are not equal. This means we would have a different topological structure between two elliptic sectors in a subset of such an „attracting region". But in this region we have the structure of a parabolic sector (attracting for $\lambda_i = -1$ and repelling for $\lambda_i = 1$).

As described in Remark 4.17, we can add this parabolic sector to one of the adjacent elliptic ones. The (in general not unique) common characteristic orbit of these two elliptic sectors is now one orbit lying in the parabolic sector. Finally, there is no more space left in $\overline{\mathrm{Int}(\Gamma)}$. Hence the FSD with the d elliptic sectors must be already minimal and has in fact order d.

$\square$

Corollary 4.25

Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F \in \mathcal{O}(\Omega)$ and $F \not\equiv 0$. Then all nodes, centers and foci have index 1.

Proof

Let $a \in \Omega$ be a node, center or focus of (4.1). Then a is an equilibrium with order $m \in \mathbb{N}$. If $m > 1$, we can apply Theorem 4.24 and Definition 4.15. Thus a is not a node, focus or center, a contradiction. Hence $m = 1$. The assertion follows with Proposition 4.19.

$\square$

Remark 4.26

a) In the proof of Theorem 4.24 we used the Poincare-Bendixson Index Formula to proof the existence of elliptic sectors. I conjecture that this is also possible „by hand": A posteriori the „attracting regions" cover the punctured circle $\overline{\mathrm{Int}(\Gamma)} \setminus \{a\}$, i.e. the radii of these regions are big enough. Maybe this can be proven without using the Index Formula. It is probably possible to derive a contradiction by supposing the following: There exists an $i \in \{i, \ldots, d\}$ such that $\theta_i + \delta_i < \theta_{i+1} - \delta_{i+1}$. Geometrically,

this would generate a set between the „attracting regions“ similar to a hyperbolic sector, which is impossible.
As already mentioned in Remark 4.16, this could be possible with the ideas in [6, Chapter VIII]. In particular, at the beginning of [6, §19] the author summarizes the possible local structures of an isolated equilibrium.

b) Some of the results in [6, Chapter VIII] are used in the proof of the Poincare-Bendixson Index Formula, cf. [6, Appendix, pp. 511]. Moreover, the properties in the definitions of sectors at the beginning of this section (cf. [1, Chapter 1.5]) are basically the results of [6, Chapter VIII]. Hence these definitions are formulated in a way such that the types of sectors in Definition 4.14 coincide with those in [6, Chapter VIII] and the Poincare-Bendixson Index Formula is indeed applicable for the sectors defined in Definition 4.14.

c) In the proof of Theorem 4.24 we used Remark 4.17 to conclude the non-existence of parabolic sectors in every minimal FSD in a. In addition we have seen that the characteristic orbits are not uniquely determined. Later we will see (under certain conditions) that there exists indeed exactly one characteristic „delimiting“ orbit with maximum interval of existence different from $\mathbb{R}$, called separatrix.

Example 4.27
In this example we want to visualize our results in this section. Define the polynomial $F : \mathbb{C} \to \mathbb{C}$ by $F(x) := (2 + 3\mathrm{i} - x)^4$, $F \in \mathcal{O}(\mathbb{C})$. This is a polynomial of degree 4 having a zero in $a := 2 + 3\mathrm{i}$ of order $m := 4$. By Theorem 4.24 there exists a minimal FSD of order $d := 2m - 2 = 8 - 2 = 6$. It is easy to calculate the definite directions

$$\mathcal{E}(F, 4) = \left\{ \frac{\ell\pi}{3} : 1 \leq \ell \leq 6 \right\}.$$

Note that $F^(m)(2 + \mathrm{i}) > 0$ in this case. Between two such adjacent directions there exists exactly one elliptic sector. Set $\theta_\ell := \frac{\ell\pi}{3}$. Moreover, we can calculate the directions of the rays and thus of all orbits. We have

$$\cos(\arg(F^{(m)}(a)) + \theta_\ell(m-1)) = \cos(\pi\ell) = \begin{cases} 1 & \text{if } i \text{ even} \\ -1 & \text{if } i \text{ odd} \end{cases}.$$

Thus the elliptic sector lying above the ray with angle zero is an sector with clockwise direction. After each ray the direction of the sector changes.

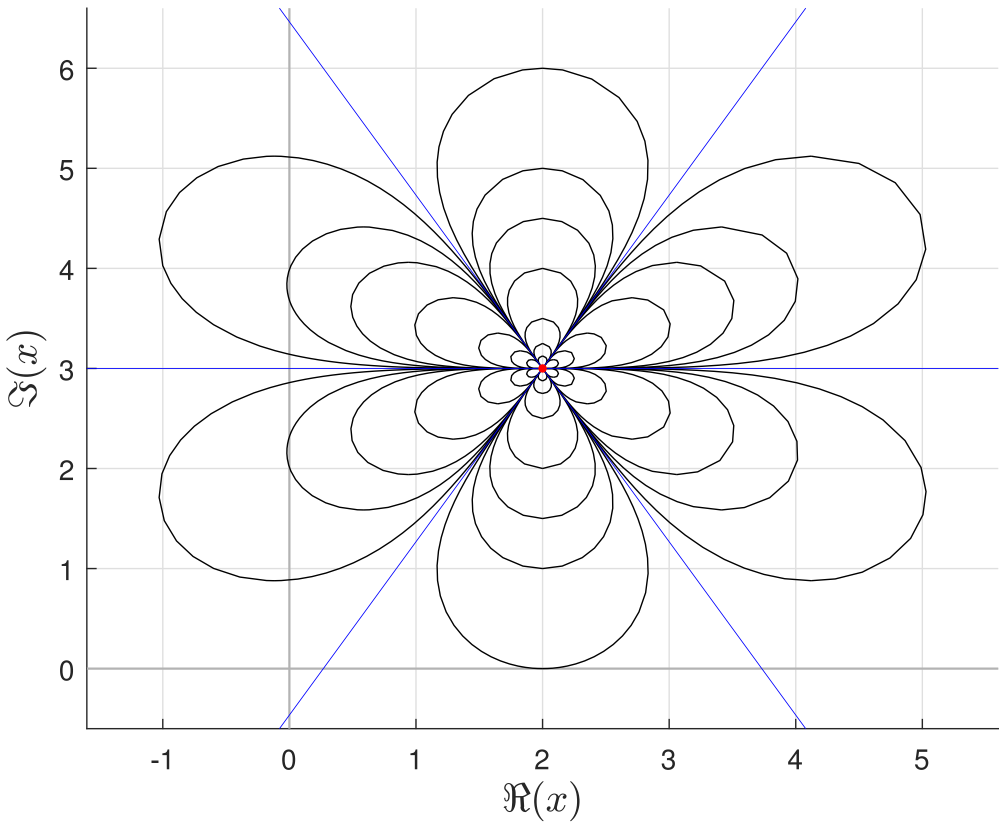

Figure 4.3: Phase portrait of system (4.1) with $F(x) = (2 + 3\mathrm{i} - x)^4$

In the phase portrait of (4.1) (cf. Figure 4.3) we see the chracteristic orbits (blue) and the equilibrium a (red). In this example the characteristic orbits are exactly the rays. We verify that there are indeed 6 elliptic sectors. The radius of the FSD can be chosen arbitrarily large in this case.

Example 4.28

In this example we want to look at a more complicated example. Define $F : \mathbb{C} \to \mathbb{C}$ by $F(x) := x^5 e^x$, $F \in \mathcal{O}(\mathbb{C})$. The unique zero $a = 0$ has order $m := 5$. By Theorem 4.24 there exists a minimal FSD of order $d := 2m - 2 = 10 - 2 = 8$. We calculate the Taylor series of F by

$$F(x) = x^5 \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{k+5}}{k!} = \sum_{k=5}^{\infty} \frac{1}{(k-5)!} x^k$$

and conclude the definite directions

$$\mathcal{E}(F,5) = \left\{ \frac{\ell\pi}{4} : 1 \leq \ell \leq 8 \right\}.$$

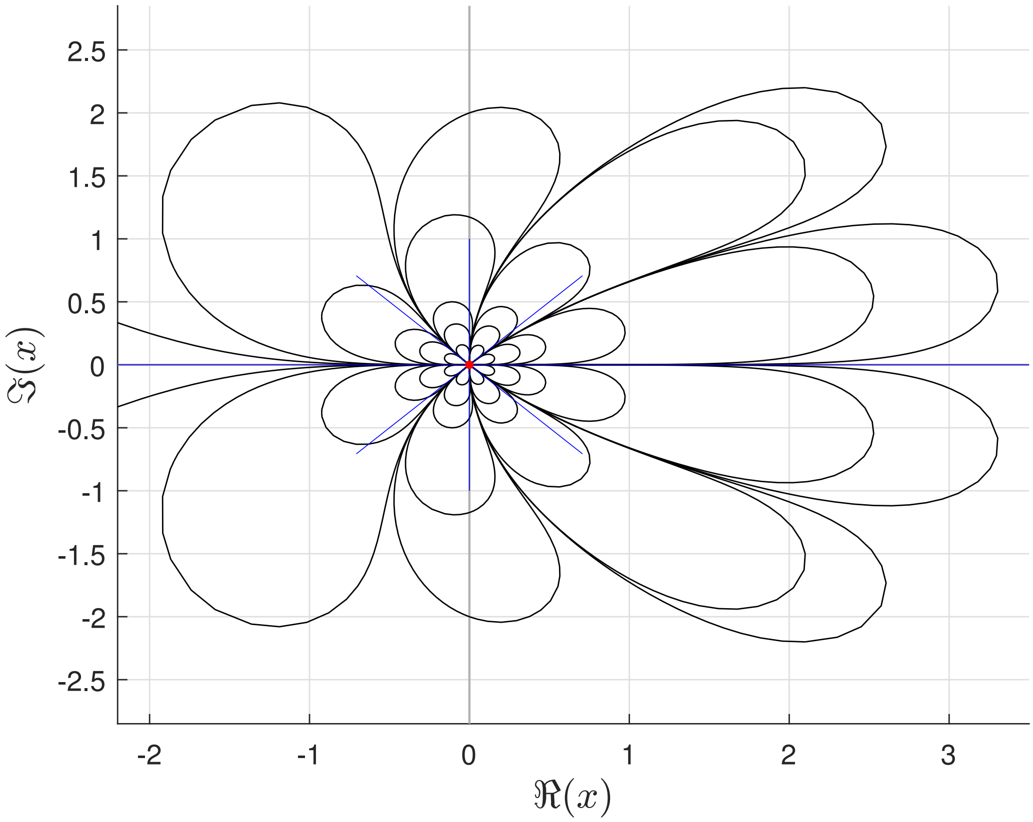

Figure 4.4: Phase portrait of system (4.1) with $F(x) = x^5 e^x$

Between two such adjacent directions there exists now exactly one elliptic sector. In this example only the characteristic orbits lying on the $\Re$-axis are rays. As we can see in the phase portrait of (4.1) (cf. Figure 4.4), the exponential function „tugs" all orbits (up to these on the $\Re$-axis) to the right. We verify that there are indeed 8 elliptic sectors.
If the radius of the FSD for this example is not sufficiently small, we have a different topological structure near the ray $\theta = \pi$. It can be shown that there exists a parabolic sector near this ray and all orbits in this sector tend to the $\Re$-axis. In addition, here we can use the argument in Remark 4.17 only locally. Globally it is possible that there is a „parabolic region" between two elliptic sectors. This geometrical structure will be discussed in section 5.2.

4.4 The non-existence of limit cycles

In this section the main aim is to prove our second fundamental result: Holomorphic flows do not have limit cycles. The ideas are based on [2, Chapter 3].

Definition 4.29
Let $\Omega \subset \mathbb{C}$ be an open domain and $F \in \mathcal{O}(\Omega)$. A periodic orbit $\Gamma \subset \Omega$ of (4.1) is called an *limit cycle* for (4.1) if there exists an $r > 0$ such that for all $x_0 \in \{x \in \Omega : \operatorname{dist}(\Gamma, x) < r\} \setminus \Gamma$ the orbit $\Gamma(x_0)$ is not periodic.

Definition 4.30
Let $\Omega \subset \mathbb{C}$, $O \subset \Omega$ be open domains and $F \in \mathcal{O}(\Omega)$. Then F is *complete* on O if for every $x_0 \in \Omega$ we have $I(x_0) = \mathbb{R}$, i.e. the solution through x_0 exists globally.

Proposition 4.31
Let $\Omega \subset \mathbb{C}$ be an open domain, $F = F_1 + \mathrm{i}F_2 \in \mathcal{O}(\Omega)$ a complete vector field on $O \subset \Omega$ and $\tau \in \mathbb{R}$. Define $G : O \to O$ by $G(z) := \Phi(\tau, z)$. Then $G \in \mathcal{O}(O)$.

Proof
If $\tau = 0$, G is the identity and thus holomorphic. Assume w.l.o.g. $\tau > 0$. The case $\tau < 0$ is analogous. Set $A := \Re(\Phi)$ and $B := \Im(\Phi)$. Then A and B are real analytic in O with fixed time, cf. [9, Chapter 2.3, Remark 1]. For all $z = (x, y) \in O$ and $t > 0$ we have

$$\Phi(t, x, y) = (x, y)^T + \int_0^t F(\Phi(s, x, y))\,\mathrm{d}s.$$

We calculate the derivative (the partial derivatives of A, B and Φ are with respect to z)

$$\begin{aligned}
\mathcal{J}_{\Phi(t,\cdot)}(z) &= \begin{pmatrix} \partial_1 A(t,z) & \partial_2 A(t,z) \\ \partial_1 B(t,z) & \partial_2 B(t,z) \end{pmatrix} \\
&= \begin{pmatrix} 1 + \int_0^t \langle \nabla F_1(\Phi(s,z)), \partial_1 \Phi(s,z) \rangle \,\mathrm{d}s & \int_0^t \langle \nabla F_1(\Phi(s,z)), \partial_2 \Phi(s,z) \rangle \,\mathrm{d}s \\ \int_0^t \langle \nabla F_2(\Phi(s,z)), \partial_1 \Phi(s,z) \rangle \,\mathrm{d}s & 1 + \int_0^t \langle \nabla F_2(\Phi(s,z)), \partial_2 \Phi(s,z) \rangle \,\mathrm{d}s \end{pmatrix}
\end{aligned}$$

and conclude by applying the notations $\overline{z} = (s, z)$, $z_t = (t, z)$

$$(\partial_1 A - \partial_2 B)(z_t) = \int_0^t \partial_1 F_1(\Phi(\overline{z}))\,(\partial_1 A(\overline{z}) - \partial_2 B(\overline{z})) + \partial_2 F_1(\Phi(\overline{z}))\,(\partial_2 A(\overline{z}) - \partial_1 B(\overline{z}))\,\mathrm{d}s.$$

Furthermore, by the Cauchy-Riemann equations

$$(\partial_1 B + \partial_2 A)(z_t) = \int_0^t \partial_1 F_1(\Phi(\overline{z}))\,(\partial_2 A(\overline{z}) - \partial_1 B(\overline{z})) - \partial_2 F_1(\Phi(\overline{z}))\,(\partial_1 A(\overline{z}) - \partial_2 B(\overline{z}))\,\mathrm{d}s.$$

Now fix a $z_0 \in O$ and define $C \in \mathcal{C}^1((0,\infty), \mathbb{R}^2)$ by

$$C(t) := \begin{pmatrix} \partial_1 A(t, z_0) - \partial_2 B(t, z_0) \\ \partial_2 A(t, z_0) + \partial_1 B(t, z_0) \end{pmatrix}.$$

We want to prove $C \equiv 0$. Set $Q := \mathcal{J}_F(z_0) \in \mathbb{R}^{2\times 2}$. Because of the above calculations, C satisfies the equation

$$C(t) = \int_0^t QC(s)\,\mathrm{d}s \quad \forall t \in (0,\infty).$$

Hence C is a solution to the linear ODE $\tilde{z}' = Q\tilde{z}$, $\tilde{z} \in \mathbb{R}^2$. Furthermore, since Φ defines the flow of (4.1), we have $\Phi(0,\cdot) = \mathrm{id}$ on O and thus

$$C(0) = \begin{pmatrix} \partial_1 A(0, z_0) - \partial_2 B(0, z_0) \\ \partial_2 A(t, z_0) + \partial_1 B(t, z_0) \end{pmatrix} = \begin{pmatrix} 1 - 1 \\ 1 - 1 \end{pmatrix} = 0.$$

Together with the initial condition $\tilde{z}(0) = 0$, by the Picard-Lindelöf theorem, there exists a unique solution of the linear ODE through z_0. Hence C is this unique solution. But obviously, the constant zero-map is also a solution to this linear ODE. So we must have $C \equiv 0$. In particular, this implies for $t = \tau$

$$\begin{pmatrix} \partial_1 A(\tau, z_0) \\ \partial_2 A(\tau, z_0) \end{pmatrix} = \begin{pmatrix} \partial_2 B(\tau, z_0) \\ -\partial_1 B(\tau, z_0) \end{pmatrix}$$

Thus $G = \Phi(\tau,\cdot) = A(\tau,\cdot) + \mathrm{i}B(\tau,\cdot)$ satisfies the Cauchy-Riemann equations in z_0. Since $z_0 \in O$ is arbitrary, we conclude $G \in \mathcal{O}(O)$.

□

Lemma 4.32
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain and $F \in \mathcal{O}(\Omega)$, $F \not\equiv 0$. Let $\Gamma \subset \Omega$ be a periodic orbit of (4.1). Then there is exactly one equilibrium a in $\mathrm{Int}(\Gamma)$ and a is either a node, a focus, or a center. In addition, every periodic orbit $\tilde{\Gamma} \subset \mathrm{Int}(\Gamma)$ satisfies $a \in \mathrm{Int}(\tilde{\Gamma})$.

Proof
Assume that Γ is parameterized as solution. Theorem 4.10 implies that there is at least one equilibrium a in $\mathrm{Int}(\Gamma)$ and

$$\sum_{\substack{b \in \mathrm{Int}(\Gamma) \\ F(b)=0}} \mathrm{ind}(F, b) = 1.$$

By Proposition 4.19 the index of every equilibrium in $\mathrm{Int}(\Gamma)$ is at least 1. Hence a is the only equilibrium in $\mathrm{Int}(\Gamma)$ and $\mathrm{ind}(F, a) = 1$, i.e. the order of a is 1. Theorem 4.3 implies that a must be either a node, a focus, or a center.
Suppose there exists a periodic orbit $\tilde{\Gamma} \subset \mathrm{Int}(\Gamma)$ with $a \notin \mathrm{Int}(\tilde{\Gamma})$. Then again by Proposition 4.19, there exists at least one equilibrium $\tilde{a} \in \mathrm{Int}(\tilde{\Gamma}) \subset \mathrm{Int}(\Gamma)$. But $\tilde{a} \neq a$ and thus there are at least two equilibria in $\mathrm{Int}(\Gamma)$, which is a contradiction to the above argument.
$\square$

Lemma 4.33
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F \in \mathcal{O}(\Omega)$, $F \not\equiv 0$ and $a \in \Omega$ be an equilibrium of (4.1). Assume there are two points $x_1, x_2 \in \Omega \setminus \{a\}$ such that $w_+(\Gamma(x_1)) = w_-(\Gamma(x_2)) = \{a\}$. Then $\mathrm{ind}(F, a) \geq 2$.

Proof
Suppose $\mathrm{ind}(F, a) < 2$, i.e. $\mathrm{ind}(F, a) = 1$. Then by Proposition 4.19 and Theorem 4.3, a is a node, focus or center. A center is geometrically impossible. Hence, a must be asymptotically stable in exactly one time direction, say w.l.o.g. $t \to \infty$. Note that the real parts of the non-zero eigenvalues of $\mathcal{J}_F(a)$ have the same sign. But we also have $w_-(\Gamma(x_2)) = \{a\}$. With this property there exists a sequence $(y_k)_{k \in \mathbb{N}} \subset \Gamma_-(x_2)$ with $y_k \to a$ for $k \to \infty$ such that for all $k \in \mathbb{N}$ sufficiently large there is a $\varepsilon > 0$ such that $|a - y_k| < |a - \Phi(\varepsilon, y_k)|$. This is a contradiction to the stability of a. Hence a cannot be a focus or node and we must have $\mathrm{ind}(F, a) \geq 2$.

$\square$

Theorem 4.34 (Non-existence of limit cycles)
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain and $F \in \mathcal{O}(\Omega)$. Then the system (4.1) does not have a limit cycle in Ω.

Proof
This idea of the proof is based on [2]. The omitted steps are performed by me. In addition, the structure of the proof is modified.
If $F \equiv 0$, there is nothing to show, since every orbit consists only of one point. Hence assume w.l.o.g $F \not\equiv 0$ and suppose there is at least one limit cycle. By the theorem of Dulac, cf. [9, Chapter 3.3], there exist at most a finite number of limit cycles.[5] Therefore there exists a limit cycle Γ with no other limit cycles in $\mathrm{Int}(\Gamma)$. By Lemma 4.32, there is exactly one equilibrium a in $\mathrm{Int}(\Gamma)$ and this equilibrium is either a node, a focus, or a center. We have $\mathrm{ind}(F, a) = 1$ and $F'(a) \neq 0$.
Since the set $\mathrm{Int}(\Gamma)$ is clearly invariant and $\overline{\mathrm{Int}(\Gamma)} \subset \Omega$, every orbit in $\mathrm{Int}(\Gamma)$ exists globally. Let $\tau \in \mathbb{R}$ be arbitrary and define $G : \mathrm{Int}(\Gamma) \to \mathrm{Int}(\Gamma)$ by $G(z) := \Phi(\tau, z)$. Then, by Proposition 4.31, $G \in \mathcal{O}(\mathrm{Int}(\Gamma))$. In addition, G is continuous on $\overline{\mathrm{Int}(\Gamma)}$. Set $D := \mathcal{B}_1(0) \subset \mathbb{C}$. By the Riemann mapping theorem, there exists a conformal map $\varphi : \mathrm{Int}(\Gamma) \to D$ such that $\varphi(a) = 0$. Define $h : D \to D$ by $h := \varphi \circ G \circ \varphi^{-1}$. Since the orbit through a is constant for all times, we conclude

$$h(0) = \varphi(G(\varphi^{-1}(0))) = \varphi(G(a)) = \varphi(a) = 0.$$

Furthermore, we have $h \in \mathcal{O}(D)$. Hence we can apply the Schwarz lemma to conclude $|h(z)| \leq |z|$ for all $z \in D$. Let $z_1 \in (0, 1) \subset D \setminus \{0\}$ be arbitrary and set $z_2 := \varphi^{-1}(z_1) \in \mathrm{Int}(\Gamma) \setminus \{a\}$. Define $\alpha_n := h^n(z_1)$ and $\beta_n := G^n(z_2)$, $n \in \mathbb{N}$.
By a short inductive argument, it can be shown that $h^n = \varphi \circ G^n \circ \varphi^{-1}$ for all $n \in \mathbb{N}$ and $(\alpha_n)_{n\in\mathbb{N}} \subset \overline{\mathcal{B}_{z_1}(0)} \subset D$, which is compact. By the Bolzano-Weierstrass theorem, there exists a subsequence $(\alpha_{n_j})_{j\in\mathbb{N}}$ and $z_3 \in \overline{\mathcal{B}_{z_1}(0)} \subset D$ such that $\alpha_{n_j} \to z_3$ for $j \to \infty$. Note that convergent sequences in a compact set have their limit in this compact set. By continuity of φ and φ^{-1}, we get

$$\lim_{j\to\infty} \beta_{n_j} = \varphi^{-1}\left(\lim_{j\to\infty} \left(\varphi \circ G^{n_j} \circ \varphi^{-1}\right)(\varphi(z_2))\right) = \varphi^{-1}\left(\lim_{j\to\infty} \underbrace{h^{n_j}(z_1)}_{=\alpha_{n_j}}\right) = \varphi^{-1}(z_3).$$

[5]The original proof by Dulac can be found in [18].

In addition, G defines the flow of a dynamical system. Hence we have

$$\varphi^{-1}(z_3) = \lim_{j\to\infty} \beta_j = \lim_{j\to\infty} G^{n_j}(z_2) = \lim_{j\to\infty} \Phi(\tau n_j, z_2).$$

Note that z_3 depends on τ and z_1 in this argument. We conclude now the following: For all $z_1 \in (0,1)$ and $\tau > 0$ ($\tau < 0$) there exists a $z_3 \in \overline{\mathcal{B}_{z_1}(0)}$ such that $\varphi^{-1}(z_3) \in w_{+(-)}(\Gamma(\varphi^{-1}(z_1)))$. In the next step, using this statement, we want to derive a contradiction.

First, suppose a is a center. Since Γ is a limit cycle, there exists an $r > 0$ such that for all $x_0 \in \{x \in \Omega : \operatorname{dist}(\Gamma, x) < r\}$ the orbit $\Gamma(x_0)$ is not periodic. Since φ^{-1} is uniformly continuous in particular, there exists a continuation of φ^{-1} on $\overline{D}$ such that $\varphi^{-1}(\partial D) = \Gamma$. This ensures the existence of a $z_1 \in (0,1)$ near $1 \in \partial D$ such that $z_2 := \varphi^{-1}(z_1) \in \{x \in \Omega : \operatorname{dist}(\Gamma, x) < r\}$ and $\Gamma(z_2) \subset \operatorname{Int}(\Gamma)$ is not a periodic orbit. We have $I(z_2) = \mathbb{R}$. Furthermore, there exists at least one periodic orbit around a and thus $\Gamma(z_2)$ cannot tend to a (orbits cannot cross each other). In addition, there are no other equilibria or limit cycles in $\operatorname{Int}(\Gamma)$. Hence, by the Poincare-Bendixson theorem, we must have $w_+(\Gamma(z_2)) = w_-(\Gamma(z_2)) = \Gamma$, i.e. the orbit converges to the limit cycle for both, $t \to \infty$ and $t \to -\infty$. But by the above statement with $\tau := 1$ there exists a $z_3 \in \overline{\mathcal{B}_{z_1}(0)}$ such that $\varphi^{-1}(z_3) \in w_+(\Gamma(z_2)) = \Gamma$, which implies $z_3 \in \overline{\mathcal{B}_{z_1}(0)} \cap \partial D = \emptyset$. Note that $z_1 < 1$. This is a contradiction. Hence a cannot be a center.

Second, suppose a is a focus or node. Assume w.l.o.g that a is asymptotically stable and set $\tau := -1$. The unstable case can be treated similarly, here we can set $\tau = 1$. By Definition 4.1, there exits a $\delta > 0$ such that for all $x \in \mathcal{B}_\delta(a)$ we have $w_+(x) = \{a\}$. Assume δ so small that $\overline{\mathcal{B}_\delta(a)} \subset \operatorname{Int}(\Gamma)$. Then, by the topological characterization of continuity of φ^{-1}, the set $B := \varphi(\mathcal{B}_\delta(a))$ is an open set containing 0 and satisfying $\overline{B} \subset D$. Hence there exists a $z_1 \in (0,1) \cap B \neq \emptyset$ satisfying $\overline{\mathcal{B}_{z_1}(0)} \subset B$ and $z_2 := \varphi^{-1}(z_1) \in \mathcal{B}_\delta(a) \subset \operatorname{Int}(\Gamma)$. We have $I(z_2) = \mathbb{R}$ and $w_+(z_2) = \{a\}$.

By the above statement, with this τ and z_1, there exists a $z_3 \in \overline{\mathcal{B}_{z_1}(0)}$ such that $\varphi^{-1}(z_3) \in w_-(\Gamma(z_2))$. Hence $z_3 \in B$ and $\varphi^{-1}(z_3) \in \varphi^{-1}(B) = \mathcal{B}_\delta(a)$. This implies $\operatorname{dist}(\Gamma, \varphi^{-1}(z_3)) > 0$, i.e. $\varphi^{-1}(z_3) \notin \Gamma$. But there are no other limit cycles or equilibria (up to a) in $\operatorname{Int}(\Gamma)$. Hence, again by a Corollary on the Poincare-Bendixson theorem, [7, Satz 9.2.4], we must have $a = \varphi^{-1}(z_3)$ and $w_+(\Gamma(z_2)) = w_-(\Gamma(z_2)) = \{a\}$ or there is another homoclinic orbit $\Lambda \subset w_-(\Gamma(z_2))$ satisfying $w_+(\Lambda) = w_-(\Lambda) = \{a\}$. Note that $a \notin \Gamma(x_2)$ and that there are no heteroclinic orbits in $w_-(\Gamma(z_2))$, since a is the only equilibrium. In both cases this is a contradiction to Lemma 4.33.

Finally, a cannot be an equilibrium, a contradiction. Thus the limit cycle Γ does indeed not exist.

□

Corollary 4.35 (Poincare-Bendixson for Analytic Dynamical Systems)
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F \in \mathcal{O}(\Omega)$, $F \not\equiv 0$ and $K \subset \Omega$ compact. Let $\Gamma \subset K$ be an orbit of (4.1) in K. Then Γ either is a periodic orbit with exactly one equilibrium, a center, in $\mathrm{Int}(\Gamma)$, or the limit sets of Γ each consist of exactly one equilibrium. Additionally, if Γ is periodic, the interior of Γ (except of the center) is filled entirely with periodic orbits all having the center in its interior.

Proof
Since $\Gamma \subset K$, the maximum interval of existence of Γ is $\mathbb{R}$. By Theorem 4.34, there are no limit cycles in Ω. Hence we can apply the Poincare-Bendixson theorem to conclude that Γ is either a periodic solution, or $w_\pm(\Gamma)$ consists of at least one equilibrium.
Assume that Γ is a periodic solution. By Lemma 4.32 there exists exactly one equilibrium $a \in \mathrm{Int}(\Gamma)$ with $\mathrm{ind}(F, a) = 1$. Assume that a is not a center. Then, by a Corollary on the Poincare-Bendixson theorem, [7, Satz 9.2.4], the limit sets of all orbits sufficiently close to a consist only of a or there is at least one homoclinic orbit Λ satisfying $w_+(\Lambda) = w_-(\Lambda) = \{a\}$. In both cases this is a contradiction to Lemma 4.33. Thus a must be a center. In addition, the same argumentation holds now for every point in $\mathrm{Int}(\Gamma) \setminus \{a\}$. Hence the set $\mathrm{Int}(\Gamma) \setminus \{a\}$ has to be filled entirely with periodic orbits all having a in its interior, cf. Lemma 4.32.
Assume that Γ is not periodic. Then, by [7, Satz 9.2.4] and [9, Chapter 3.7, Theorem 3], the following case occurs: The set $w_\pm(\Gamma)$ is either a single equilibrium of (4.1), or $w_\pm(\Gamma)$ consists of a finite number of equilibria connected by a finite number of limit orbits together with a finite number of homoclinic orbits each in one of these equilibria. One can say, that each equilibrium in $w_\pm(\Gamma)$ forms a „rose“ with finitely many „petals“, cf. [9, p. 245]. We want to prove that the latter case cannot occur.
Suppose w.l.o.g. there is at least one homoclinic orbit $S \subset w_+(\Gamma)$ in the equilibrium $a \in w_+(\Gamma)$, i.e. $w_+(S) = w_-(S) = \{a\}$. The case $S \subset w_-(\Gamma)$ is analogous. Then, by Lemma 4.33 and Theorem 4.24, the equilibrium has a finite sectorial decomposition with only elliptic sectors. Let $x \in \Gamma, y \in S$ be arbitrary. There also exist sequences $(t_k)_{k\in\mathbb{N}}, (\tilde{t}_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ with $t_k, \tilde{t}_k \to \infty$, $\Phi(t_k, x) \to a$ and $\Phi(\tilde{t}_k, x) \to y$ for $k \to \infty$. Hence the orbit comes arbitrary close to a, but does not tend to a. This is impossible for such

an equilibrium, cf. Definition 4.14. Hence $w_+(\Gamma)$ can only consist of heteroclinic orbits, i.e. the „roses“ have no „petals“.
Suppose there are at least two equilibria $a_1, a_2 \in w_+(\Gamma)$, $a_1 \neq a_2$, with at least one heteroclinic limit orbit $S \subset w_+(\Gamma)$ satisfying $w_+(S) = \{a_1\}$ and Theorem $w_-(S) = \{a_2\}$. The case $S \subset w_-(\Gamma)$ is analogous. By the proof of the Generalized Poincare-Bendixson Theorem for Analytic Systems, cf. [9, p. 249], there must be at least one limit orbit $\tilde{S} \subset w_+(\Gamma) \setminus S$ such that $w_-(\tilde{S}) = \{a_1\}$. Choose $x_1 \in S$ and $x_2 \in \tilde{S}$. Then, by Lemma 4.33 and 4.24, the equilibrium has again a finite sectorial decomposition with only elliptic sectors. But we have already proven that none of these sectors (these are heteroclinic orbits) can be part of $w_+(\Gamma)$. But in both cyclic directions, there must be at least one elliptic sector between S and $\tilde{S}$. Hence this sector must be part of $w_+(\Gamma)$, a contradiction. We conclude that there are neither homoclinic nor heteroclinic orbits in $w_\pm(\Gamma)$, i.e. $w_\pm(\Gamma)$ must consist of only one equilibrium.

□

5 Topological structure of global neighbourhoods and separatrices

We have already seen that in a small neighbourhood of an equilibrium there is a specific topological and geometrical structure. For example that of a center. Around a center are only periodic orbits having exactly one equilibrium, the center, in the interior. A natural question is now the following: How does the set consisting of all of these periodic orbits around the center look like? What topological properties does this set have? What happens with the orbits on the boundary of such a set? How does the maximum interval of existence look like for such boundary orbits? These questions, not only for centers, will be answered in the following considerations.
The underlying literature is [2] and [3]. The topological basis is provided by [11] and [19]. In particular, the Jordan curve theorem will be used many times.
In the theorems and proofs in [2] and [3], there are many omitted arguments. The following sections provide a more detailed proof structure of these results. All added arguments are made by myself. In particular, the considerations and solution findings in section 5.2 are completely worked out by myself.

5.1 The global neighbourhood of equilibria

Definition 5.1
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F \in \mathcal{O}(\Omega)$ and $a \in \Omega$ an equilibrium of (4.1). If a

a) is a center, the *global neighbourhood* $\mathcal{U}_c(a)$ of F in a is defined as the set

$$\mathcal{U}_c(a) := \{a\} \cup \{x \in \Omega : \Gamma(x) \text{ is a periodic orbit with } a \in \mathrm{Int}(\Gamma)\}\,.$$

b) is a stable (unstable) node or focus, the *global neighbourhood* $\mathcal{U}_n(a)$ of F in a is defined as the set

$$\mathcal{U}_n(a) := \Big\{ x \in \Omega : w_{+(-)}(\Gamma(x)) = \{a\} \Big\}.$$

c) has order $m \geq 2$, the *global neighbourhood* or *global sector* $\mathcal{U}_s(a, \theta_+, \theta_-)$ of F in a with respect to the adjacent directions $\theta_+, \theta_- \in \mathcal{E}(F, a)$ is defined as the set

$$\mathcal{U}_s(a, \theta_+, \theta_-) := \Big\{ x \in \Omega : w_\pm(\Gamma(x)) = \{a\}, \lim_{t \to \pm\infty} \arg(\Phi(t, x) - a) = \theta_\pm \Big\}.$$

Theorem 4.24 ensures the existence of the elliptic sector with corresponding adjacent directions given by $\mathcal{E}(F, a)$.

Remark 5.2

a) Definition 4.1 and 4.14 ensures that the global neighbourhood of an equilibrium always consists of more than just the equilibrium itself, i.e. the topological interior of these sets is always nonempty.

b) In the case of nodes and foci, the equilibrium a always lies in the global neighbourhood $\mathcal{U}_n(a)$. Every global sector is defined in such a way that the equilibrium a itself is not contained in the global sector. Note that $\arg(a - a) = \arg(0)$ is not defined.

c) Note that the global sector in the case $m = 2$ (by Theorem 4.24, there exists only one elliptic sector) does actually not look like a sector envisioned as a „piece of cake“ between two adjacent definite directions. In fact, there are locally only two elliptic sectors and the global sector looks like a „filled eight“ as subset of $\mathbb{C}$. Note that in this case there are only two definite directions and on both „sides“ of these directions the conditions in Definition 5.1 c) are fulfilled, cf. Theorem 4.24. If $m \geq 3$, this effect cannot occur, since there are at least 3 definite directions and every elliptic sector lies between two adjacent directions.

(d) If $m \geq 3$, a global sector is in some sense a global point of view of (local) elliptic sectors, cf. Definition 4.14: For an arbitrary orbit $\Lambda \subset (U)_s(a, \theta_+, \theta_-)$ the radius of the elliptic sector can be set as

$$r := \max_{x \in \Lambda} |a - x| \in (0, \infty).$$

Note, that all orbits in $\Lambda \subset \mathcal{U}_s(a,\theta_+,\theta_-)$ are bounded. The point E in Definition 4.14 b) is then given by a point in $\partial\mathcal{B}_r(a) \cap \Lambda$. By the continuous dependence on initial conditions applied in E, there exist suitable characteristic boundary orbits Λ_1 and Λ_2 such that these orbits form indeed an elliptic sector in a. Considering these orbits, the properties (iii) and (iv) in Definition 4.14 b) are satisfied.
Additionally, in this chapter we are going to show that $\mathrm{Int}(\Gamma(E) \cup \{a\}) \subset \mathcal{U}_s(a,\theta_+,\theta_-)$. Hence, such orbits $\Gamma(E)$ form a so-called *elliptic region* and satisfy property (v) of Definition 4.14 b).

Definition 5.3
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F \in \mathcal{O}(\Omega)$ and $a \in \Omega$ an equilibrium of (4.1). Then the *boundary* of a is defined as the boundary of the global neighbourhood of F in a (with respect to adjacent directions given by $\mathcal{E}(F,a)$).

Proposition 5.4
Let $F \in \mathcal{O}(\mathbb{C})$ be entire and $a \in \mathbb{C}$ an equilibrium of (4.1). Let $\mathcal{U}$ be a global neighbourhood of F in a. Then

$$\mathcal{U} = \bigcup_{x \in \mathcal{U}} \Gamma(x). \tag{5.1}$$

Furthermore, if $x \in \partial\mathcal{U}$, we have $\Gamma(x) \subset \partial\mathcal{U}$. In particular, $\mathcal{U}$ and $\partial\mathcal{U}$ are both invariant. Additionally, in the case of a center, we even have

$$\mathcal{U}_c(a) = \bigcup_{x \in \mathcal{U}_c(a)} \overline{\mathrm{Int}(\Gamma(x))}. \tag{5.2}$$

Proof
Let $y \in \mathcal{U}$ and $z \in \Gamma(y)$. Then $\Gamma(y) = \Gamma(z)$ and $\Gamma(z)$ has the same properties as $\Gamma(y)$. This proves equation (5.1) and that $\mathcal{U}$ is invariant.
In the case $\partial\mathcal{U} = \emptyset$, $\mathcal{U}$ is open, closed and nonempty, thus completely $\mathbb{C}$ and nothing is to show. Suppose there exists a $\tau \in \mathbb{R}$ such that $\xi := \Phi(\tau, x) \notin \partial\mathcal{U}$, i.e. $\xi \in \mathring{\mathcal{U}} \cup \mathbb{C} \setminus \overline{\mathcal{U}}$, which is open. The case $\tau = 0$ is not possible, since $\Phi(0,x) = x \in \partial\mathcal{U}$. Since $\mathring{\mathcal{U}} \neq \emptyset$, there exists a $\varepsilon > 0$ such that either $\mathcal{B}_\varepsilon(\xi) \subset \mathcal{U}$, or $\mathcal{B}_\varepsilon(\xi) \subset \mathbb{C} \setminus \mathcal{U}$. By the continuous dependence on initial conditions, [9, Chapter 2.4, Theorem 4], there exists a $\delta > 0$ such that $|\Phi(\tau, y) - \xi| < \varepsilon$ for all $y \in \mathcal{B}_\delta(x)$. Since $x \in \partial\mathcal{U}$, there exist points $y_1 \in \mathcal{B}_\delta(x) \cap \mathcal{U}$

and $y_2 \in \mathcal{B}_\delta(x) \cap (\mathbb{C} \setminus \mathcal{U})$. If $\xi \in \mathbb{C} \setminus \overline{\mathcal{U}}$, we conclude $\Phi(\tau, y_1) \notin \mathcal{U}$ with $y_1 \in \mathcal{U}$. If $\xi \in \mathring{\mathcal{U}}$, we conclude $\Phi(\tau, y_2) \in \mathcal{U}$ with $y_2 \notin \mathcal{U}$. But $\mathcal{U}$ is invariant, this is a contradiction. Hence such a τ cannot exists, $\Gamma(x) \subset \partial\mathcal{U}$ and $\partial\mathcal{U}$ is also invariant.
If a is a center, then every $y \in \operatorname{Int}(\Gamma(x))$ either satisfies $y = a$, or, by Corollary 4.35, is a periodic orbit with a in its interior. This proves equation (5.2).

□

Proposition 5.5
Let $F \in \mathcal{O}(\mathbb{C})$ be entire and $a \in \mathbb{C}$ an equilibrium of (4.1). If a is a center, then $\partial\mathcal{U}_c(a) \cap F^{-1}(\{0\}) = \emptyset$, i.e. there are no equilibria on the boundary of a. If a has a global sector with adjacent directions $\theta_+, \theta_- \in \mathcal{E}(F, a)$, then $\partial\mathcal{U}_s(a, \theta_+, \theta_-) \cap F^{-1}(\{0\}) = \{a\}$, i.e. a is the only equilibrium on the boundary of a.

Proof
First, let a be a center and $x \in \partial\mathcal{U}_c(a)$. We show, that x cannot be an equilibrium. Suppose this would be the case. Then for all $\varepsilon > 0$ sufficiently small there exists a $y \in \mathcal{B}_\varepsilon(x) \cap \mathcal{U}_c(a)$ such that $\Gamma(y)$ neither has x in its limit sets nor is a periodic orbit with x in its interior. Note, that a is the only equilibrium in $\mathcal{U}_c(a)$. Hence x cannot be a center, focus or node and, by Theorem 4.3, the order of x must be at least 2, i.e. $F'(x) = 0$. By Theorem 4.24, x has a FSD with only elliptic sectors. But the orbits in $\mathcal{U}_c(a)$ all do not converge to x, this is a contradiction to Definition 4.14. In particular, the orbits in $\mathcal{U}_c(a)$ even form a hyperbolic sector in x, which is also impossible, since all sectors have to be elliptic. Thus x cannot be an equilibrium.
Second, let the order of a be at least 2 with a FSD in a. Set $\theta := (\theta_+, \theta_-)$ and let $x \in \partial\mathcal{U}_s(a, \theta) \setminus \{a\}$. Suppose x would be an equilibrium. Since the orbits in $\mathcal{U}_s(a, \theta)$ all converge to $a \neq x$ in both time directions and are not periodic, x cannot be a center, focus or node. With the same argumentation as above (by applying Theorem 4.24), we again derive a contradiction to Definition 4.14. Also in this case the orbits in $\mathcal{U}_s(a, \theta)$ form a hyperbolic sector in x, which is impossible. Thus x cannot be an equilibrium. This proves $\partial\mathcal{U}_s(a, \theta) \cap F^{-1}(\{0\}) \subset \{a\}$.
It remains to show that a lies indeed on the boundary of a. On the one hand, there are orbits (sequences of points) in $\mathcal{U}_s(a, \theta)$ converging to a, hence we must have $a \in \overline{\mathcal{U}_s(a, \theta)}$. One the other hand, $a \notin \mathcal{U}_s(a, \theta)$. Thus $a \in \partial\mathcal{U}_s(a, \theta)$.

□

Theorem 5.6
Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $F \not\equiv 0$ and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m \in \mathbb{N}$. Then the global neighbourhood (or global sector) of F in a is open. Furthermore, if $m = 1$, the global neighbourhood is connected and path-connected. If a is a center, the global neighbourhood is even simply connected. If $m \geq 2$, the set consisting of the global sector and a is connected and path-connected.

Proof
We proof the assertion by distinguishing the following three cases:

(i) a is a center.
Suppose there would exist a $x \in \mathcal{U}_c(a) \cap \partial\mathcal{U}_c(a) \neq \emptyset$. Then also $\Gamma(x) \in \mathcal{U}_c(a) \cap \partial\mathcal{U}_c(a)$, since both sets are invariant, by Proposition 5.4. The case $x = a$ can be excluded, since there are no equilibria on the boundary of a, cf. Proposition 5.5. Hence $\Gamma(x) \subset \mathcal{U}_c(a)$ is a periodic orbit with $a \in \mathrm{Int}(\Gamma(x)) \neq \emptyset$. Suppose there also exists a $y \in \partial\mathcal{U}_c(a) \setminus \Gamma(x) \neq \emptyset$. If $y \in \mathrm{Int}(\Gamma(x))$, by openness of $\mathrm{Int}(\Gamma(x))$, there exists an $r > 0$ such that $\mathcal{B}_r(y) \subset \mathrm{Int}(\Gamma(x))$. By equation (5.2), we get $\mathcal{B}_r(y) \subset \mathcal{U}_c(a)$, i.e. y is not a boundary point of $\mathcal{U}_c(a)$. This a contradiction. Furthermore, if $y \in \mathrm{Ext}(\Gamma(x))$, which is also open, there exists an $r > 0$ such that $\mathcal{B}_r(y) \subset \mathrm{Ext}(\Gamma(x))$. Since y lies on the boundary of a, there exists a $z \in \mathcal{B}_r(y) \cap \mathcal{U}_c(a)$, i.e. $\Gamma(z)$ is a periodic orbit with $a \in \mathrm{Int}(\Gamma(z))$. Note that we must have $z \neq x$. Since $\Gamma(z)$ is path-connected and $z \in \mathrm{Ext}(\Gamma(x))$, we must have $x \in \Gamma(x) \subset \mathrm{Int}(\Gamma(z))$. By equation (5.2) and openness of $\mathrm{Int}(\Gamma(z))$, we conclude $x \in \mathring{\mathcal{U}}_c(a)$, which is a contradiction to the fact that x lies on the boundary of a. Thus such a y does not exist and $\partial\mathcal{U}_c(a) \setminus \Gamma(x) = \emptyset$, i.e. the boundary of a exists exactly of $\Gamma(x)$. In addition, in this case $\mathcal{U}_c(a)$ would be compact and $\Gamma(x)$ would be the unique outermost periodic orbit in $\mathcal{U}_c(a)$.
By equation (5.2), the periodic orbit $\Gamma(x)$ is not isolated. Hence we can apply a result on limit cycles of analytic dynamical systems, [6, §12.1, Lemma 1, p. 203], to conclude the existence of a neighbourhood $U := \{y \in \mathbb{C} : \mathrm{dist}(\Gamma(x), y) < \rho\} \cap \mathrm{Ext}(\Gamma(x))$ with $\rho > 0$ such that all orbits in this neighbourhood are periodic. This result is based on the theory of succession functions for arcs without contact[6], cf. [6, §3.8]. Reduce ρ such that $U \cap F^{-1}(\{0\}) = \emptyset$. This is possible, since $F \not\equiv 0$. Let $T > 0$ be the period of $\Gamma(x)$. By the continuous dependence on initial conditions, [9, Chapter 2.4, Theorem 4], there exists a $\delta > 0$ such that $\Phi(T, z) \in \mathcal{B}_\rho(x)$ for all $z \in \mathcal{B}_\delta(x)$.

[6]In the literature, arcs without contact are sometimes called „transversals“, cf. [7, Chapter 9.1].

We conclude the existence of at least one periodic orbit $\Gamma \subset U$. Since $\Gamma(x)$ is the outermost periodic orbit in $\mathcal{U}_c(a)$, we must have $a \in \operatorname{Ext}(\Gamma)$, i.e. $\Gamma \subset \mathbb{C} \setminus \mathcal{U}_c(a)$. But by Lemma 4.32, there exists an equilibrium $\tilde{a} \in \operatorname{Int}(\Gamma) \subset \operatorname{Int}(\Gamma(x)) \cup U$ with $\tilde{a} \neq a$. This is a contradiction to the choice of ρ, since there are no more equilibria in $\operatorname{Int}(\Gamma(x)) \cup U$ than a. Hence such a x does not exists and $\mathcal{U}_c(a) \cap \partial\mathcal{U}_c(a) = \emptyset$. It follows the openness of $\mathcal{U}_c(a)$.
Next, we want to proof that $\mathcal{U}_c(a)$ is connected. There are two possibilities to prove this, with a contradiction argument or directly.
The first proof is by contradiction: Suppose $\mathcal{U}_c(a)$ is not connected. Then there exists a separation (U, V) of $\mathcal{U}_c(a)$. But a cannot lie in both sets, U and V. Hence all periodic orbits with a in its interior must lie either in U, or in V. Note that all orbits are connected as image of an interval under a continuous function (the solution). It follows that one of the two sets has empty intersection with $\mathcal{U}_c(a)$, i.e. (U, V) is not a separation and $\mathcal{U}_c(a)$ is connected.
The second proof can be formulated directly: The interior of any periodic orbit is connected. Hence, by Proposition 5.4, $\mathcal{U}_c(a)$ can be written as union of connected sets, all having the point a in common. Thus, also the union must be connected, cf. [11, Theorem 23.3].
The global neighbourhood $\mathcal{U}_c(a)$ is even path-connected. A simple proof follows by applying [19, Proposition 12.25]. An alternative direct proof with our obtained results is as follows. For $x, y \in \mathcal{U}_c(a)$ there exists a $z \in \mathcal{U}_c(a) \cap (\operatorname{Ext}(\Gamma(x)) \cup \operatorname{Ext}(\Gamma(y))) \neq \emptyset$ such that $\Gamma(x) \cup \Gamma(y) \subset \operatorname{Int}(\Gamma(z))$. This follows from the openness of $\mathcal{U}_c(a)$, the Jordan curve theorem and equation (5.2). The set $\operatorname{Int}(\Gamma(z))$ is clearly path-connected (even simply connected), hence there exists a path from x to y in $\operatorname{Int}(\Gamma(z))$. Equation (5.2) ensures that this path lies completely in $\mathcal{U}_c(a)$. Actually, this proves already that $\mathcal{U}_c(a)$ is connected. All in all, these considerations give a good imagination of the topological and geometrical structure of $\mathcal{U}_c(a)$.
In the last step we proof that $\mathcal{U}_c(a)$ is even simply connected. Let $L \subset \mathcal{U}_c(a)$ be a loop, i.e. a closed path. Let $x \in L$ be arbitrary. As above, there exists a $y_x \in \operatorname{Ext}(\Gamma(x)) \cap \mathcal{U}_c(a) \neq \emptyset$ such that $x \in \Gamma(x) \subset \operatorname{Int}(\Gamma(y_x)) \subset \mathcal{U}_c(a)$. Hence the set $\{\operatorname{Int}(\Gamma(y_x)) : x \in L\}$ is an open cover of L. Furthermore, L is compact, i.e. there exist finitely many points $x_1, \ldots, x_\ell \in L$, $\ell \in \mathbb{N}$, such that

$$L \subset \bigcup_{k=1}^{\ell} \operatorname{Int}(\Gamma(y_{x_k})) \subset \mathcal{U}_c(a).$$

Since the orbits $\Gamma(y_x)$, $x \in L$, are path-connected and ℓ is finite, there must exist a $y \in \{y_{x_k} : 1 \leq k \leq \ell\}$ such that $\Gamma(y)$ is the outermost periodic orbit having the other $\ell - 1$ periodic orbits in its interior. By equation (5.2), we conclude $L \subset \mathrm{Int}(\Gamma(y)) \subset \mathcal{U}_c(a)$ with an $y \in \mathcal{U}_c(a)$. Since $\mathrm{Int}(\Gamma(y))$ is clearly simply connected, L can be continuously deformed to a point $p \in L \subset \mathcal{U}_c(a)$. This proves that $\mathcal{U}_c(a)$ is simply connected.

(ii) a is a (w.l.o.g. stable) focus or node (the proof of the unstable case is analogous).
By Definition 4.1, there exists a $\varepsilon_1 > 0$ such that for all $y \in \mathcal{B}_{\varepsilon_1}(a)$ we have $w_+(\Gamma(y)) = \{a\}$. This proves $\mathcal{B}_{\varepsilon_1}(a) \subset \mathcal{U}_n(a)$ and $a \in \mathring{\mathcal{U}}_n(a)$. Let $x \in \mathcal{U}_n(a) \setminus \{a\}$ be arbitrary. Then there exists a $\tau > 0$ such that $\xi := \Phi(\tau, x) \in \mathcal{B}_{\varepsilon_1}(a)$. Choose $\varepsilon_2 > 0$ so small that $\mathcal{B}_{\varepsilon_2}(\xi) \subset \mathcal{B}_{\varepsilon_1}(a)$, e.g. $\varepsilon_2 := \frac{1}{2}\min\{|\xi - a|, \varepsilon_1 - |\xi - a|\}$. By the continuous dependence on initial conditions, [9, Chapter 2.4, Theorem 4], there exists a $\delta > 0$ such that $|\Phi(\tau, z) - \xi| < \varepsilon_2$ for all $z \in \mathcal{B}_\delta(x)$. Thus $w_+(\Gamma(z)) = \{a\}$ for all $z \in \mathcal{B}_\delta(x)$ and $\mathcal{U}_n(a)$ is open.
All orbits converging to a are path-connected. Hence, by the same contradiction argument as in (i), $\mathcal{U}_n(a)$ is connected. The direct proof is also possible in this case: We can write $\mathcal{U}_n(a)$ as union of sets, each consisting of an orbit converging to a and a itself. The equilibrium a lies on the boundary of each of these orbits. Hence, by [11, Theorem 23.4], these sets are connected and so is the union.
Again by [19, Proposition 12.25] we conclude the path-connectedness of $\mathcal{U}_n(a)$. Also an alternative proof is possible. Let $x, y \in \mathcal{U}_n(a)$. Then there are $\tau_1, \tau_2 > 0$ such that $\{\Phi(\tau_1, x), \Phi(\tau_2, y)\} \subset \mathcal{B}_{\varepsilon_1}(a) \subset \mathcal{U}_n(a)$, which is clearly path-connected (even convex). Hence there exists a path from $\Phi(\tau_1, x)$ to $\Phi(\tau_2, y)$ lying in $\mathcal{B}_{\varepsilon_1}(a) \subset \mathcal{U}_n(a)$, e.g. the convex combination of these two points. The orbits $\Gamma(x)$ and $\Gamma(y)$ are also path-connected. This leads to the existence of a path from x to y in $\mathcal{U}_n(a)$.

(iii) a has order at least 2.
By Theorem 4.24, there is a FSD in a. Let $\theta_+, \theta_- \in \mathcal{E}(F, a)$ be two adjacent directions and set $\theta := (\theta_+, \theta_-)$. By Proposition 4.22, there exist $r, \delta_+, \delta_- > 0$ such that for every $x_0 \in \{x \in \mathbb{R}^2 : |x - a| < r, |\arg(x - a) - \theta_\pm| < \delta_\pm\} := A_\pm$ the orbit though x_0 tends to a in the definite direction $\theta_\pm$ for $t \to \pm\infty$.
Let $x \in \mathcal{U}_s(a, \theta)$. Then there exists a $\tau_\pm$ such that $\xi_\pm := \Phi(\tau_\pm, x) \in A_\pm$. Choose $\varepsilon_\pm > 0$ small enough such that $\mathcal{B}_{\varepsilon_\pm}(\xi_\pm) \subset A_\pm$. By the continuous dependence on initial conditions, [9, Chapter 2.4, Theorem 4], there exists a $\delta_\pm > 0$ such that $|\Phi(\tau_\pm, y) - \xi_\pm| < \varepsilon_\pm$ for all $y \in \mathcal{B}_{\delta_\pm}(x)$. Thus with $\delta := \min\{\delta_+, \delta_-\}$ we have

$\mathcal{B}_\delta(x) \subset \mathcal{U}_s(a,\theta)$. This proves the openness of $\mathcal{U}_s(a,\theta)$.
Define $A := \mathcal{U}_s(a,\theta) \cup \{a\}$. We have to show that A is connected and path-connected. We have $a \in A$. Hence, with the same contradiction argument as in (i) and (ii), one can show that A is connected. The direct version is also possible. The set A can be written as union of connected sets, each consisting of an orbit of $\mathcal{U}_s(a,\theta)$ and a.
In this case we cannot use [19, Proposition 12.25] to conclude the path-connectedness of A, since A is not open. The alternative proof also does not work. In fact, for arbitrary $x, y \in \mathcal{U}_s(a,\theta)$ we do find $\tau_1, \tau_2 > 0$ such that $\{\Phi(\tau_1, x), \Phi(\tau_2, y)\} \subset A_+$, which is clearly path-connected (even convex). Note that this set looks like a „piece of cake“. But we cannot ensure $A_\pm \subset \mathcal{U}_s(a,\theta)$. Hence it is not straightforward to construct a path from $\Phi(\tau_1, x)$ to $\Phi(\tau_2, y)$ lying in $\mathcal{U}_s(a,\theta)$. Thus the arguments and ideas in (i) and (ii) cannot be applied analogously. Especially in the case $m = 2$ the global sector does not have to be (path-)connected, cf. Remark 5.2.
We proof the path-connectedness by constructing a path $\gamma_z : [0,1] \to A$ from $z \in \mathcal{U}_s(a,\theta)$ to a. Note that the existence of such a path is not clear, since paths always map from a compact interval in $\mathbb{R}$ to the underlying set, in our case A. Define for a fixed $z \in \mathcal{U}_s(a,\theta)$ the path

$$\gamma_z(t) := \begin{cases} \Phi\left(\frac{1}{1-t} - 1, z\right) & \text{if } t \in [0,1) \\ a & \text{if } t = 1 \end{cases}.$$

The global neighbourhood is invariant, cf. Proposition 5.4, hence $\gamma_z([0,1]) \subset A$ and γ is indeed well-defined. We have $\gamma_z(0) = \Phi(0,z) = z$. Moreover, the map $t \mapsto \frac{1}{1-t} - 1$ is strictly monotonously increasing on $[0,1]$ and converges to infinity for $t \to 1$. Thus, by [7, Proposition 8.4.1], we have

$$\lim_{t \to 1^-} \gamma_z(t) \in w_+(\Gamma(z)) = \{a\}$$

and Γ is continuous. Here we used the fact that the existential quantifier in the definition of limit sets on p. 3 can be replaced by a universal quantifier, if the limit set does only have one element. Note that continuity in 1 means that we have convergence of the function values *for all* sequences converging to 1.
For arbitrary $x, y \in \mathcal{U}_s(a,\theta)$ we now find a path γ_x from x to a and a path γ_y from y to a. This leads to the existence of a path from x to y in A.

□

Remark 5.7
Part (i) of the proof of Theorem 5.6 also shows that the following case cannot occur for holomorphic flows: There exists a periodic orbit Γ such that the obits on at least „one side“ of Γ, i.e. on $\mathrm{Int}(\Gamma)$ or $\mathrm{Ext}(\Gamma)$, converge to Γ. In fact, the interior of every periodic orbit forms a subset of a global neighbourhood $\mathcal{U}_c(a)$ of a center a and this set is open, i.e. there exists a neighbourhood $U \subset \mathcal{U}_c(a)$ of Γ. In addition, the convergence in $\mathrm{Int}(\Gamma)$ was excluded by a contradiction argument in the proof of Corollary 4.35. Afterwards, the convergence in $\mathrm{Ext}(\Gamma)$ was excluded by [6, §12.1, Lemma 1, p. 203].
Furthermore, there is not even a single solution converging to a periodic orbit $\tilde{\Gamma}$. This can also be proven by openness of $\mathcal{U}_c(a)$ (cf. Theorem 5.6). An alternative option is to use the proof of [6, §4.9, Theorem 19] separately, once for the interior and once for the exterior of the orbit $\tilde{\Gamma}$.

Theorem 5.8
Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $a \in \mathbb{C}$ an equilibrium of (4.1) and $x \in \mathbb{C}$ a point on the boundary of a. If a is a center, then $\Gamma(x)$ is unbounded. If a is a focus or node, then $\Gamma(x)$ is either unbounded, a focus, or a node. If a has order at least 2, then either $x = a$ or $\Gamma(x)$ is unbounded.

Proof
We proof this result by distinguishing again the following three cases:

(i) a is a center.
Suppose $\Gamma(x) \subset \partial\mathcal{U}_c(a)$ (cf. Proposition 5.4) is bounded. Then $K := \overline{\Gamma(x)}$ is compact and $\Gamma(x) \subset K$. Furthermore, $K \subset \partial\mathcal{U}_c(a)$ and $w_\pm(\Gamma(x)) \subset K$. Note that boundaries are always closed. By Proposition 5.5, there are no equilibria on the boundary of a, hence also not in $w_\pm(\Gamma(x)) \subset K$. Thus, by Corollary 4.35, $\Gamma(x)$ must be a periodic orbit with exactly one equilibrium, a center $\tilde{a}$, in $\mathrm{Int}(\Gamma(x))$.
Suppose $a \neq \tilde{a}$. By Theorem 5.6, the set $\mathcal{U}_c(\tilde{a})$ is open, i.e. there exists an $r > 0$ such that $\mathcal{B}_r(x) \subset \mathcal{U}_c(\tilde{a})$. But x lies also on the boundary of a, thus there exists at least one point $z \in \mathcal{B}_r(x) \cap \mathcal{U}_c(a) \subset \mathcal{U}_c(\tilde{a}) \cap \mathcal{U}_c(a) \neq \emptyset$. This is impossible, since there can only be exactly one equilibrium in the interior of every periodic orbit, cf. Lemma 4.32. We conclude $a = \tilde{a}$ and thus $x \in \mathcal{U}_c(a)$, which is a contradiction to the openness of $\mathcal{U}_c(a)$, cf. Theorem 5.6. Hence $\Gamma(x)$ must be unbounded.

(ii) a is a (w.l.o.g. stable) focus or node (the proof of the unstable case is analogous). Suppose there exists an equilibrium $\tilde{a}$ on the boundary of a being a center or having order at least 2. Then $\tilde{a}$ has a global neighbourhood (or global sector) $\mathcal{U}$ that is open, cf. Theorem 5.6. Hence there exists $r > 0$ such that $\mathcal{B}_r(\tilde{a}) \subset \mathcal{U}$. But $\tilde{a}$ lies also on the boundary of a. Hence there exists a $y \in \mathcal{U}_n(a) \cap \mathcal{B}_r(\tilde{a}) \neq \emptyset$. This is impossible, since $\tilde{a} \neq a \in \mathcal{U}_n(a)$ ($\mathcal{U}$ cannot be a global sector) and $w_+(\Gamma(y)) = \{a\}$ ($\tilde{a}$ cannot be a center). Thus all equilibria on the boundary of a have to be nodes and foci. In particular, this holds for $\Gamma(x) \subset \partial\mathcal{U}_c(a)$, if this orbit is an equilibrium.
Suppose $\Gamma(x) \subset \partial\mathcal{U}_n(a)$ is bounded and not an equilibrium. As in (i), $K := \overline{\Gamma(x)}$ is compact and $\Gamma(x) \cup w_\pm(\Gamma(x)) \subset K \subset \partial\mathcal{U}_n(a)$. By Corollary 4.35, $\Gamma(x)$ either is a periodic orbit with exactly one equilibrium, a center $\overline{a} \neq a$, in $\mathrm{Int}(\Gamma(x))$, or $w_+(\Gamma(x))$ and $w_-(\Gamma(x))$ each consist of exactly one equilibrium. If the first case occurs, there exists an $r > 0$ such that $\mathcal{B}_r(x) \subset \mathcal{U}_c(\overline{a})$, cf. Theorem 5.6. But x lies also on the boundary of a, i.e. there exists a $z \in \mathcal{B}_r(x) \cap \mathcal{U}_n(a) \neq \emptyset$. Since $w_+(\Gamma(z)) = \{a\}$, $\Gamma(z)$ ca not be a periodic orbit, which is a contradiction. Hence the second case must occur: There exist two equilibria $a_+, a_- \in \partial\mathcal{U}_n(a) \cap F^{-1}(\{0\})$ such that $w_\pm(\Gamma(x)) = \{a_\pm\}$, i.e. $\Gamma(x)$ is either a heteroclinic orbit or a homoclinic orbit. In both cases we want to derive a contradiction.
Suppose $\Gamma(x)$ is a homoclinic, i.e. $a_+ = a_-$. By Lemma 4.33 and Proposition 4.19, the order of a_+ is at least 2, i.e. a_+ is neither a node nor a focus, cf. Corollary 4.25. But a_+ is an equilibrium on the boundary of a. This is a contradiction (cf. above).
Suppose, $\Gamma(x)$ is heteroclinic, i.e. $a_+ \neq a_-$. Since there are only foci and nodes on the boundary of a, a_+ must be a stable node or focus. If we investigated the case that a is unstable, we would choose a_- at this point. By Definition 4.1, there exists a $\delta > 0$ such that for all $y \in \mathcal{B}_\delta(a_+)$ the solution through y satisfies $w_+(\Gamma(y)) = \{a_+\}$. But a_+ lies also on the boundary of a, i.e. there exists a $z \in \mathcal{U}_n(a) \cap \mathcal{B}_\delta(a_+) \neq \emptyset$ with $w_+(\Gamma(z)) = \{a\}$. We conclude $a = a_+$, i.e. $a \in \partial\mathcal{U}_n(a)$. This is a contradiction to the openness of $\mathcal{U}_n(a)$, cf. Theorem 5.6. Thus $\Gamma(x)$ must be unbounded.

(iii) a has order at least 2.
By Theorem 4.24, there is a FSD in a. Let $\theta_+, \theta_- \in \mathcal{E}(F, a)$ be two adjacent directions and set $\theta := (\theta_+, \theta_-)$. Suppose $\Gamma(x) \subset \partial\mathcal{U}_s(a, \theta) \setminus \{a\}$ is bounded. As in (i), $K := \overline{\Gamma(x)}$ is compact and $\Gamma(x) \cup w_\pm(\Gamma(x)) \subset K \subset \partial\mathcal{U}_s(a, \theta)$. By Corollary 4.35, $\Gamma(x)$ either is a periodic orbit with exactly one equilibrium, a center $\tilde{a}$, in $\mathrm{Int}(\Gamma(x))$, or $w_+(\Gamma(x))$ and $w_-(\Gamma(x))$ each consist of exactly one equilibrium. If the first case occurs, $a \neq \tilde{a}$ and $a \in \mathrm{Ext}(\Gamma(x))$. By path-connectedness of $\mathcal{U}_s(a, \theta) \cup \{a\}$, cf.

Theorem 5.6, we must have $\mathcal{U}_s(a,\theta) \subset \mathrm{Ext}(\Gamma(x))$. Since $\mathcal{U}_c(\tilde{a})$ is open, cf. Theorem 5.6, there exists an $r > 0$ such that $\mathcal{B}_r(x) \subset \mathcal{U}_c(\tilde{a})$. But x lies on the boundary of $\mathcal{U}_s(a,\theta)$, i.e. there must also exist a point $y \in \mathcal{U}_s(a,\theta) \cap \mathcal{B}_r(x) \neq \emptyset$. This is a contradiction. Thus $\Gamma(x)$ is not periodic and the second case occurs.
By Proposition 5.5, a is the only equilibrium on the boundary of a. Hence $\Gamma(x)$ must be a homoclinic orbit satisfying $w_{\pm}(\Gamma(x)) = \{a\}$. We want to derive a contradiction also in this case. By Theorem 4.24, $\Gamma(x)$ converges to a in adjacent definite directions $\eta_{\pm} \in \mathcal{E}(F,a)$ for $t \to \pm\infty$. Hence $x \in \mathcal{U}_s(a,\eta_+,\eta_-)$. By Theorem 5.6, this set is open, i.e. there exists a $\rho > 0$ such that $\mathcal{B}_\rho(x) \subset \mathcal{U}_s(a,\eta_+,\eta_-)$. But x lies also on the boundary of $\mathcal{U}_s(a,\theta)$, i.e. there exists a $z \in \mathcal{U}_s(a,\theta) \cap \mathcal{B}_\rho(x)$. The orbit cannot converge to a in more than two directions, thus we must have $\mathcal{U}_s(a,\theta) = \mathcal{U}_s(a,\eta_+,\eta_-)$, i.e. $\theta_+ = \eta_+$ and $\theta_- = \eta_-$. We conclude $x \in \partial\mathcal{U}_s(a,\theta) \cap \mathcal{U}_s(a,\theta)$, which is a contradiction to the openness of $\mathcal{U}_s(a,\theta)$. Thus $\Gamma(x)$ cannot be a homoclinic orbit and must be unbounded.

□

Corollary 5.9
Let $F \in \mathcal{O}(\mathbb{C})$, $F \not\equiv 0$, be entire and $a \in \mathbb{C}$ an equilibrium of (4.1) with order 1. Then the global neighbourhood in a is unbounded. In particular, the boundary of a node or focus is either empty, or consists of at least one unbounded orbit.

Proof
Let $\mathcal{U}$ be the global neighbourhood in a. If the boundary of a is empty, then $\mathcal{U}$ is open, closed and nonempty, i.e. completely $\mathbb{C}$ and thus unbounded. Hence assume $\partial\mathcal{U} \neq \emptyset$.
If a is a center, the boundary of a consists of unbounded orbits, cf. Theorem 5.8. Hence $\mathcal{U}$ must also be unbounded.
Suppose that a is a node or focus, the global neighbourhood is bounded and its boundary consists only of equilibria. Then $\mathbb{C} \setminus \partial\mathcal{U}$ is the union of two disjoint nonempty open sets (the global neighbourhood itself and the complement of its closure). Hence $\mathbb{C} \setminus \partial\mathcal{U}$ cannot be connected, which is a contradiction. In fact, since the boundary of a has to be a discrete set, $\mathbb{C} \setminus \partial\mathcal{U}$ is path-connected and thus connected. Note that $F \not\equiv 0$. Hence, by Theorem 5.8, the boundary consists of at least one unbounded orbit and the global neighbourhood is unbounded also in this case.

□

Theorem 5.10
Let $F \in \mathcal{O}(\mathbb{C})$, $F \not\equiv 0$, be entire and $a \in \mathbb{C}$ a focus or node of (4.1). Assume that $\partial\mathcal{U}_n(a) \cap F^{-1}(\{0\})$ does no have isolated points in $\partial\mathcal{U}_n(a)$, i.e. for all $\tilde{a} \in \partial\mathcal{U}_n(a) \cap F^{-1}(\{0\})$ and all $\rho > 0$ it holds that $(\mathcal{B}_\rho(\tilde{a}) \cap \partial\mathcal{U}_n(a)) \setminus \{\tilde{a}\} \neq \emptyset$. Then $\mathcal{U}_n(a)$ is simply connected.

Proof
Let $L \subset \mathcal{U}_n(a)$ be a loop. Since L is compact and $\mathcal{U}_n(a)$ is open (cf. Theorem 5.6), there exist points $y_1, \ldots, y_\ell \in L$ and radii $r_1, \ldots, r_\ell > 0$, $\ell \in \mathbb{N}$, such that

$$L \subset \bigcup_{k=1}^{\ell} \mathcal{B}_{r_k}(y_k) =: O.$$

and $\overline{O} \subset \mathcal{U}_n(a)$. In addition, there exists a closed Jordan curve $J \subset \partial O \subset \mathcal{U}_n(a)$ such that $L \subset \mathrm{Int}(J)$. Suppose now that $A := \mathrm{Int}(J) \setminus \mathcal{U}_n(a) \neq \emptyset$.
There holds $\overline{A} \subset \mathrm{Int}(J)$. In fact, we have $A \subset \mathrm{Int}(J)$ and thus $\overline{A} \subset \overline{\mathrm{Int}(J)}$. If there exists a point $z \in \partial A \cap J$, then there is an $r > 0$ such that $\mathcal{B}_r(z) \subset \mathcal{U}_n(a)$, cf. Theorem 5.6. But then we would have $A \cap \mathcal{U}_n(a) \neq \emptyset$, since $z \in \partial A$. This is a contradiction to the definition of A.
Suppose $\partial A = \emptyset$. We clearly have $\mathrm{Int}(J) \subset A \cup \mathcal{U}_n(a)$ and $A \cap \mathcal{U}_n(a) = \emptyset$. Since $A \neq \emptyset$ is open and $\mathrm{Int}(J)$ is connected, we must have $\mathcal{U}_n(a) \cap \mathrm{Int}(J) = \emptyset$, i.e. $A = \mathrm{Int}(J)$. This implies $\partial A = \partial\mathrm{Int}(J) = J$, i.e. $\overline{A} \not\subset \mathrm{Int}(J)$. This is a contradiction to our above result and thus $\partial A \neq \emptyset$.
Let $x \in \partial A \neq \emptyset$ be arbitrary. Then clearly $x \notin \mathcal{U}_n(a)$, since this set is open, cf. Theorem 5.6. In particular, $x \in \mathrm{Int}(J)$, since $\overline{A} \subset \mathrm{Int}(J)$. Suppose $x \notin \partial\mathcal{U}_n(a)$. Then there exists a $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(x) \subset \mathbb{C} \setminus \mathcal{U}_n(a)$ and $\mathcal{B}_\varepsilon(x) \subset \mathrm{Int}(J)$. Note that $\mathrm{Int}(J)$ is open. We conclude $\mathcal{B}_\varepsilon(x) \subset \mathrm{Int}(J) \cap (\mathbb{C} \setminus \mathcal{U}_n(a))$, i.e. $x \in \mathring{A}$. Since $\partial A \cap A = \emptyset$, this is a contradiction. Thus $x \in \partial\mathcal{U}_n(a)$. Since orbits cannot cross each other, we must have $\Gamma(x) \subset \mathrm{Int}(J)$, i.e. $\Gamma(x)$ is bounded. By Theorem 5.8, $\Gamma(x)$ must be an equilibrium.
We conclude that $\partial A \subset F^{-1}(\{0\})$ is a nonempty, discrete and bounded set, i.e. ∂A is finite and $\mathrm{Int}(J) \setminus \partial A$ is connected (even path-connected). Note that every path going through a point of ∂A can be deformed such that the path „bypasses“ this point, e.g. the bypass could be on the boundary of a sufficiently small circle around this point. In addition, $\mathrm{Int}(J) \setminus \partial A$ is the union of the two disjoint open sets $\mathcal{U}_n(a) \cap \mathrm{Int}(J) \neq \emptyset$ and $\mathring{A} \subset \mathbb{C} \setminus \mathcal{U}_n(a)$. Hence we must have $\mathring{A} = \emptyset$, i.e. $A = \partial A$ and A consists only of finitely many equilibria. In particular, $A \subset \partial\mathcal{U}_n(a) \cap F^{-1}(\{0\})$.
We derive now a contradiction by the following argument: We can choose $\tilde{a} \in A$, since

$A \neq \emptyset$. Then $\tilde{a} \in \partial\mathcal{U}_n(a) \cap F^{-1}(\{0\})$. But $\mathrm{Int}(J)$ is open and A is discrete, which implies the existence of a $\rho > 0$ such that $\mathcal{B}_\rho(\tilde{a}) \subset \mathrm{Int}(J)$ and $\mathcal{B}_\rho(\tilde{a}) \cap A = \{\tilde{a}\}$, i.e.

$$(\mathcal{B}_\rho(\tilde{a}) \cap \partial\mathcal{U}_n(a)) \setminus \{\tilde{a}\} = (\mathcal{B}_\rho(\tilde{a}) \cap \underbrace{\mathrm{Int}(J) \cap \partial\mathcal{U}_n(a))}_{\subset \mathrm{Int}(J)\setminus\mathcal{U}_n(a)=A} \setminus \{\tilde{a}\} = \{\tilde{a}\} \setminus \{\tilde{a}\} = \emptyset$$

and $\tilde{a}$ is isolated in $\partial\mathcal{U}_n(a)$. Hence we must have $A = \emptyset$ and thus $\mathrm{Int}(J) \subset \mathcal{U}_n(a)$. Since $\mathrm{Int}(J)$ is simply connected, L can be deformed to a point. Since L is arbitrary, $\mathcal{U}_n(a)$ is simply connected.

□

Remark 5.11

In Theorem 5.10 we assumed that the boundary of the global neighbourhood does not consist of equilibria $\tilde{a}$ being isolated on the boundary. Since $F^{-1}(\{0\})$ is a discrete set, this is equivalent to the fact that there always exists at least one unbounded (cf. Theorem 5.8) orbit Γ on the boundary satisfying $\tilde{a} \in w_+(\Gamma) \cup w_-(\Gamma)$. I conjecture that this assumption is redundant, i.e. it holds the following:

Every equilibrium $\tilde{a}$ one the boundary is attached by at least one unbounded orbit on the boundary of a, i.e. there exists at least one point $x_0 \in \partial\mathcal{U}_n(a) \setminus \{\tilde{a}\}$ satisfying $\tilde{a} \in w_+(\Gamma(x_0)) \cup w_-(\Gamma(x_0))$.

A sketch of the proof of this conjecture can be formulated as follows: Let $a_1, a_2 \in F^{-1}(\{0\})$ be two nodes (or foci) with a_1 being stable and a_2 unstable. Set $O := (\mathcal{U}_n(a_1) \cap \mathcal{U}_n(a_2)) \cup \{a_1, a_2\}$. This set consists of a_1, a_2 and all heteroclinic orbits connecting a_1 with a_2. Suppose $\{a_1, a_2\} \subset \mathring{O}$. One has to proof that this is impossible. More precisely, we do have either $a_1 \in \mathring{O}$, or $a_2 \in \mathring{O}$, but not both.

First, we can choose a fixed orbit Γ_0 satisfying $w_+(\Gamma_0) = \{a_1\}$ and $w_-(\Gamma_0) = \{a_2\}$ and a point $z_0 \in \Gamma_0$. Second, we can choose a straight line L through z_0 not being tangent to Γ (cf. [7, Chapter 9.1]). In addition, this line can be chosen such that $\{a_1, a_2\} \cap L = \emptyset$. By the Jordan curve theorem on S^2, this line separates the plane in two connected pieces O_1 and O_2 such that $a_1 \in O_1$ and $a_2 \in O_2$. Hence, every orbit $\Gamma \in O$ must cross L at least once. At this point, the most difficult part must be shown: Every orbit $\Gamma \in O$ must cross L an odd number of times. But what means „crossing“ in this sense? It means that the orbit passes through L, i.e. there exists an intersection point $p \in \Gamma \cap L$ and a small $\tau > 0$ such that $\Phi(-\tau, p) \in O_1$ and $\Phi(\tau, p) \in O_2$. At this point of the proof one has to check if Γ „lies“ on L, i.e. there exists a $\tilde{\tau} > 0$ with $\Phi((-\tilde{\tau}, \tilde{\tau}), p) \subset L$. In addition, one has to check, if there are other possible „crossings“ with L.

After constructing such a definition of „crossings“, we can define $N(x)$ for all $x \in O$ as the number of such „crossings“ of $\Gamma(x)$ with L. Of course, we should ensure that $\mathbb{N}(x) = 0$ if and only if $x \in \{a_1, a_2\}$. Here one has to check, if more than finitely many „crossings“ can occur or if we always have $N < \infty$. Moreover, these „crossings“ and the associated number $N : O \to \mathbb{N} \cup \{0\}$ should be defined in a way such that N is continuous. With this N, we can define analogously N_1 and N_2 as the number of crossings with the two rays $L_1 \subset L$ and $L_2 \subset L$ with origin z_0, i.e. $L = L_1 \cup \{z_0\} \cup L_2$ and $L_1 \cap L_2 = \emptyset$. Since z_0 is an interior point of O, it is easy to show that $N_1^{-1}(\mathbb{N}) \neq \emptyset$ and $N_2^{-1}(\mathbb{N}) \neq \emptyset$, i.e. both rays are crossed by at least one orbit. Moreover, not both rays can be crossed an odd number of times by one orbit.
Define $M : O \to \{1, 2\}$ such that $M(x) = 1$ if and only if $\Gamma(x)$ crosses L_1 odd times. We should verify that M is continuous. In addition, if $\{a_1, a_2\} \subset \mathring{O}$, then the set $\tilde{O} := O \setminus \Gamma_0 \cup \{a_1, a_2\}$ is open and connected. After defining the two sets $A_j := \{x \in \tilde{O} : M(x) = j\}$, $j \in \{1, 2\}$, one should check that these sets are both, nonempty and open. Now we have $\tilde{O} = A_1 \cup A_2$. Then, by connectedness of $\tilde{O}$, we conclude that $A_1 = \emptyset$ or $A_2 = \emptyset$, which should be a contradiction.
The proof could be easier, if we deform L in such a way that $L \cap \Gamma_0 = \{z_0\}$, i.e. there exists only one intersection point of Γ_0 with L. In fact, in this case we have (z_0 is an interior point of O) $N_1^{-1}(\{1\}) \neq \emptyset$ and $N_2^{-1}(\{1\}) \neq \emptyset$. But now the definition of “crossing„ could be more difficult, since L does not necessarily have to be a straight line.

5.2 The local structure of global sectors

We have already seen at various points that the structure of local elliptic sectors is difficult to generalize to global sectors, see e.g. Remark 5.2 and the proof of Theorem 5.6, case (iii). In particular, it is not clear yet what the boundary looks like near an equilibrium with order at least 2. How many boundary orbits are there, near such an equilibrium? Is the union of all global sectors near an equilibrium already „everything“ or is there space left for other topological structures, cf. Remark 4.26 and the outlook at the end of Example 4.28? How many global sectors exists in an equilibrium with order at least 2? What happens in the special case $m = 2$? All these questions will be answered in this section.
All results and ideas are my own as there is no reference that I know of, which provides a rigorous analysis of the global sectors near an equilibrium. Only in [6, Chapter VIII], the

author analyses this geometrical structure in detail. But as already mentioned in Remark 4.16, this approach is different from mine.

Proposition 5.12
Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $F \not\equiv 0$ and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m \in \mathbb{N}\backslash\{1,2\}$. Then for every global sector $\mathcal{U}$ in a there exist two unique unbounded orbits $\Gamma_1, \Gamma_2 \in \partial\mathcal{U}$ with the property $w_+(\Gamma_1) = w_-(\Gamma_2) = \{a\}$. In addition, there exists an $r > 0$ and two points $p_1 \in \Gamma_1 \cap \partial\mathcal{B}_r(a)$, $p_2 \in \Gamma_2 \cap \partial\mathcal{B}_r(a)$ such that

$$\overline{\mathcal{B}_r(a)} \cap \partial\mathcal{U} = \{a\} \cup \Gamma_+(p_1) \cup \Gamma_-(p_2). \tag{5.3}$$

In particular, Γ_1 and Γ_2 form the boundary of an elliptic sector, i.e. they are possible characteristic orbits of this elliptic sector.
Furthermore, for all $\theta \in \mathcal{E}(F,a)$ there exists an unbounded orbit Γ_θ tending to a in the definite direction θ. Moreover, for all orbits Γ satisfying $w_+(\Gamma) = w_-(\Gamma) = \{a\}$ there exists a global sector $\mathcal{U}_\Gamma$ in a with $\Gamma \subset \mathcal{U}_\Gamma$, i.e. Γ tends to a in adjacent definite directions given by $\mathcal{E}(F,a)$.

Proof
By Theorem 4.24, there exists a minimal FSD in a with $d := 2m-2 \geq 4$ elliptic sectors. Let $\theta_1, \theta_2, \theta_3 \in \mathcal{E}(F,a)$ be three adjacent directions, i.e. θ_1 and θ_2 as well as θ_2 and θ_3 are adjacent. Set $\mathcal{U}_1 := \mathcal{U}_s(a,\theta_1,\theta_2)$ and $\mathcal{U}_2 := \mathcal{U}_s(a,\theta_2,\theta_3)$. Assume w.l.o.g. that $\cos(\arg(F^{(m)}(a)) + \theta_2(m-1)) > 0$, i.e. the orbits near the ray with angle θ_2 tend to a for $t \to \infty$, cf. Proposition 4.22. The other case is analogous. Since $d > 2$, we have $\theta_3 \neq \theta_1$ and thus $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$.
By Definition 4.14, there exists an $r > 0$ and two points $E_1, E_2 \in \mathcal{B}_r(a)$ such that $\Gamma(E_1) \cup \Gamma(E_2) \subset \mathcal{B}_r(a) \cup \{E_1, E_2\}$, $\Gamma(E_1) \subset \mathcal{U}_1$ and $\Gamma(E_2) \subset \mathcal{U}_2$. Additionally, we have $E_1 \neq E_2$ and $\Gamma(E_1) \cap \Gamma(E_2) = \emptyset$. Define Ξ as the curve piece of $\partial\mathcal{B}_r(a)$ from E_1 to E_2, i.e. $\Xi = (B_r(a))(E_1, E_2)$. Since the orbits $\Gamma(E_1)$ and $\Gamma(E_2)$ cannot cross each other, the curve

$$J := \Xi \cup \Gamma_+(E_1) \cup \Gamma_+(E_2) \cup \{a\}$$

is a closed Jordan curve. Let $\nu : \partial\mathcal{B}_r(a) \to S^1$ be the outer unit normal of $\partial\mathcal{B}_r(a)$. By Definition 4.14 b) (iii) and (iv), we have $\langle F(y), \nu(y)\rangle < 0$ for all $y \in \Xi$, i.e. between E_1 and E_2 the vector field points inwards. If we had $\cos(\arg(F^{(m)}(a)) + \theta_2(m-1)) < 0$, the vector field would point outwards and a would be reached for $t \to -\infty$. Additionally, the

radius $r > 0$ can be chosen such that all orbits through a point in $\mathrm{Int}(J)$ tend to a for $t \to \infty$ (with direction θ_2).
Furthermore, by openness of $\mathcal{U}_1$ and $\mathcal{U}_2$ (cf. Theorem 5.6), we have $\mathcal{U}_1 \cap \mathrm{Int}(J) \neq \emptyset$ and $\mathcal{U}_2 \cap \mathrm{Int}(J) \neq \emptyset$. Note that $\Gamma_+(E_1) \subset J \cap \mathcal{U}_1$ and $\Gamma_+(E_2) \subset J \cap \mathcal{U}_2$. Since $\mathrm{Int}(J)$ is connected, we conclude that $A := \mathrm{Int}(J) \setminus (\mathcal{U}_1 \cup \mathcal{U}_2)$ is closed with respect to $\mathrm{Int}(J)$ and nonempty. Moreover, for all $x \in A$ we have $w_+(\Gamma(x)) = \{a\}$.
Suppose that $\partial\mathcal{U}_1 \cap \mathrm{Int}(J) = \emptyset$. Since $\mathcal{U}_1 \cap \mathrm{Int}(J) \neq \emptyset$, this implies $\mathcal{U}_1 = \mathrm{Int}(J)$ and thus the contradiction $A = \mathcal{U}_1 \setminus (\mathcal{U}_1 \cup \mathcal{U}_2) = \emptyset$. Thus $\partial\mathcal{U}_1 \cap \mathrm{Int}(J) \neq \emptyset$. Of course, the same argumentation holds for $\partial\mathcal{U}_2$, i.e. we also have $\partial\mathcal{U}_2 \cap A \neq \emptyset$. In particular, we have $\partial\mathcal{U}_1 \cap \mathrm{Int}(J) \subset A$ and $\partial\mathcal{U}_2 \cap \mathrm{Int}(J) \subset A$. Choose $\xi_1 \in \partial\mathcal{U}_1 \cap \mathrm{Int}(J)$ and $\xi_2 \in \partial\mathcal{U}_2 \cap \mathrm{Int}(J)$. Then, by our above argumentation, $w_+(\Gamma(\xi_1)) = w_+(\Gamma(\xi_2)) = \{a\}$. By Theorem 5.8, $\Gamma(\xi_1)$ and $\Gamma(\xi_2)$ are unbounded. Hence $a \notin w_-(\Gamma(\xi_1)) \cup w_-(\Gamma(\xi_2))$.
Since θ_1, θ_2 and θ_3 are arbitrary, we can use this argument now between all adjacent elliptic sectors. We conclude that the two orbits $\Gamma(\xi_1)$ and $\Gamma(\xi_2)$ can indeed be chosen as characteristic orbits lying on the boundary of these adjacent elliptic sectors and tend to a in the definite direction θ_2. With these two orbits between each adjacent elliptic sectors, one can construct a FSD in a. The points p_1 and p_2 are the intersection points of $\Gamma(\xi_1)$ and $\Gamma(\xi_2)$ with $\partial\mathcal{B}_r(a)$.
We conclude the following: For every global sector $\mathcal{U}_s(a\theta_+, \theta_-)$ with two adjacent directions $\theta_+, \theta_- \in \mathcal{E}(F, a)$ there exist two unbounded orbits $\Gamma_1, \Gamma_2 \subset \partial\mathcal{U}_s(a\theta_+, \theta_-)$ satisfying $w_+(\Gamma_1) = w_-(\Gamma_2) = \{a\}$ with definite direction θ_+ and θ_-. It remains to show that the following two geometrical structures can not occur:
First, suppose there exists a $x \in \mathbb{C}$ with $w_+(\Gamma(x)) = w_-(\Gamma(x)) = \{a\}$ with two definite directions given by $\mathcal{E}(F, a)$ not being adjacent. Then, by Proposition 4.22, in $\mathrm{Int}(\Gamma(x) \cup \{a\})$ there is at least one whole (local) elliptic sector S. In addition, this elliptic sector must be part of a connected global sector with the same adjacent definite directions. This elliptic sector has now at least one unbounded orbit on its boundary. But $\mathrm{Int}(J)$ is bounded. This is a contradiction to the connectedness of this global sector. Thus there exist indeed d global sectors in a, each having exactly one (local) elliptic sector as subset. Furthermore, all orbits Γ satisfying $w_+(\Gamma) = w_-(\Gamma) = \{a\}$ tend to a in adjacent definite directions given by $\mathcal{E}(F, a)$.
Second, suppose there exist more than two orbits on the boundary of an arbitrary global sector in a having nonempty intersection with $\mathcal{B}_r(a)$. Then there exist at least three unbounded boundary orbits tending to a for $t \to \infty$ or $t \to -\infty$. In addition, two of these three orbits must tend to a in the same definite direction $\eta \in \mathcal{E}(F, a)$ and the third orbit

tends to a in a direction $\tilde{\eta} \in \mathcal{E}(F,a)\setminus\{\eta\}$. Since the Jordan curve theorem holds also on S^2 (cf. [11, Lemma 61.1]), these two orbits separate $\mathbb{C}$ in exactly two unbounded nonempty connecting components. Note that orbits cannot cross each other. Let O be the (unique) component, where the third orbit does not lie. Now there exists at least one orbit Γ in the global sector lying completely in O. Note that orbits are always connected. We conclude that Γ must tend to a in the same direction η for both time directions. But this is impossible, since the number $\cos(\arg(F^{(m)}(a)) + \eta(m-1))$ is either positive, or negative, cf. Proposition 4.22 and the proof Lemma 4.23. In particular, this proves the uniqueness of these two boundary orbits in equation 5.3.

$\square$

Definition 5.13
Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $F \not\equiv 0$ and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m = 2$. Let $\mathcal{U}$ be the global sector of F in a. A pair of points $(q_1, q_2) \in \mathbb{C}^2$ are *separating the global sector* $\mathcal{U}$, if there exist two orbits $\Gamma_1, \Gamma_2 \subset \mathcal{U}$ satisfying $q_1 \in \mathrm{Int}(\Gamma_1 \cup \{a\})$, $q_2 \in \mathrm{Int}(\Gamma_2 \cup \{a\})$, $\Gamma_1 \subset \mathrm{Ext}(\Gamma_2)$ and $\Gamma_2 \subset \mathrm{Ext}(\Gamma_1)$.

Definition and Proposition 5.14
Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $F \not\equiv 0$ and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m = 2$. Let $\mathcal{U}$ be the global sector in a. Then there exists a pair $(q_1, q_2) \in \mathbb{C}^2$ separating $\mathcal{U}$ with corresponding orbits $\Gamma_1, \Gamma_2 \subset \mathcal{U}$. Define for $j \in \{1, 2\}$ the j^{th} *sector component* of $\mathcal{U}$ (with respect to (q_1, q_2)) as

$$\mathcal{U}^{[j]} := \{x \in \mathcal{U} : q_j \in \mathrm{Int}(\Gamma(x) \cup \{a\})\} \cup \mathrm{Int}(\Gamma_j \cup \{a\}) \cup \Gamma_j.$$

Then it holds that $\mathcal{U} = \mathcal{U}^{[1]} \cup \mathcal{U}^{[2]}$ and $\mathcal{U}^{[1]} \cap \mathcal{U}^{[2]} = \emptyset$. Furthermore, the separating points can be chosen such that $\mathcal{U}^{[j]}$ is open and $\mathcal{B}_\rho(a) \cap \mathcal{U}^{[j]} \neq \emptyset$ for all $\rho > 0$ and $j \in \{1, 2\}$.

Proof
Set $J_x := \Gamma(x) \cup \{a\}$, $x \in \mathcal{U}$. By Theorem 4.24, there exists a FSD in a with two (local) elliptic sectors. By Definition 4.14, there exist $r > 0$ and two points $E_1, E_2 \in \partial\mathcal{B}_r(a) \cap \mathcal{U}$ satisfying $\Gamma(E_1) \neq \Gamma(E_2)$ and $\Gamma(E_1) \cup \Gamma(E_2) \subset \overline{\mathcal{B}_r(a)}$. This implies $\mathrm{Int}(J_{E_1}) \cup \mathrm{Int}(J_{E_2}) \subset \overline{\mathcal{B}_r(a)}$.
Suppose $\Gamma(E_1) \subset \mathrm{Int}(J_{E_2})$. Since $E_1 \in \Gamma(E_1)$, there exists a $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(E_1) \subset \mathrm{Int}(J_{E_2})$. Since $E_1 \in \partial\mathcal{B}_r(a)$, there exists a $y \in \mathcal{B}_\varepsilon(E_1) \setminus \overline{\mathcal{B}_r(a)}$, e.g. $y := E_1 + \frac{\varepsilon}{2r}(E_1 - a)$.

We conclude the contradiction $y \in \mathrm{Int}(J_{E_2}) \setminus \overline{\mathcal{B}_r(a)} = \emptyset$. Thus we have $\Gamma(E_1) \subset \mathrm{Ext}(J_{E_2})$. Note that $\Gamma(E_1) \neq \Gamma(E_2)$. Of course, the same argumentation holds for $\Gamma(E_2)$, i.e. we also have $\Gamma(E_2) \subset \mathrm{Ext}(J_{E_1})$. Here, we only have to change the roles of E_1 and E_2.
Furthermore, we have $\mathrm{Int}(J_{E_1}) \neq \emptyset$ and $\mathrm{Int}(J_{E_2}) \neq \emptyset$. Note that E_1 and E_2 are not equilibria of (4.1). Hence we can choose $q_1 \in \mathrm{Int}(\Gamma(E_1))$ and $q_2 \in \mathrm{Int}(J_{E_2})$. By setting $\Gamma_1 := \Gamma(E_1)$ and $\Gamma_2 := \Gamma(E_2)$, we have a pair of separating points. Set $A := \mathrm{Int}(J_{E_1}) \cup \mathrm{Int}(J_{E_2})$.
Suppose there exists an orbit $\Gamma \subset \mathcal{U}$ satisfying $\{q_1, q_2\} \subset \mathrm{Int}(\Gamma \cup \{a\})$. Then clearly $\mathrm{Int}(J_{E_1}) \cup \mathrm{Int}(J_{E_2}) \subset \mathrm{Int}(\Gamma \cup \{a\})$. But with this property we must have $\mathcal{B}_r(a) \subset \mathrm{Int}(\Gamma \cup \{a\})$. This implies the contradiction $a \in \mathrm{Int}(\Gamma \cup \{a\})$.
Suppose there exists an orbit $\Gamma \subset \mathcal{U} \setminus A$ satisfying $\{q_1, q_2\} \subset \mathrm{Ext}(\Gamma \cup \{a\})$. Then, by the same argumentation, we would have $\mathcal{B}_r(a) \subset \mathrm{Ext}(\Gamma \cup \{a\})$, i.e. $a \in \mathrm{Ext}(\Gamma \cup \{a\})$. This is also a contradiction.
Hence all orbits $\Gamma \subset \mathcal{U} \setminus A$ satisfy either $q_1 \in \mathrm{Int}(\Gamma \cup \{a\})$, or $q_2 \in \mathrm{Int}(\Gamma \cup \{a\})$. This implies $\mathcal{U} = \mathcal{U}^{[1]} \cup \mathcal{U}^{[2]}$ and $\mathcal{U}^{[1]} \cap \mathcal{U}^{[2]} = \emptyset$. Note that $x \in \mathcal{U}^{[j]}$ clearly implies $\Gamma(x) \subset \mathcal{U}^{[j]}$ with $j \in \{1, 2\}$.
Set $\rho_0 := \frac{1}{2}\min\{|a - q_1|, |a - q_2|\} \in (0, r)$. Fix $\rho > 0$ and $j \in \{1, 2\}$. We proved that $\Gamma(E_j) \subset \mathcal{U}^{[j]}$. Since $\Gamma(E_j) \cap \mathcal{B}_\rho(a) \neq \emptyset$, we also have $\mathcal{B}_\rho(a) \cap \mathcal{U}^{[j]} \neq \emptyset$. Let $x \in \mathcal{U}^{[j]} \setminus \overline{\mathrm{Int}(J_{E_j})}$ be arbitrary. By applying Definition 4.14 b) (iv), there exist two points $\xi_+, \xi_- \in \partial\mathcal{B}_{\rho_0}(a) \cap \Gamma(x) \subset \mathcal{B}_r(a) \setminus \overline{A}$. Let $\gamma_x : [0, 1] \to \mathbb{C}$ be the closed Jordan curve consisting of the curve pieces $(\partial\mathcal{B}_{\rho_0}(a))(\xi_+, \xi_-)$ and $(\Gamma(x))(\xi_-, \xi_+)$, i.e. γ_x traverses $\Gamma(x)$ except of the parts in $\mathcal{B}_{\rho_0}$ and closes by using the boundary of the circle $\mathcal{B}_{\rho_0}(a)$. We choose the path on $\partial\mathcal{B}_{\rho_0}(a)$ in such a way that $q_j \in \mathrm{Int}(\gamma_x([0, 1]))$, i.e. $(\partial\mathcal{B}_{\rho_0}(a))(\xi_+, \xi_-) \subset \mathrm{Int}(J_x)$. Since $w_+(\Gamma(x)) = w_-(\Gamma(x)) = \{a\}$, there exists, in particular, a $T > 0$ such that $\{\Phi(T, x), \Phi(-T, x)\} \subset \mathcal{B}_{\rho_0}(a) \setminus \overline{A}$. By applying Definition 4.14 b) (iii), we can choose this T such that $\Phi((-\infty, -T], x) \cup \Phi([T, \infty), x) \subset \mathcal{B}_r(a) \setminus \overline{A}$. This means that the orbit through x lies in $\mathcal{B}_r(a)$ at sufficiently large times $t \in \mathbb{R}$. Moreover, by openness of the set $\mathcal{B}_{\rho_0}(a) \setminus \overline{A}$, there exists a $\eta > 0$ such that

$$\{\mathcal{B}_\eta(\Phi(T, x)), \mathcal{B}_\eta(\Phi(-T, x))\} \subset \mathcal{B}_{\rho_0}(a) \setminus \overline{A}.$$

Note that $J_{E_1} \cup J_{E_2}$ separates $\overline{\mathcal{B}_r(a)}$ in exactly four connecting components. By construction, $\Phi(T, x)$ and $\Phi(-T, x)$ lie in different connecting components. Let $\tilde{\varepsilon} > 0$ be arbitrary. By Theorem 5.6 and the continuous dependence on initial conditions, [9, Chapter 2.4, Theorem 4], there exists a $\delta > 0$ such that $\mathcal{B}_\delta(x) \subset \mathcal{U}$ and $|\Phi(t, y) - \Phi(t, x)| < \min\{\eta, \tilde{\varepsilon}\}$

for all $t \in [-T, T]$ and $y \in \mathcal{B}_\delta(x)$. We conclude

$$\sup_{t\in[0,1]} |\gamma_x(t) - \gamma_y(t)| < \min\{\eta, \tilde{\varepsilon}\} \quad \forall\, y \in \mathcal{B}_\delta(x). \tag{5.4}$$

Actually, on a piece of $\partial\mathcal{B}_{\rho_0}(a)$ the two curves γ_x and γ_y, $y \in \mathcal{B}_\delta(x)$, even coincide. Note that γ_y is well-defined, since $y \in \mathcal{U}$.
Let $y \in \mathcal{B}_\delta(x)$ be arbitrary. By applying Definition 4.14 b) (iii), we have the following equivalence: The curve γ_y satisfies $q_j \in \mathrm{Int}(\gamma_y([0,1]))$ if and only if $y \in \mathcal{U}^{[j]} \setminus \overline{\mathrm{Int}(J_{E_j})}$. Note that we always have $\mathrm{Int}(\gamma_y([0,1])) \subset \mathrm{Int}(J_y)$. Furthermore, $q_j \in \mathrm{Int}(\gamma_y([0,1]))$ if and only if the winding number $w(y,j) := \mathrm{wind}(\gamma_y, q_j)$ of γ_y with respect to q_j is ± 1, i.e. $|w(y,j)| = 1$, cf. [11, Theorem 66.2] and [11, Lemma 66.3]. Choose $\tilde{\varepsilon}$ sufficiently small such that there are no equilibria in $\{z \in \mathbb{C} : \mathrm{dist}(\gamma_y([0,1]), z) < \tilde{\varepsilon}\}$ and such that $\{z \in \mathbb{C} : \mathrm{dist}(\Phi([-T,T],x), z) < \tilde{\varepsilon}\} \subset \mathcal{U}$. The latter property can be ensured by using the compactness of $\Phi([-T,T],x)$. Then, by using (5.4), we can find a path homotopy between γ_x and γ_y. Hence, by [11, Lemma 66.1 (b)], we get $w(y,j) = w(x,j)$ and thus $y \in \mathcal{U}^{[j]}$. This shows $\mathcal{B}_\delta(x) \subset \mathcal{U}^{[j]}$. If $x \in \mathrm{Int}(J_{E_j})$, we clearly have a $\delta > 0$ such that $\mathcal{B}_\delta(x) \subset \mathcal{U}^{[j]}$. Note that $\overline{\mathrm{Int}(J_{E_j})} \subset \mathcal{U}^{[j]}$. Hence $\mathcal{U}^{[j]}$ is open.

□

Corollary 5.15
Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $F \not\equiv 0$ and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m = 2$. Let $\mathcal{U}$ be the global sector in a. Then $\mathcal{U}$ has at least two connecting components. In particular, $\mathcal{U}$ is not connected.

Proof
By Proposition 5.14, there exists a pair $(q_1, q_2) \in \mathbb{C}^2$ separating $\mathcal{U}$ such that $\mathcal{U}^{[1]}$ and $\mathcal{U}^{[2]}$ are open, $\mathcal{U} = \mathcal{U}^{[1]} \cup \mathcal{U}^{[2]}$ and $\mathcal{U}^{[1]} \cap \mathcal{U}^{[2]} = \emptyset$. Moreover, $\mathcal{U}^{[1]} \neq \emptyset$ and $\mathcal{U}^{[2]} \neq \emptyset$. Hence $(\mathcal{U}^{[1]}, \mathcal{U}^{[2]})$ is a separation of $\mathcal{U}$ and thus $\mathcal{U}$ cannot be connected, i.e. $\mathcal{U}$ has at least two connecting components.

□

Proposition 5.16
Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $F \not\equiv 0$ and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m = 2$. Let $\mathcal{U} := \mathcal{U}_s(a, \theta_+, \theta_-)$, $\mathcal{E}(F, a) = \{\theta_+, \theta_-\}$, be the global sector in a and $(q_1, q_2) \in \mathbb{C}^2$ a pair of points separating $\mathcal{U}$. Then for every $j \in \{1, 2\}$ there exist two unique unbounded orbits

$\Gamma_\pm \in \partial\mathcal{U}^{[j]}$ and $r > 0$ with two points $p_\pm \in \Gamma_\pm \cap \partial\mathcal{B}_r(a)$ such that $w_+(\Gamma_+) = w_-(\Gamma_-) = \{a\}$ and

$$\overline{\mathcal{B}_r(a)} \cap \partial\mathcal{U}^{[j]} = \{a\} \cup \Gamma_+(p_+) \cup \Gamma_-(p_-). \tag{5.5}$$

In particular, Γ_+ and Γ_- form the boundary of an elliptic sector, i.e. they are possible characteristic orbits of this elliptic sector.

Proof

The proof of this Proposition is quite similar to that of Proposition 5.12. But we have to change the „separating property“ for the connectedness-argument, since we only have two definite directions. We construct this by using Definition 5.13.

By Theorem 4.24, we have a FSD in a with two (local) elliptic sectors. Furthermore, there exist $r > 0$ and two points $E_1, E_2 \in \partial\mathcal{B}_r(a)$ such that $\Gamma(E_1) \cup \Gamma(E_2) \subset \mathcal{B}_r(a) \cup \{E_1, E_2\}$, $\Gamma(E_1) \subset \mathcal{U}_1$ and $\Gamma(E_2) \subset \mathcal{U}_2$. Additionally, we have $E_1 \neq E_2$ and $\Gamma(E_1) \cap \Gamma(E_2) = \emptyset$. Set $A := \mathcal{B}_r(a) \setminus \left(\overline{\mathrm{Int}(J_{E_1})} \cup \overline{\mathrm{Int}(J_{E_2})}\right)$. This set has two connecting components $A_\pm$ having nonempty intersection with the ray with angle $\theta_\pm$. The set A_+ (and A_-, respectively) has the role of the set $\mathrm{Int}(J)$ in the proof of Proposition 5.12. By Proposition 5.14 and Corollary 5.15, we can use the same argumentation (using the definition of connectedness) as in the proof of Proposition 5.12 with obvious changes. Note that we clearly have $\partial\mathcal{U}^{[j]} \subset \partial\mathcal{U}$ for all $j \in \{1, 2\}$, since the boundary of every connecting component of $\mathcal{U}$ is part of the boundary of $\mathcal{U}$. We can conclude that each connecting component cannot be covered by the two nonempty open sector components of $\mathcal{U}$, i.e. there exist closed sets $\tilde{A}_\pm \subset A_\pm$ (they have the role of the set A in the proof of Proposition 5.12). The existence of $\Gamma_\pm \in \partial\mathcal{U}^{[j]}$ and $p_\pm \in \Gamma_\pm \cap \partial\mathcal{B}_r(a)$ can be constructed in the same way as in the proof of Proposition 5.12.

The two excluded geometrical structures at the end of the proof of Proposition 5.12 can also not occur in our case, since we only have two directions, i.e. all directions are trivially adjacent in our case.

□

Corollary 5.17

Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $F \not\equiv 0$ and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m \in \mathbb{N} \setminus \{1\}$. Then all global sectors in a are unbounded.

Proof
Let $\mathcal{U}$ be an arbitrary global sector in a. If $m = 2$, there exists at least one unbounded orbit $\Gamma \subset \partial\mathcal{U}$, cf. Proposition 5.12. By Proposition 5.16, the same holds for $m \geq 3$. Hence $\partial\mathcal{U}$ is unbounded. This implies that $\mathcal{U}$ is also unbounded.

□

Proposition 5.18
Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $F \not\equiv 0$ and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m \in \mathbb{N} \setminus \{1\}$. Let $\theta_+, \theta_- \in \mathcal{E}(F, a)$ be two adjacent directions. Set $\mathcal{U} := \mathcal{U}(a, \theta_+, \theta_-)$. Then $\operatorname{Int}(\Gamma(x) \cup \{a\}) \subset \mathcal{U}$ for all $x \in \mathcal{U}$. If $m = 2$, we even have $\operatorname{Int}(\Gamma(x) \cup \{a\}) \subset \mathcal{U}^{[j]}$ for all $x \in \mathcal{U}^{[j]}$, $j \in \{1, 2\}$, with respect to a chosen pair $(q_1, q_2) \in \mathbb{C}^2$ separating $\mathcal{U}$ with corresponding orbits $\Gamma_1, \Gamma_2 \subset \mathcal{U}$.

Proof
Let $x \in \mathcal{U}$ be arbitrary. Set $J := \Gamma(x) \cup \{a\}$ and $A := F^{-1}(\{0\}) \cap \operatorname{Int}(J)$. Suppose $A \neq \emptyset$, i.e. there exists at least one equilibrium $\tilde{a}$ in $\operatorname{Int}(J)$.
Suppose $\tilde{a}$ is a node, focus or center, i.e. has order 1. The orbit $\Gamma(x)$ is neither periodic nor tends to $\tilde{a}$, i.e. $\Gamma(x)$ has empty intersection with the global neighbourhood $\mathcal{U}$ in $\tilde{a}$. In addition, if a lay in $\mathcal{U}$, we would have a contradiction to the openness of $\mathcal{U}$, cf. Theorem 5.6. Note that $a \in \partial\Gamma(x)$. Hence, by connectedness of $\mathcal{U}$, we have $\mathcal{U} \subset \operatorname{Int}(J)$, i.e. $\mathcal{U}$ is bounded. This a contradiction to Corollary 5.9.
Suppose $\tilde{a}$ has order at least 2 and let S be a global sector in $\tilde{a}$. Then ∂S consists of $\tilde{a}$ and unbounded orbits, cf. Theorem 5.8. But $\Gamma(x)$ and $\Gamma(a) = \{a\}$ are both bounded. By connectedness of $S \cup \tilde{a}$ (cf. Theorem 5.6), we must have $S \subset \operatorname{Int}(J)$. This is a contradiction.
We conclude $A = \emptyset$. Let $y \in \operatorname{Int}(J)$ be arbitrary. Then $\Gamma(y)$ is bounded. If $\Gamma(y) \subset \operatorname{Int}(J)$ was periodic, then there would be an equilibrium in $\operatorname{Int}(\Gamma(y))$. But we clearly have $\operatorname{Int}(\Gamma(y)) \subset \operatorname{Int}(J)$, i.e. a contradiction to $A = \emptyset$. Hence we can apply Corollary 4.35 to conclude $w_+(\Gamma(y)) \cup w_-(\Gamma(y)) \subset F^{-1}(\{0\})$. Since $A = \emptyset$, we also have

$$\overline{\operatorname{Int}(J)} \cap F^{-1}(\{0\}) = J \cap F^{-1}(\{0\}) = \{a\}.$$

Thus we must have $w_+(\Gamma(y)) = w_-(\Gamma(y)) = \{a\}$.
If $m = 2$, we only have two definite directions and thus automatically $y \in \mathcal{U}$. Note that there exists only one global sector in this case. If $m \geq 3$, we can apply Proposition 5.12 to

conclude the existence of an unbounded orbit Γ_θ attached to a for all $\theta \in \mathcal{E}(F,a)$. These orbits separate the plane in disjoint connected sets all having a on its boundary. Since $\mathrm{Int}(J)$ is connected, this set lies in exactly one of these connecting components between two adjacent definite directions. This implies that there are no other possible definite directions for $\Gamma(y)$ in $\mathrm{Int}(J)$, i.e. $y \notin \mathcal{U}$. We conclude $\mathrm{Int}(\Gamma(x) \cup \{a\}) \subset \mathcal{U}$.
Fix $j \in \{1,2\}$ and assume $x \in \mathcal{U}^{[j]}$. Let $y \in \mathrm{Int}(J)$ be again arbitrary. We have to show $y \in \mathcal{U}^{[j]}$. If $y \in \overline{\mathrm{Int}(\Gamma_j)}$, there is nothing to show, see the definition in Proposition 5.14. If $y \in \mathrm{Ext}(\Gamma_j)$, we must have $x \in \mathrm{Ext}(\Gamma_j)$, i.e. $y \in \mathrm{Int}(J) \setminus \overline{\mathrm{Int}(\Gamma_j)}$. Set $\tilde{j} := 1 - j^2$, i.e. $\{\tilde{j}\} = \{1,2\} \setminus \{j\}$. If we had $q_{\tilde{j}} \in \mathrm{Int}(\Gamma(y) \cup \{a\})$, then we would get $q_{\tilde{j}} \in \mathrm{Int}(J)$, i.e. $x \in \mathcal{U}^{[\tilde{j}]}$. This is a contradiction to $\mathcal{U}^{[1]} \cap \mathcal{U}^{[2]} = \emptyset$, cf. Proposition 5.14. Since we also have $\mathcal{U} = \mathcal{U}^{[1]} \cup \mathcal{U}^{[2]}$, we conclude $q_j \in \mathrm{Int}(\Gamma(y) \cup \{a\})$ and thus $y \in \mathcal{U}^{[j]}$.

$\square$

Remark 5.19
Proposition 5.18 shows the assertion stated in Remark 5.2 d). Moreover, it is now clear why the global sectors are indeed a global point of view of (local) elliptic sectors, if the order of the equilibrium is at least 3. If the order is 2, the two sector components can be viewed as the global version of the (local) elliptic sectors. We summarize the following results:
If $m \geq 3$, for every elliptic sector exists a global sector having the elliptic sector as subset and every orbit in a global sector forms an elliptic sector (or an elliptic region). The global sector with its two unique boundary orbits near the equilibrium is unique. The elliptic sector depends on the choice of the characterisic orbits and is not unique. The same holds for $m = 2$, if we consider the two sector components of the unique global sector instead of the $2m - 2$ (cf. Theorem 4.24) global sectors.
Furthermore, we can now characterize, whether there exists a „global parabolic region", see in particular the remarks at the end of Example 4.28.
If the space near a ray with angle θ between two adjacent elliptic sectors without the corresponding global sectors (the set A in the proof of Proposition 5.12) consists of exactly one orbit Γ, then this orbit is the (common) boundary orbit of these two global sectors near this ray. This orbit is unique and unbounded. In that case, there is no space left between these two sectors and we do not have such a „global parabolic region". For example, this can be observed in Example 4.27. The boundary orbits (the rays) are the common boundary orbits of any two adjacent global sectors.
If this set A, between two adjacent global sectors, consists of more than one orbit, then

there does not exist a common boundary orbit. Note that the global sectors (with the equilibrium) are connected and that the boundary orbits separate the plane in disjoint connecting components. We have seen that there cannot be another global sector in this set A. Hence, the space between these two boundary orbits is in some sense a „global parabolic region“. The orbits in A tend only in one time direction to the equilibrium. For example, this is the case in Example 4.28 near the ray on the negative $\Re$-axis.

Theorem 5.20
Let $F \in \mathcal{O}(\mathbb{C})$ be entire and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m \in \mathbb{N} \setminus \{1\}$. Let $\theta_+, \theta_- \in \mathcal{E}(F, a)$ be two adjacent directions. If $m \geq 3$, then $\mathcal{U} := \mathcal{U}_s(a, \theta_+, \theta_-)$ is connected, path-connected and simply connected. If $m = 2$, then the sector components of $\mathcal{U}$ are connected, path-connected and simply connected.

Proof
First, assume $m \geq 3$. By Proposition 5.12, there exist two unbounded orbits $\Gamma_1, \Gamma_2 \subset \mathcal{U}$ satisfying $w_+(\Gamma_1) = w_-(\Gamma_2) = \{a\}$. In addition, there exists an $r > 0$ and two points $p_1 \in \Gamma_1 \cap \partial\mathcal{B}_r(a)$, $p_2 \in \Gamma_2 \cap \partial\mathcal{B}_r(a)$ such that

$$A := \overline{\mathcal{B}_r(a)} \cap \partial\mathcal{U} = \{a\} \cup \Gamma_+(p_1) \cup \Gamma_-(p_2). \tag{5.6}$$

Hence the set A looks like a (continuously deformed) “piece of cake„. This set is clearly connected. Moreover, for all $x \in \mathcal{U}$ there exists a $T > 0$ such that $\Phi((-\infty, -T], x) \cup \Phi([T, \infty), x)\} \subset A$, i.e. every orbit in $\mathcal{U}$ reaches the set A for sufficiently large times and stays there.
Suppose now there exists a separation (U, V) of $\mathcal{U}$. Since A is connected, we must have either $A \subset U$, or $A \subset V$ (but not both), cf. [11, Lemma 23.2]. Assume w.l.o.g. $A \subset U$. Since $V \neq \emptyset$, there exists a $y \in V$ and a $T > 0$ such that $\Phi(T, y) \in A$. Since $\Gamma(y)$ is connected, we conclude, by [11, Theorem 23.3], that also $A \subset \Gamma(y)$ is connected, i.e. we must have $A \subset \Gamma(y) \subset U$. This leads to the contradiction $U \cap V \neq \emptyset$. Thus there does not exist a separation of $\mathcal{U}$ and $\mathcal{U}$ is connected.
Since $\mathcal{U}$ is open (cf. Theorem 5.6), we can apply [19, Proposition 12.25] to conclude the path-connectedness of $\mathcal{U}$. Constructing a path between two arbitrary points in $\mathcal{U}$ is not straightforward. This is only possible, if we add the point a, cf. part (iii) of the proof of Theorem 5.6.
Let $L \subset \mathcal{U}$ be a loop. For every $x \in L$ the set $J_x := \Gamma(x) \cup \{a\}$ is a closed Jordan

curve. Furthermore, by openness of $\mathcal{U}$, for every $x \in L$ there exists an $r > 0$ such that $\mathcal{B}_r(x) \subset \mathcal{U}$. Analogous to the argumentation at the end of the proof of Theorem 5.6 (i), by the Jordan curve theorem, there exists a $y_x \in \mathrm{Ext}(J_x) \cap \mathcal{B}_r(x)$ satisfying $\Gamma(x) \subset \mathrm{Int}(J_{y_x})$ and $J_x \cap J_{y_x} = \{a\}$. Note that orbits (the equilibrium a included) cannot cross each other. Hence we can use exactly the same idea as in the proof of Theorem 5.6 (i). One replaces the periodic orbits $\Gamma(y_x)$, $x \in \mathcal{U}$, by the curves J_{y_x}, $x \in \mathcal{U}$. Thus we can find a finite open cover of L and a outermost curve $J \subset \mathcal{U}$ satisfying $a \in J$ and $L \subset \mathrm{Int}(J)$. By Proposition 5.18, we have $\mathrm{Int}(J) \subset \mathcal{U}$, which is clearly simply connected. Hence L can be continuously deformed to a point $p \in L \subset \mathcal{U}$.
Second, assume $m = 2$. By Proposition 5.16, we can use the same argumentation for $\mathcal{U}^{[1]}$ and $\mathcal{U}^{[2]}$ (instead of $\mathcal{U}$). It follows that the two sector components of $\mathcal{U}$ are also connected, path-connected and simply connected. For the path-connectedness, one uses [19, Proposition 12.25] also in this case, since the construction of a specific path is not straightforward.

□

5.3 Separatrices as boundary orbits of global neighbourhoods

The last two sections laid the foundation for our third major result: Every orbit on the boundary of a center or global sector is a so-called separatrix. The boundary of foci and nodes consist of separatrices and equilibria with attached separatrices. This result is proven by using the concept of transit times, which are introduced in the following.

Definition 5.21
Let $\Omega \subset \mathbb{C}$ be an open domain and $F \in \mathcal{O}(\Omega)$. Let $\Gamma \subset \Omega$ be an arbitrary orbit (not an equilibrium) of (4.1) and $a, b \in \Gamma$.

a) The *transit time* $\tau(\Gamma)$ of Γ is defined as the Lebesgue measure of the maximum interval of existence of Γ, i.e.

$$\tau(\Gamma) := \lambda(I(x))$$

for an arbitrary $x \in \Gamma$. The definition does not depend on x. If $I(x) \neq \mathbb{R}$, we have $\tau(\Gamma) = \sup I(x) - \inf I(x)$.

b) The *transit time* $\tau(a,b)$ from a to b is defined by

$$\tau(a,b) := \int_{\Gamma(a,b)} \frac{1}{F(z)}\,\mathrm{d}z.$$

In this notation $\Gamma(a,b)$ is the piece of Γ from a to b parameterized as solution of (4.1). Note that if $z_0 \in \Gamma$, $a = \Phi(t_1, z_0)$ and $b = \Phi(t_2, z_0)$, then $\tau(a,b) = t_2 - t_1$. This follows from the definition of the complex line integral.

Lemma 5.22
Let $\Omega \subset \mathbb{C}$ be an open domain, $F \in \mathcal{O}(\Omega)$ and $\Gamma \subset \Omega$ an orbit of (4.1). Assume that Γ is not periodic. Then

$$\tau(\Gamma) = \sup_{x,y\in\Gamma} \tau(x,y).$$

Proof
Fix $x \in \Gamma$. Since Γ is not periodic, the function $\varphi_x : I(x) \to \Gamma$, $\varphi_x(t) := \Phi(t,x)$, is a bijection. The inverse function is given by $\varphi_x^{-1}(y) = \tau(x,y)$, $y \in \Gamma$.
By using this, for all $y \in \Gamma$ there exists $t_y := \tau(x,y) \in I(x)$ such that $\varphi_x(t_y) = y$. Thus

$$\tau(\Gamma) = \lambda(I(x)) \geq \lambda\left([0, |t_y|]\right) = |\tau(x,y)|.$$

Since x is arbitrary, we conclude the inequality

$$\tau(\Gamma) \geq \sup_{x,y\in\Gamma} \tau(x,y).$$

Suppose, this inequality is strict and assume that $\tau(\Gamma) < \infty$, i.e. the maximum interval of existence of Γ is bounded in $\mathbb{R}$. By assumption, there exists a $\varepsilon > 0$ such that for all $x, y \in \Gamma$ we have $\tau(\Gamma) - \varepsilon > \tau(x,y) = \varphi_x^{-1}(y)$. For fixed $z \in \Gamma$ there are $\alpha < 0$ and $\beta > 0$ such that $I(z) = (\alpha, \beta)$. Choose $x := \varphi_z\left(\alpha + \frac{\varepsilon}{2}\right) \in \Gamma$ and $x := \varphi_z\left(\beta - \frac{\varepsilon}{2}\right) \in \Gamma$. Since the flow defines a dynamical system, we conclude the contradiction

$$\varphi_x^{-1}(y) = \varphi_x^{-1}(z) + \varphi_z^{-1}(y) = -\left(\alpha + \frac{\varepsilon}{2}\right) + \beta - \frac{\varepsilon}{2} = \beta - \alpha - \varepsilon = \tau(\Gamma) - \varepsilon > \varphi_x^{-1}(y).$$

Thus such a ε does not exist and the inequality is not strict in the case $\tau(\Gamma) < \infty$.

Assume now $\tau(\Gamma) = \infty$ and set

$$M := \sup_{x,y\in\Gamma} |\tau(x,y)|.$$

By assumption, $0 \leq M < \infty$. Fix a $x \in \Gamma$. Since $I(x)$ is unbounded and connected (it is an interval), there exists $t \in \{M+1, -(M+1)\} \cap I(x) \neq \emptyset$. But now we cleary have $|\tau(x, \varphi_x(t))| = |t| = M+1 > M$. Hence $\varphi_x(t) \notin \Gamma$, which is a contradiction to the fact that φ_x is a surjection. Thus $M = \infty$ and the above inequality is not strict also in this case.

$\square$

Lemma 5.23
Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $F \not\equiv 0$, $a \in \mathbb{C}$ a center of (4.1) and $\Gamma \subset \partial\mathcal{U}_c(a)$ an arbitrary orbit on the boundary of a. Fix $x, y \in \Gamma$ with $\tau(x,y) > 0$ and $\varepsilon > 0$. Then there exists a $\delta \in (0, \varepsilon]$ such that $\mathcal{B}_\delta(x) \cap \mathcal{B}_\delta(y) = \emptyset$ and for all orbits $\Lambda \subset \mathcal{U}_c(a)$ satisfying $\mathcal{B}_\delta(x) \cap \Lambda \neq \emptyset$ and $\mathcal{B}_\delta(y) \cap \Lambda \neq \emptyset$ it hold for all $x' \in \mathcal{B}_\delta(x) \cap \Lambda$ and $y' \in \mathcal{B}_\delta(y) \cap \Lambda$ the inequalities

$$|\tau(x', y') - \tau(x,y)| < \varepsilon \tag{5.7}$$

and

$$|\Phi(t,x) - \Phi(t,x')| < \varepsilon \quad \forall\, t \in [0, \tau(x,y)]. \tag{5.8}$$

Furthermore, if $\mathcal{B}_\delta(x) \cap \Lambda \neq \emptyset$ and $\mathcal{B}_\delta(y) \cap \Lambda \neq \emptyset$, then $\mathcal{B}_\delta(x) \cap \Gamma(z) \neq \emptyset$ and $\mathcal{B}_\delta(y) \cap \Gamma(z) \neq \emptyset$ for all $z \in \operatorname{Ext}(\Lambda) \cap \mathcal{U}_c(a)$.

Proof
Set $K := \Gamma(x,y) \subset \partial\mathcal{U}_c(a)$ as the curve piece of the orbit Γ from x to y. Since K is compact and $F \not\equiv 0$, we have

$$\overline{\varepsilon_0} := \frac{\inf\{\operatorname{dist}(K,b) : F(b) = 0\}}{2} > 0.$$

Note that the zeros of F cannot lie arbitrarily close to K, see also the argumentation in the proof of Proposition 4.6.

Choose $\varepsilon_0 \in (0, \overline{\varepsilon_0})$ sufficiently small and set

$$O := \bigcup_{\xi \in K} \mathcal{B}_{\varepsilon_0}(\xi)$$

as a small simply connected open neighbourhood of K. The set O looks like an „elongated tube" away from the zeros of F. It is homeomorphic to a rectangle in C, if ε_0 is sufficiently small. In addition, K is not periodic and thus homeomorphic to a compact interval in $\mathbb{R}$. Hence O is indeed simply connected. With this choice, F has no zeros on $\overline{O}$ and $F^{-1} \in \mathcal{O}(O)$.

Furthermore, we find a $\delta_1 > 0$ such that $\mathcal{B}_{\delta_1}(x) \cap \mathcal{B}_{\delta_1}(y) = \emptyset$. Note that $x \neq y$, since $\tau(x, y) \neq 0$. By the continuous dependence on initial conditions, [9, Chapter 2.4, Theorem 4], there exists a $\delta_2 > 0$ such that $|\Phi(t, z) - \Phi(t, x)| < \min\{\varepsilon, \varepsilon_0\}$ for all $t \in [0, \tau(x, y)]$ and $z \in \mathcal{B}_{\delta_2}(x)$. Define $M := \min_{z \in \overline{O}} |f(z)| > 0$. Then the number

$$\delta := \min\left\{\varepsilon, \varepsilon_0, \delta_1, \delta_2, \frac{M\varepsilon}{2}\right\} > 0.$$

is sufficiently small for our assertions. In fact, let $\Lambda \subset \mathcal{U}_c(a)$ be an arbitrary periodic orbit with a in its interior such that $\mathcal{B}_\delta(x) \cap \Lambda \neq \emptyset$ and $\mathcal{B}_\delta(y) \cap \Lambda \neq \emptyset$. Let $x' \in \mathcal{B}_\delta(x) \cap \Lambda$ and $y' \in \mathcal{B}_\delta(y) \cap \Lambda$ be arbitrary. Then equation (5.8) is already satisfied, since $\delta \leq \delta_2$. Set $\tilde{K} := \Lambda(x', y') \subset \partial\mathcal{U}_c(a)$ as the curve piece of the orbit Λ from x' to y'. Let Ξ_1 be the convex combination (straight connection line) of x and x'. Let Ξ_2 be the convex combination of y' and y. Since $\delta \leq \delta_2$, these convex combinations lie both in $\mathcal{B}_\delta(x)$ and $\mathcal{B}_\delta(y)$, respectively. Note that balls in $\mathbb{C}$ are always convex with respect to the Euclidean metric. By construction and the choice of δ_2, the path $\Xi := \Xi_1 + \tilde{K} + \Xi_2 + K$ is a closed Jordan curve lying completely in O. Since O is simply connected, Ξ is null-homotopic in O. By applying Definition 5.21 and the homotopy version of Cauchy's Integral Theorem, we conclude

$$\tau(x', y') - \tau(x, y) = \int_{\tilde{K}} F^{-1}\,\mathrm{d}z - \int_K F^{-1}\,\mathrm{d}z = \underbrace{\int_{\Xi} F^{-1}\,\mathrm{d}z}_{=0} - \int_{\Xi_1} F^{-1}\,\mathrm{d}z - \int_{\Xi_2} F^{-1}\,\mathrm{d}z$$

and thus

$$|\tau(x', y') - \tau(x, y)| = \left| \int_{\Xi_1} F^{-1}\,\mathrm{d}z + \int_{\Xi_2} F^{-1}\,\mathrm{d}z \right| \leq \frac{|x - x'|}{M} + \frac{|y - y'|}{M} < \frac{2\delta}{M} \leq \frac{M\varepsilon}{M} = \varepsilon.$$

This proves equation (5.7).
Moreover, let $z \in \mathrm{Ext}(\Lambda) \cap \mathcal{U}_c(a)$ be arbitrary. Then $K \subset \mathrm{Ext}(\Gamma(z))$ and $\tilde{K} \subset \mathrm{Int}(\Gamma(z))$. Since Ξ is periodic and path-connected (as subset of $\mathbb{C}$), there must exist two points $p_1, p_2 \in \Gamma(z) \cap (\Xi_1 \cup \Xi_2)$. Since $a \in \mathrm{Int}(\Gamma(z))$ and $\partial\mathcal{U}_c(a) \subset \mathrm{Ext}(\Gamma(z))$, p_1 and p_2 do not lie both on Ξ_1 or Ξ_2, i.e. w.l.o.g. $p_1 \in \Xi_1$ and $p_2 \in \Xi_2$. Since $\Xi_1 \subset \mathcal{B}_\delta(x)$ and $\Xi_2 \subset \mathcal{B}_\delta(y)$, we have $p_1 \in \mathcal{B}_\delta(x) \cap \Gamma(z) \neq \emptyset$ and $p_2 \in \mathcal{B}_\delta(y) \cap \Gamma(z) \neq \emptyset$.

□

Lemma 5.24
Let $F \in \mathcal{O}(\mathbb{C})$ be entire, $F \not\equiv 0$ and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m \in \mathbb{N} \setminus \{1\}$. Assume that either a is a node/focus or $m \geq 2$. Let $\mathcal{U}$ be the global neighbourhood, a global sector (with definite directions $\theta_+, \theta_- \in \mathcal{E}(F, a)$ in the case $m \geq 3$) or a sector component (in the case $m = 2$). Let $\Gamma \subset \partial\mathcal{U}$ be an arbitrary orbit of (4.1) not being an equilibrium. Fix $x, y \in \Gamma$ with $\tau(x, y) > 0$ and $\varepsilon > 0$. Then there exists a $\delta \in (0, \varepsilon]$ such that $\mathcal{B}_\delta(x) \cap \mathcal{B}_\delta(y) = \emptyset$ and for all orbits $\Lambda \subset \mathcal{U}_s(a, \theta_+, \theta_-)$ satisfying $\mathcal{B}_\delta(x) \cap \Lambda \neq \emptyset$ and $\mathcal{B}_\delta(y) \cap \Lambda \neq \emptyset$ it hold for all $x' \in \mathcal{B}_\delta(x) \cap \Lambda$ and $y' \in \mathcal{B}_\delta(y) \cap \Lambda$ the inequalities

$$|\tau(x', y') - \tau(x, y)| < \varepsilon \tag{5.9}$$

and

$$|\Phi(t, x) - \Phi(t, x')| < \varepsilon \quad \forall\, t \in [0, \tau(x, y)]. \tag{5.10}$$

Proof
The proof of this Lemma is equal to the proof of Lemma 5.23 with obvious changes. By Theorem 5.8, it is indeed possible in this case that we have equilibria on the boundary of $\mathcal{U}$. The assumption that Γ is not an equilibrium ensures $\overline{\varepsilon_0} > 0$. In particular, $\Gamma \neq \{a\}$ in the case $m \geq 2$. Finally, the same δ is sufficiently small such that the equations (5.9) and (5.10) hold. For a proof one uses also the homotopy version of Cauchy's Integral Theorem for a suitably constructed null-homotopic curve.

□

Proposition 5.25
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F \in \mathcal{O}(\Omega)$ and $a \in \Omega$ a center of (4.1). Let $\Gamma \subset \mathcal{U}_c(a)$ be a periodic orbit of (4.1) with $a \in \operatorname{Int}(\Gamma)$. Then the period T of Γ is equal to $\frac{2\pi\mathrm{i}}{F'(a)}$. In particular, T does not depend on the chosen orbit Γ.

Proof
The period T is the transit time over one trip around Γ. We calculate

$$T = \int\limits_{\Gamma} \frac{1}{F(z)}\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}\left(\frac{1}{F}, a\right) = 2\pi\mathrm{i}\lim_{z\to a}\frac{z-a}{F(z)} = \frac{2\pi\mathrm{i}}{\lim_{z\to a}\frac{F(z)-F(a)}{z-a}} = \frac{2\pi\mathrm{i}}{F'(a)}.$$

In the penultimate transformation we used $F(a) = 0$ and $F'(a) \neq 0$, cf. Corollary 4.25.

$\square$

Definition 5.26
Let $\Omega \subset \mathbb{C}$ be a simply connected open domain, $F \in \mathcal{O}(\Omega)$ and $a \in \Omega$ a center of (4.1). Then the *period* $T(a)$ of a is defined as the period of an arbitrary orbit in $\mathcal{U}_c(a)$, i.e. $T(a) := \frac{2\pi\mathrm{i}}{F'(a)}$. By Proposition 5.25, this number is well-defined.

Definition 5.27
Let $F \in \mathcal{O}(\mathbb{C})$ be an entire vector field, Γ an arbitrary orbit of (4.1) and $x_0 \in \Gamma$. If $I(x_0) \cap [0,\infty) \subset \mathbb{R}$ is bounded, Γ is called a positive separatrix. If $I(x_0) \cap (-\infty, 0] \subset \mathbb{R}$ is bounded, Γ is called a negative separatrix. Γ is a separatrix, if it is a positive or negativ separatrix.

Remark 5.28

a) Whether an orbit is a separatrix or not, does not depend on the chosen point x_0 in Definition 5.27. In fact, the boundedness of the maximum interval of existence is independent of the chosen point on the orbit.

b) Every separatirx must be unbounded and forms a blow-up. In fact, the vector field is entire and the maximum interval of existence is bounded in at least one time direction. Hence, the orbit cannot be bounded in this unbounded time direction, cf. Remark 3.3. Note that $\partial\mathbb{C} = \emptyset$.

c) The definition of separatrix lacks consistency in the literature. For our complex analytic case, Definition 5.27 is suitable. Generally, separatrices are known as the specific orbits forming the boundary between two regions with different geometry and topology. As we will demonstrate later in this chapter, our definition aligns with this commonly accepted notion. A more detailed analysis of this topic and the term „separatrix“ can be found in [20, Chapter II] and [9, Chapter 3.11]. Additionally, in [1, Chapter 1.9], the author introduces the definition of different types of sectors (cf. Definition 4.14) and describes the separatrices as the boundaries of these sectors.

Lemma 5.29

Let $F \in \mathcal{O}(\mathbb{C})$ be an entire vector field and Γ an arbitrary orbit of (4.1). Then it holds:

a) Γ is a positive and negative separatrix if and only if $\tau(\Gamma) < \infty$.

b) Γ is a positive separatrix if and only if there exists a $x \in \Gamma$ such that

$$\sup_{y \in \Gamma_+(x)} \tau(x, y) < \infty.$$

c) Γ is a negative separatrix if and only if there exists a $\tilde{x} \in \Gamma$ such that

$$\sup_{y \in \Gamma_-(\tilde{x})} \tau(\tilde{x}, y) > -\infty.$$

Proof

The Lebesgue measure of an interval in $\mathbb{R}$ is finite if and only if it is bounded. Hence, the last assertion follows directly by Definition 5.27.

The proof of the first and second assertion works with the same argumentation as in the proof of Lemma 5.22 with obvious changes. Note that only these $y \in \Gamma$ are relevant, where $\tau(x, y) > (<)\, 0$, if we want to show the assertion for positive (negative) separatrices.

$\square$

Lemma 5.30

Let $\Omega \subset \mathbb{C}$ be an open domain and $F \in \mathcal{O}(\Omega)$, $F \not\equiv 0$. Then the set $A := \{\Gamma(x) : x \in \Omega\}$ of all orbits of (4.1) in Ω is countable.

Proof

At first, note that A is a set of sets. For all $x \in \Omega$ and $y \in \Gamma(x)$ we have $\Gamma(x) = \Gamma(y) \subset \Omega$. Define the set

$$\tilde{A} := \left\{\Gamma(x) : x \in \Omega \cap \mathbb{Q}^2\right\} \cup \left\{\{a\} : a \in F^{-1}(\{0\})\right\}.$$

Since $F \not\equiv 0$, there exist at most countably many equilibria in Ω. Furthermore, $\mathbb{Q}^2$ is countable. Hence $\tilde{A}$ is countable as union of two countable sets. It remains to show that $A = \tilde{A}$.

Since $\Omega \cap \mathbb{Q}^2 \subset \Omega$, the first subset relation $\tilde{A} \subset A$ is obvious. For the second relation, let $\Gamma \in \tilde{A}$ and suppose that $\Gamma \notin \tilde{A}$. In particular, this implies $\Gamma \subset \Omega \cap (\mathbb{R} \setminus \mathbb{Q})^2$. Let $\pi_j(\Gamma)$ be the projection of Γ onto the the j^{th} coordinate, $j \in \{1, 2\}$. Since π_j is continuous and Γ is connected, we can apply [11, Theorem 23.5] to conclude that $\pi_j(\Gamma)$ must also be connected, $j \in \{1, 2\}$. But for all $j \in \{1, 2\}$ the set $\pi_j(\Gamma) \subset \mathbb{R} \setminus \mathbb{Q}$ is only connected, if $\pi_j(\Gamma)$ consists of only one point. Hence there exists a point $a \in \Omega$ such that $\Gamma = \{a\}$. This implies that Γ must be an equilibrium and $a \in F^{-1}(\{0\})$, i.e. $\Gamma \in \tilde{A}$. This is a contradiction.

$\square$

Theorem 5.31 (Separatrix configuration on the boundary of centers)
Let $F \in \mathcal{O}(\mathbb{C})$, $F \not\equiv 0$, be entire and $a \in \mathbb{C}$ a center of (4.1). Then the boundary of a consists of at least countably many separatrices, i.e. there exists a index set $\mathcal{Q} \subset \mathbb{N}$ and separatrices $C_n \subset \partial\mathcal{U}_c(a)$, $n \in \mathcal{Q}$, such that

$$\partial\mathcal{U}_c(a) = \bigcup_{n \in \mathcal{Q}} C_n.$$

Furthermore, the sum of the transit times on the boundary is bounded by the period of a. More precisely,

$$\sum_{n \in \mathcal{Q}} \tau(C_n) \leq T(a) = \frac{2\pi \mathrm{i}}{F'(a)}. \tag{5.11}$$

In particular, every orbit on the boundary of a is a positive and negative separatrix.

Proof

The basic idea follows the proof of [3, Theorem 4.1, step 2].
If $\partial\mathcal{U}_c(a) = \emptyset$, nothing is to show. So assume that the boundary of a is not empty. Then Proposition 5.4 ensures that the boundary of a is the union of orbits (without equilibria). By Lemma 5.30, these orbits can be indexed by a set $\mathcal{Q} \subset \mathbb{N}$, i.e. the boundary of a consists of orbits C_n, $n \in \mathcal{Q}$. By Theorem 5.8, these orbits are unbounded. In the following, we proof that they are even positive and negative separatrices. Actually, this result already implies Theorem 5.8. For this implication one uses [9, Chapter 2.4, Theorem 2].
Let $n \in \mathcal{Q}$, $\varepsilon > 0$ and $x, y \in C_n$ with $\tau(x, y) > 0$ be fixed. The last condition is possible, since there are no equilibria on $\partial\mathcal{U}_c(a)$, cf. Proposition 5.5. By Lemma 5.23, there exists a $\delta \in (0, \varepsilon]$ such that for all orbits $\Lambda \subset \mathcal{U}_c(a)$ satisfying $\mathcal{B}_\delta(x) \cap \Lambda \neq \emptyset$ and $\mathcal{B}_\delta(y) \cap \Lambda \neq \emptyset$ it holds that

$$|\tau(x', y') - \tau(x, y)| < \varepsilon \quad \forall\, x' \in \mathcal{B}_\delta(x) \cap \Lambda,\ \forall\, y' \in \mathcal{B}_\delta(y) \cap \Lambda.$$

By the continuous dependence on initial conditions, [9, Chapter 2.4, Theorem 4], there exists a $\tilde{\delta} \in (0, \delta]$ such that $|\Phi(\tau(x, y), z) - y| < \delta$ for all $z \in \mathcal{B}_{\tilde{\delta}}(x)$. Since x lies on the boundary of a, there exists indeed a point $z_0 \in \mathcal{B}_{\tilde{\delta}}(x) \cap \mathcal{U}_c(a)$, i.e. for $\Lambda := \Gamma(z_0) \subset \mathcal{U}_c(a)$, $x' := z_0 \in \mathcal{B}_\delta(x)$ and $y' := \Phi(\tau(x, y), z_0) \in \mathcal{B}_\delta(y)$ we can apply Lemma 5.23. Since C_n is unbounded, it cannot be periodic and thus $\tau(x', y') \leq T(a)$, cf. Proposition 5.25. We conclude

$$|\tau(x, y)| \leq |\tau(x', y')| + |\tau(x, y) - \tau(x', y')| \leq T(a) + \varepsilon.$$

Since x and y are arbitrary, it follows by Lemma 5.22

$$\tau(C_n) = \sup_{x,y \in C_n} \tau(x, y) \leq \sup_{x,y \in C_n} |\tau(x, y)| \leq \sup_{x,y \in C_n} T(a) + \varepsilon = T(a) + \varepsilon$$

Since ε is arbitrary, we get $\tau(C_n) \leq T(a) < \infty$, i.e. C_n is a positive and negative separatrix, cf. Lemma 5.29. And since n is arbitrary, we have proven that the boundary of a consists only of separatrices.
It remains to show equation (5.11). In the proof of [3, Theorem 4.1] are many inaccuracies and some mistakes. In particular, it is not ensured that the δ_i in this proof are small enough such that the sum of the transit times on the outermost „approximating“ periodic orbit is indeed bounded by the period of a. Because of this, the following proof is modified and made by myself.

Fix $\tilde{\varepsilon} > 0$, $N \in \mathbb{N} \setminus \{1\}$ and $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ with $|\tilde{\mathcal{Q}}| = N$, i.e. we fix $N \geq 2$ pairwise disjoint separatrices C_n, $n \in \tilde{\mathcal{Q}}$. Fix for all $n \in \tilde{\mathcal{Q}}$ points $x_n, y_n \in C_n$ such that $\tau(x_n, y_n) > 0$. Set $K_n := C_n(x_n, y_n)$, $n \in \tilde{\mathcal{Q}}$, and

$$\varepsilon_0 := \min_{i,j \in \tilde{\mathcal{Q}}} \operatorname{dist}(K_i, K_j) > 0.$$

Note that every metric space (like the Euclidean $\mathbb{C}$ in our case) satisfies the normality axiom[7], cf. [11, Theorem 32.2], i.e. closed disjoint sets can be separated from each other by open sets. Because of this, we indeed have $\varepsilon_0 > 0$.
For fixed $n \in \tilde{\mathcal{Q}}$ and $\varepsilon := \min\left\{\frac{\varepsilon_0}{4}, \frac{\tilde{\varepsilon}}{N}\right\}$ we can use our above result: There exist $\delta_n \in (0, \varepsilon]$, a orbit $\Lambda_n \subset \mathcal{U}_c(a)$ and points $x'_n \in \mathcal{B}_{\delta_n}(x_n) \cap \Lambda_n$ and $y'_n \in \mathcal{B}_{\delta_n}(y) \cap \Lambda_n$ such that Lemma 5.23 can be applied. Since $N < \infty$, we can apply equation (5.2) to conclude the existence of a $n_0 \in \tilde{\mathcal{Q}}$ such that Λ_{n_0} is the outermost orbit, i.e. $\Lambda_n \subset \overline{\operatorname{Int}(\Lambda_{n_0})} \subset \mathcal{U}_c(a)$ for all $n \in \tilde{\mathcal{Q}}$. By applying the last result in Lemma 5.23, for all $n \in \tilde{\mathcal{Q}}$ one can even find two points $x'_n \in \mathcal{B}_{\delta_n}(x_n) \cap \Lambda_{n_0}$ and $y'_n \in \mathcal{B}_{\delta_n}(y) \cap \Lambda_{n_0}$ such that Lemma 5.23 can be applied, i.e. the equations (5.7) and (5.8) hold.
Set $\tilde{K}_n := \Lambda_{n_0}(x'_n, y'_n)$ and suppose that there exist two indices $i, j \in \tilde{\mathcal{Q}}$ such that $\tilde{K}_i \cap \tilde{K}_j \neq \emptyset$. Then $\{x'_i, y'_i\} \cap \tilde{K}_j \neq \emptyset$. Assume $x'_i \in \tilde{K}_j$. The case $y'_i \in \tilde{K}_j$ can be proven in the same way. Set $t := \tau(x'_j, x'_i) > 0$ and $\eta_1 := \Phi(t, x_j) \in K_j$. Then we can use equation (5.8) to conclude

$$\operatorname{dist}(x'_i, K_j) \leq |\eta_1 - x'_i| = |\Phi(t, x_j) - \Phi(t, x'_j)| < \varepsilon \leq \frac{\varepsilon_0}{4}.$$

Set $\eta_2 := \Phi(t, x_i) \in K_i$. Again by equation (5.8), we have

$$\operatorname{dist}(x'_i, K_i) \leq |\eta_2 - x'_i| = |\Phi(t, x_i) - \Phi(t, x'_i)| < \varepsilon \leq \frac{\varepsilon_0}{4}.$$

By the choice of ε_0, both inequalities lead then to the contradiction

$$\varepsilon_0 \leq \operatorname{dist}(K_i, K_j) \leq \operatorname{dist}(x'_i, K_j) + \operatorname{dist}(x'_i, K_i) = \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{2}.$$

Thus $\tilde{K}_i \cap \tilde{K}_j = \emptyset$ and (cf. Proposition 5.25)

$$\sum_{n \in \tilde{\mathcal{Q}}} \tau(x'_n, y'_n) \leq \tau(\Lambda_{n_0}) \leq T(a).$$

[7] In the context of topological separation theory and T-spaces this property is sometimes also-called the T_4-axiom.

By equation (5.7), it follows

$$\sum_{n\in\tilde{\mathcal{Q}}}\tau(x_n,y_n)\leq\sum_{n\in\tilde{\mathcal{Q}}}|\tau(x_n',y_n')|+\underbrace{|\tau(x_n,y_n)-\tau(x_n',y_n')|}_{<\frac{\tilde{\varepsilon}}{N}}<T(a)+\frac{N\tilde{\varepsilon}}{N}=T(a)+\tilde{\varepsilon}.$$

Since x_n and y_n, $n\in\tilde{\mathcal{Q}}$, are arbitrary, it follows by Lemma 5.22

$$\sum_{n\in\tilde{\mathcal{Q}}}\tau(C_n)=\sum_{n\in\tilde{\mathcal{Q}}}\sup_{\substack{x_n\in C_n\\ y_n\in C_n}}\tau(x_n,y_n)=\sup\left\{\sum_{n\in\tilde{\mathcal{Q}}}\tau(x_n,y_n):x_n,y_n\in C_n\ \forall\, n\in\tilde{\mathcal{Q}}\right\}\leq T(a)+\tilde{\varepsilon}$$

In the second equality we used the fact that for all $n\in\tilde{\mathcal{Q}}$ the number $\tau(x_n,y_n)$ does not depend on the choice of $x_m,y_m\in C_m$, $m\in\tilde{\mathcal{Q}}\setminus\{n\}$.
Since $\tilde{\varepsilon}$ and N are arbitrary, we conclude equation (5.11) for finite index sets. A priori, if $\mathcal{Q}$ is countable (and not finite), the sum of the transit times could depend on the summation order. Let $\mathcal{Q}_N$ be the set of the first N indices in $\mathcal{Q}$ and define

$$\mu_N:=\sum_{n\in\mathcal{Q}_N}\tau(C_n).$$

Then $(\mu_N)_{N\in\mathbb{N}}\subset[0,T(a)]$ is bounded and strictly monotonously increasing, since transit times of orbits are always positive. Thus there exists a $\mu\in[0,T(a)]$ such that $\mu_N\to\mu$ for $N\to\infty$. Since all terms of this series are positive, this convergence is absolute. Hence, by the Levy-Steinitz theorem, the value of the series does not depend on the summation order and the notation in (5.11) is indeed well-defined. Finally, we conclude

$$\sum_{n\in\mathcal{Q}}\tau(C_n)=\lim_{N\to\infty}\mu_N=\mu\leq T(a).$$

We summarize, that the choice of ε and ε_0 ensures the correctness of [3, Theorem 4.1].

□

Theorem 5.32 (Separatrix configuration on the boundary of nodes and foci)
Let $F\in\mathcal{O}(\mathbb{C})$, $F\not\equiv 0$, be entire and $a\in\mathbb{C}$ a node or focus of (4.1). Assume that $\partial\mathcal{U}_n(a)\cap F^{-1}(\{0\})$ does no have isolated points in $\partial\mathcal{U}_n(a)$, i.e. for all $\tilde{a}\in\partial\mathcal{U}_n(a)\cap F^{-1}(\{0\})$ and all $\rho>0$ it holds that $(\mathcal{B}_\rho(\tilde{a})\cap\partial\mathcal{U}_n(a))\setminus\{\tilde{a}\}\neq\emptyset$. Then the path-connecting components of $\partial\mathcal{U}_n(a)$ can be indexed by a at most countable index set $\mathcal{Q}\subset\mathbb{N}$, i.e. the path-connecting

components $\{C_n\}_{n\in\mathcal{Q}}$ satisfy

$$\partial\mathcal{U}_n(a) = \bigcup_{n\in\mathcal{Q}} C_n. \tag{5.12}$$

Furthermore, for all $n \in \mathcal{Q}$ the set $C_n \subset \partial\mathcal{U}_n(a)$ is of one of the following types:

(i) The set C_n consists of exactly one equilibrium a_n and one attached separatrix $C_n^{[1]}$. This separatrix is positive (negative) if and only if a is stable (unstable).

(ii) The set C_n consists of exactly one equilibrium a_n and two attached separatrices $C_n^{[1]}$ and $C_n^{[2]}$. Both separatrices are positive (negative) if and only if a is stable (unstable).

(iii) The set C_n consists of exactly one separatrix $C_n^{[1]}$. This separatrix is positive (negative) if and only if a is stable (unstable).

Proof

In the following, many ideas and arguments are similar to these in the proof of Theorem 5.31. The basic idea follows the proof of [3, Theorem 4.3, step 6].

By Theorem 5.8 and Lemma 5.30, the boundary of a consists of at least countably many unbounded orbits and countably many nodes and foci. Since all orbits are clearly path-connected, the number of path-connecting components of $\partial\mathcal{U}_n(a)$ does not exceed the number of all orbits and equilibria in $\mathbb{C}$. This proves the existence of a at most countable index set $\mathcal{Q} \subset \mathbb{N}$ such that the path-connecting-components $\{C_n\}_{n\in\mathcal{Q}}$ indeed satisfy equation (5.12).

From now on, assume that a is stable. The unstable case can be proven analogously by reversing the direction of time. Let $n \in \mathcal{Q}$ be arbitrarily fixed.

First, assume that there exists an equilibrium $a_n \in C_n$. Since a is stable, $a_n \in \partial\mathcal{U}_n(a)$ must be unstable, i.e. a_n is reached with negative time. Since all orbits in C_n attached to a_n are unbounded, there cannot be heteroclinic orbits, i.e. a_n must be the only equilibrium in C_n. In addition, since orbits cannot cross each other, all orbit in C_n must be attached to C_n. Suppose that there are at least three unbounded orbits lying on $\partial\mathcal{U}_n(a)$ and reaching a_n with $t \to -\infty$. Since the Jordan curve theorem holds also on S^2 (cf. [11, Lemma 61.1]), these orbits separate $\mathbb{C}$ in exactly three unbounded nonempty connecting components. In addition, $\mathcal{U}_n(a)$ is connected (cf. Theorem 5.6), i.e. lies in exactly one of these connecting components. But now at least one of these three unbounded orbits does not lie on the boundary of a. This is a contradiction. Since $\partial\mathcal{U}_n(a) \cap F^{-1}(\{0\})$ does no have isolated points in $\partial\mathcal{U}_n(a)$, this case corresponds to (i) and (ii).

Second, assume that $C_n \cap F^{-1}(\{0\}) = \emptyset$, i.e. there are no equilibria in C_n. Since orbits cannot cross each other, $C_n \neq \emptyset$ must consist of exactly one unbounded orbit. This leads to (iii). Thus is remains to show that the orbits in the cases (i), (ii) and (iii) are not only unbounded, but even separatrices.
Assume that C_n is of type (i), i.e. $C_n = \{a_n\} \cup C_n^{[1]}$, where a_n is an unstable node or focus and $C_n^{[1]} \subset \partial\mathcal{U}_n(a)$ an unbounded orbit satisfying $w_-(C_n^{[1]}) = \{a_n\}$. We have to show that $C_n^{[1]}$ is a positive separatrix. Choose $r_1, r_2 > 0$ small enough such that $\overline{\mathcal{B}_{2r_1}(a)} \subset \mathcal{U}_n(a)$, $\overline{\mathcal{B}_{r_2}(a_n)} \cap F^{-1}(\{0\}) = \emptyset$, $\mathcal{B}_{r_1}(a) \cap \mathcal{B}_{r_2}(a_n) = \emptyset$, $w_+(\Gamma(z_1)) = \{a\}$ for all $z_1 \in \mathcal{B}_{r_1}(a)$ and $w_+(\Gamma(z_2)) = \{a_n\}$ for all $z_2 \in \mathcal{B}_{2r_2}(z_2)$, cf. Definition 4.1. Moreover, assume that r_1 and r_2 are sufficiently small such that $\mathcal{B}_{r_1}(a)$ and $\mathcal{B}_{r_2}(a_n)$ are positively and negatively invariant, respectively. Set

$$b_1 := \min\{|F(z)| : z \in \partial\mathcal{B}_{r_1}(a)\} > 0$$

and

$$b_2 := \min\{|F(z)| : z \in \partial\mathcal{B}_{r_2}(a_n)\} > 0.$$

Since $a_n \in \partial\mathcal{U}_n(a)$, there exists a $z_2 \in \mathcal{U}_n(a) \cap \mathcal{B}_{r_2}(a_n) \neq \emptyset$. Set $\Xi := \Gamma(z_2)$. Then Ξ is a heteroclinic orbit connecting a_n and a, i.e. $w_+(\Xi) = \{a\}$ and $w_-(\Xi) = \{a_n\}$. In particular, there exist points $p_1 \in \Xi \cap \partial\mathcal{B}_{r_1}(a) \neq \emptyset$ and $p_2 \in \Xi \cap \partial\mathcal{B}_{r_2}(a_n) \neq \emptyset$. Define the number

$$M := \tau(p_2, p_1) + 2\pi \frac{b_2 r_1 + b_1 r_2}{b_1 b_2} > 0.$$

Choose $x \in \partial\mathcal{B}_{r_2}(a_n) \cap C_n^{[1]} \neq \emptyset$ and let $y \in \Gamma_+(x) \setminus \{x\}$ be arbitrary. We will show that $\tau(x, y) \leq M$. Let $\varepsilon \in (0, \min\{r_1, r_2\})$ be arbitrary. By Lemma 5.24, there exists a $\delta \in (0, \varepsilon]$ such that $\mathcal{B}_\delta(x) \cap \mathcal{B}_\delta(y) = \emptyset$ and for all orbits $\Lambda \subset \mathcal{U}_n(a)$ satisfying $\mathcal{B}_\delta(x) \cap \Lambda \neq \emptyset$ and $\mathcal{B}_\delta(y) \cap \Lambda \neq \emptyset$ it holds that

$$|\tau(x', y') - \tau(x, y)| < \varepsilon \quad \forall\, x' \in \mathcal{B}_\delta(x) \cap \Lambda,\ \forall\, y' \in \mathcal{B}_\delta(y) \cap \Lambda. \tag{5.13}$$

and

$$|\Phi(t, x) - \Phi(t, x')| < \varepsilon \quad \forall\, t \in [0, \tau(x, y)]. \tag{5.14}$$

By the continuous dependence on initial conditions, [9, Chapter 2.4, Theorem 4], there exists a $\tilde{\delta} \in (0, \delta]$ such that $|\Phi(\tau(x, y), z) - y| < \delta$ for all $z \in \mathcal{B}_{\tilde{\delta}}(x)$. Since $x \in \partial\mathcal{U}_n(a)$, there

exists indeed a point $z_0 \in \mathcal{B}_{\tilde{\delta}}(x) \cap \mathcal{U}_n(a) \cap \partial\mathcal{B}_{r_2}(a_n)$.[8] Hence, by choosing $\Lambda := \Gamma(z_0) \subset \mathcal{U}_n(a)$, $x' := z_0 \in \mathcal{B}_\delta(x)$ and $y' := \Phi(\tau(x,y), z_0) \in \mathcal{B}_\delta(y)$, we can apply the above result and have the property $z_0 \in \mathcal{B}_{2r_2}(a_n) \cap \mathcal{U}_n(a)$, i.e. Λ is also a heteroclinic orbit connecting a_n and a. In particular, there exists a point $\tilde{p} \in \partial\mathcal{B}_{r_1}(a) \cap \Lambda \neq \emptyset$.
Let Λ_1 be the piece of $\partial\mathcal{B}_{r_1}(a)$ connecting $\tilde{p}$ with p_1 and Λ_2 be the piece of $\partial\mathcal{B}_{r_2}(a_n)$ connecting p_2 with x'. For both curves, there are two possible directions (clockwise and counterclockwise). The direction of Λ_1 does not matter, since both pieces lie completely in $\mathcal{U}_n(a)$. We choose the direction of Λ_2 in such a way that $x \notin \Lambda_2$, i.e. $\Lambda_2 \subset \mathcal{U}_n(a)$. We conclude, by construction, that $\tilde{\Lambda} := \Lambda + \Lambda_1 - \Xi + \Lambda_2$ is a closed Jordan-Curve lying completely in $\mathcal{U}_n(a)$. Moreover, since $\overline{\mathcal{B}_{2r_1}(a)} \subset \mathcal{U}_n(a)$ and $|y - y'| \leq \varepsilon < r_1$, we have $y' \in \Lambda$. Note that we ensure this without the radius r_1 depending on the choice of y. By Theorem 5.10, $\mathcal{U}_n(a)$ is simply connected and thus $\tilde{\Lambda}$ is null-homotopic in $\mathcal{U}_n(a)$. By the homotopy version of Cauchy's Integral Theorem, we conclude the estimation

$$\begin{aligned}
\tau(x', y') \leq \tau(x', \tilde{p}) &= \int_{\Lambda} F^{-1}\,\mathrm{d}z = \underbrace{\int_{\tilde{\Lambda}} F^{-1}\,\mathrm{d}z}_{=0} - \int_{\Lambda_1} F^{-1}\,\mathrm{d}z + \int_{\Xi} F^{-1}\,\mathrm{d}z - \int_{\Lambda_2} F^{-1}\,\mathrm{d}z \\
&\leq |\tau(p_1, p_2)| + L(\Xi_1) \max_{z \in \partial\mathcal{B}_{r_1}(a)} \underbrace{\frac{1}{|F(z)|}}_{\leq b_1^{-1}} + L(\Xi_2) \max_{z \in \partial\mathcal{B}_{r_2}(a_n)} \underbrace{\frac{1}{|F(z)|}}_{\leq b_2^{-1}} \\
&\leq \tau(p_2, p_1) + \frac{2\pi r_1}{b_1} + \frac{2\pi r_2}{b_2} = M.
\end{aligned}$$

and thus

$$\tau(x, y) \leq |\tau(x', y')| + |\tau(x, y) - \tau(x', y')| \leq M + \varepsilon.$$

Here we used that $\Lambda_1 \subset \partial\mathcal{B}_{r_1}(a)$ and $\Lambda_2 \subset \partial\mathcal{B}_{r_2}(a_n)$ imply $L(\Lambda_1) \leq 2\pi r_1$ and $L(\Lambda_2) \leq 2\pi r_2$, respectively, where $L(\Lambda_j)$ is the length of Λ_j, $j \in \{1, 2\}$. Since y and ε are arbitrary, it follows

$$\sup_{y \in \Gamma_+(x)} \tau(x, y) \leq \sup_{y \in \Gamma_+(x)} M = M < \infty.$$

Note that M does not depend on the choice of ε and y. By Lemma 5.29, we conclude that $C_n^{[1]}$ is indeed a positive separatrix.

[8]The property $z_0 \in \partial\mathcal{B}_{r_2}(a_n)$ can be ensured, if we choose $x \in \partial\mathcal{B}_{r_2}(a_n) \cap C_n^{[1]}$ in such a way that for all $\rho \in (0, r_2)$ we have $(\mathcal{B}_\rho(x) \cap \partial\mathcal{B}_{r_2}(a_n)) \setminus C_n^{[1]} \neq \emptyset$. This can be ensured, if r_2 is small enough.

If C_n is of type (ii), we can use this argument twice, once for $C_n^{[1]}$ and once for $C_n^{[2]}$. Hence it remains to show, that the orbit $C_n^{[1]}$ is a positive and negative separatrix, if C_n is of type (iii).
Since we do not have a second equilibrium lying on C_n, we have do modify our above argument. As in our above argumentation, there exists an $r > 0$ such that $\overline{\mathcal{B}_{2r}(a)} \subset \mathcal{U}_n(a)$ and $w_+(\Gamma(z)) = \{a\}$ for all $z \in \mathcal{B}_r(a)$. Fix $x \in C_n^{[1]}$ and let $y \in \Gamma_+(x) \setminus \{x\}$ be arbitrary. Set

$$\varepsilon_0 := \inf \{\operatorname{dist}(\mathcal{B}_{2r}(a), \zeta) : \zeta \in C_n\} > 0.$$

Choose a transversal L through x with length $\ell < \varepsilon_0$, cf. [7, Bemerkung 9.1.2, 1.]. Since $x \in \partial\mathcal{U}_n(a)$, there exists a $p_2 \in L \cap \mathcal{U}_n(a) \neq \emptyset$. By the choice of ε_0, there exists a $p_1 \in \Gamma_+(p_2) \cap \partial\mathcal{B}_r(a)$. Let $\varepsilon \in (0, \ell)$ be arbitrary. As in our above argumentation, there exist $\delta \in (0, \varepsilon]$ and $z_0 \in \mathcal{B}_\delta(x) \cap \mathcal{U}_n(a) \cap L$ such that we can apply Lemma 5.24 by choosing $\Lambda := \Gamma(z_0) \subset \mathcal{U}_n(a)$, $x' := z_0 \in \mathcal{B}_\delta(x)$ and $y' := \Phi(\tau(x,y), z_0) \in \mathcal{B}_\delta(y)$, i.e. the equations (5.9) and (5.10) are satisfied. In particular, there exists a point $\tilde{p} \in \Gamma_+(z_0) \cap \partial\mathcal{B}_r(a)$.
Let Λ_1 be the piece of $\partial\mathcal{B}_{r_1}(a)$ connecting $\tilde{p}$ with p_1 and Λ_2 be the piece of L connecting p_2 with x'. The direction of Λ_1 does not matter and the direction of Λ_2 is already unique, since L is a straight line. Again by construction, the curve $\tilde{\Lambda} := \Lambda + \Lambda_1 - \Xi + \Lambda_2$ is a closed Jordan-Curve lying completely in $\mathcal{U}_n(a)$. Hence we can use exactly the same argument as above. The upper bound is in this case given by

$$M := \tau(p_2, p_1) + \frac{2\pi r}{b_1} + \frac{\ell}{b_2}$$

with $b_1 := \min\{|F(z)| : z \in \partial\mathcal{B}_r(a)\} > 0$ and $b_2 := \min\{|F(z)| : z \in L\} > 0$. Note that $F^{-1}(\{0\}) \cap L = \emptyset$, cf. [7, Bemerkung 9.1.2, 2.]. Our bound M does not depend on the choice of ε and y also in this case. Since y and ε are arbitrary, we conclude again, by applying Lemma 5.29, that $C_n^{[1]}$ is a positive separatrix.

□

Remark 5.33

a) There are several possibilities to connect the chosen „approximating" orbit Λ with Ξ by a curve lying completely in the simply connected set $\mathcal{U}_n(a)$. I used only two geometric objects: The boundary of circles and straight lines (transversals). If the length is small enough, one could also use transversals also for type (i) in Theorem

5.32. Using circles for type (iii) is more difficult, since we have no information about the behavior of the orbits lying outside of $\mathcal{U}_n(a)$ and going through this circle. The boundary orbit may belong to the boundary of a center, an elliptic sector, or to other as yet unknown geometric structures. Note that the global neighbourhoods are the maximum extent of influence of an equilibrium.

b) In [3, Theorem 4.3], the author additionally claims that the separatrix in case (iii) in Theorem 5.32 is positive as well as negative, independent of the stability of a. But he does not provide a proof of this result. In fact, it is not straightforward to modify the last part of our proof to show the same property for $y \in \Gamma_-(x)$. Note that Λ tends only for exactly one time direction to the equilibrium a.

Example 5.34
In this example we want to visualize the results of Theorem 5.32. Define the polynomial $F : \mathbb{C} \to \mathbb{C}$ by $F(x) := x^2 - 1 = (x-1)(x+1)$, $F \in \mathcal{O}(\mathbb{C})$. This is a polynomial of degree 2 having two zeros $a_\pm := \pm 1$ of order $m := 1$. These zeros are both nodes, this can be proven by applying Theorem 4.3.

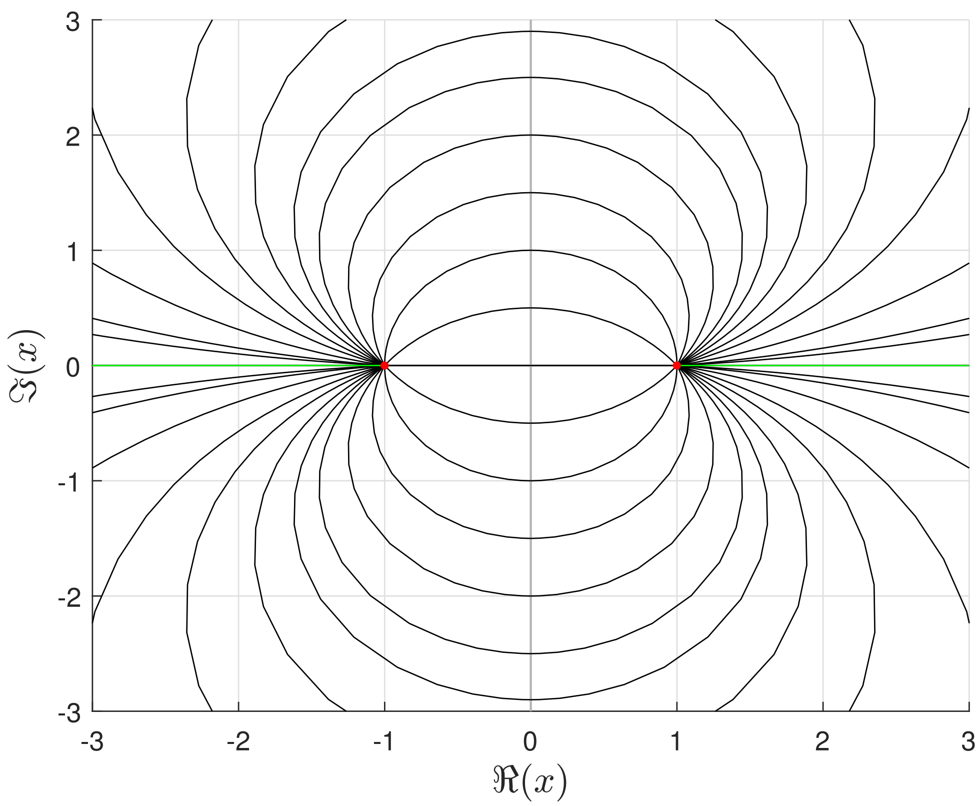

Figure 5.1: Phase portrait of system (4.1) with $F(x) = x^2 - 1$

We verify that the boundary configuration of both equilibria satisfies in this case $\mathcal{Q} = \{1\}$. Moreover, we have $\partial\mathcal{U}_n(1) = \{-1\} \cup (-\infty, -1)$ and $\partial\mathcal{U}_n(-1) = \{1\} \cup (1, \infty)$, respectively. Hence the boundary orbit is in both cases the green straight line lying on the other side of the $\Re$-axis. By Theorem 5.32, we conclude, that the green orbits are separatrices.

Theorem 5.35 (Separatrix configuration on the boundary of global sectors)
Let $F \in \mathcal{O}(\mathbb{C})$, $F \not\equiv 0$, be entire and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m \in \mathbb{N} \setminus \{1, 2\}$. Let $\theta_+, \theta_- \in \mathcal{E}(F, a)$ be two arbitrary adjacent directions. Then the boundary of a with respect to $\theta := (\theta_+, \theta_-)$ consists of a, two characteristic unbounded separatrices Γ_1, Γ_2 satisfying $w_+(\Gamma_1) = w_-(\Gamma_2) = \{a\}$ and at least countably many separatrices, i.e. there exists a index set $\mathcal{Q} \subset \mathbb{N}$ and separatrices $C_n \subset \partial\mathcal{U}_s(a, \theta)$, $n \in \mathcal{Q}$, such that

$$\partial\mathcal{U}_s(a, \theta) = \{a\} \cup \Gamma_1 \cup \Gamma_2 \cup \bigcup_{n \in \mathcal{Q}} C_n. \tag{5.15}$$

In particular, Γ_1 is a negative and Γ_2 is a positive separatrix. Furthermore, for every $n \in \mathcal{Q}$ the orbit C_n is a positive and negative separatrix.

Proof
In the following, many ideas and arguments are similar to these in the proof of Theorem 5.31. The basic idea follows the proof of [3, Theorem 4.2, step 8].
As in the proof of Theorem 5.31, we can indeed find a countable index set $\mathcal{Q}$ such that (5.15) holds. In particular, equation (5.3) ensure that Γ_1 and Γ_2 exist and are unique. It remains to show that Γ_1 is a negative, Γ_2 a positive and for every $n \in \mathcal{Q}$ the orbit C_n a positive and negative separatrix.
By Proposition 5.12, there exists $r > 0$ and $p_1, p_2 \in \partial\mathcal{B}_r(a)$ such that (5.3) holds. In particular, in $\partial\mathcal{U}_s(a, \theta) \cap \mathcal{B}_r(a)$ there are no other unbounded orbits than Γ_1 and Γ_2 and we have $\overline{\mathcal{B}_r(a)} \cap F^{-1}(\{0\}) = \{a\}$. Set

$$b := \min\left\{|F(z)| : z \in \partial\mathcal{B}_{\frac{r}{2}}(a)\right\} > 0.$$

Note that $z \mapsto |F(z)|$ is continuous and $\partial\mathcal{B}_{\frac{r}{2}}(a)$ compact. By Lemma 5.29, for the first two assertions it remains to show that

$$\sup_{y \in \Gamma_-(p_1)} \tau(p_1, y) > -\infty$$

and

$$\sup_{y\in\Gamma_+(p_2)} \tau(p_2,y) < \infty.$$

Let $y \in \Gamma_+(p_2) \setminus \{p_2\}$ and $\varepsilon \in (0, \frac{r}{2})$ be arbitrary. The argument for $\Gamma_-(p_1))$ is analogous, one only has to switch the time direction. By Lemma 5.24, there exists a $\delta \in (0,\varepsilon]$ such that $\mathcal{B}_\delta(p_2) \cap \mathcal{B}_\delta(y) = \emptyset$ and for all orbits $\Lambda \subset \mathcal{U}_s(a,\theta)$ satisfying $\mathcal{B}_\delta(p_2) \cap \Lambda \neq \emptyset$ and $\mathcal{B}_\delta(y) \cap \Lambda \neq \emptyset$ it holds that

$$|\tau(x',y') - \tau(p_2,y)| < \varepsilon \quad \forall\, x' \in \mathcal{B}_\delta(p_2) \cap \Lambda,\ \forall\, y' \in \mathcal{B}_\delta(y) \cap \Lambda.$$

By the continuous dependence on initial conditions, [9, Chapter 2.4, Theorem 4], there exists a $\tilde{\delta} \in (0,\delta]$ such that $|\Phi(\tau(p_2,y),z) - y| < \delta$ for all $z \in \mathcal{B}_{\tilde{\delta}}(x)$. Since $p_2 \in \partial\mathcal{U}_s(a,\theta)$, there exists indeed a point $z_0 \in \mathcal{B}_{\tilde{\delta}}(p_2) \cap \mathcal{U}_s(a,\theta)$, i.e. for $\Lambda := \Gamma(z_0) \subset \mathcal{U}_c(a)$, $x' := z_0 \in \mathcal{B}_\delta(x)$ and $y' := \Phi(\tau(p_2,y),z_0) \in \mathcal{B}_\delta(y)$ we can apply the above result.
Since $\delta \leq \varepsilon \leq \frac{r}{2}$, we have $x', y' \notin \mathcal{B}_{\frac{r}{2}}(a)$, i.e. Λ crosses $\partial\mathcal{B}_{\frac{r}{2}}(a)$ at two points $\tilde{p}_1$ and $\tilde{p}_2$ such that $\Lambda(x',y') \subset \Lambda(\tilde{p}_1,\tilde{p}_2)$. Here we use the properties (iii) and (iv) of Definition 4.14 b). Let Ξ be the curve piece of $\partial\mathcal{B}_{\frac{r}{2}}(a)$ from $\tilde{p}_1$ to $\tilde{p}_2$ lying completely in $\mathcal{U}_s(a,\theta)$, i.e. $\Xi = (\partial\mathcal{B}_{\frac{r}{2}}(a))(\tilde{p}_2,\tilde{p}_1)$. More precisely, Ξ is passed counterclockwise, if $\mathcal{U}_s(a,\theta)$ has clockwise direction and vice versa. Now, the curve $\tilde{\Xi} := \Lambda(\tilde{p}_1,\tilde{p}_2) + \Xi$ is a closed Jordan curve lying completely in $\mathcal{U}_s(a,\theta)$. By Theorem 5.20, $\mathcal{U}_s(a,\theta)$ is simply connected and thus $\tilde{\Xi}$ is null-homotopic in $\mathcal{U}_s(a,\theta)$. By the homotopy version of Cauchy's Integral Theorem, we conclude

$$\tau(x',y') \leq \tau(\tilde{p}_1,\tilde{p}_2) = \underbrace{\int_{\tilde{\Xi}} F^{-1}\,\mathrm{d}z}_{=0} - \int_{\Xi} F^{-1}\,\mathrm{d}z \leq \int_{\Xi} |F^{-1}|\,\mathrm{d}z \leq L(\Xi) \max_{z\in\Xi} \frac{1}{|F(z)|} \leq \frac{\pi r}{b}.$$

and thus

$$|\tau(p_2,y)| \leq |\tau(x',y')| + |\tau(p_2,y) - \tau(x',y')| \leq \frac{\pi r}{b} + \varepsilon.$$

Here we used that $\Xi \subset \partial\mathcal{B}_{\frac{r}{2}}(a)$ implies $L(\Xi) \leq \frac{2\pi r}{2} = \pi r$, where $L(\Xi)$ is the length of Ξ. Since y and ε are arbitrary, it follows

$$\sup_{y\in\Gamma_+(p_2)} \tau(p_2,y) \leq \sup_{y\in\Gamma_+(p_2)} \frac{\pi r}{b} = \frac{\pi r}{b} < \infty.$$

This proves the first and second assertion. By Lemma 5.22, for the third assertion it remains to show that

$$\sup_{x,y\in C_n} |\tau(x,y)| < \infty \quad \forall\, n \in \mathcal{Q}.$$

Let $n \in \mathcal{Q}$ and $x, y \in C_n$ be arbitrary. We have to show that there exists a constant $M > 0$ independent of n, x and y such that $|\tau(x,y)| \leq M$. We use exactly the same idea as before. We apply Lemma 5.24 and construct a closed Jordan curve that approximates on the one hand the part of the orbit C_n from x to y and lies on the other hand on $\partial\mathcal{B}_{\frac{r}{2}}(a)$. In particular, by applying equation (5.3), the properties $x, y \notin \mathcal{B}_r(a)$ and thus $x', y' \notin \mathcal{B}_{\frac{r}{2}}(a)$ are satisfied. We conclude that we can choose the same upper bound $M := \frac{\pi r}{b}$.

□

Theorem 5.36

Let $F \in \mathcal{O}(\mathbb{C})$, $F \not\equiv 0$, be entire and $a \in \mathbb{C}$ an equilibrium of (4.1) with order $m = 2$. Let $\mathcal{U}$ be the unique global sector and $(q_1, q_2) \in \mathbb{C}^2$ a pair of points separating $\mathcal{U}$ with corresponding orbits $\Gamma_1, \Gamma_2 \subset \mathcal{U}$. Let $j \in \{1,2\}$ be arbitrary. Then $\mathcal{U}^{[j]}$ consists of a, two characteristic unbounded separatrices Γ_+, Γ_- satisfying $w_+(\Gamma_+) = w_-(\Gamma_-) = \{a\}$ and at least countably many separatrices, i.e. there exists a index set $\mathcal{Q} \subset \mathbb{N}$ and separatrices $C_n \subset \partial\mathcal{U}^{[j]}$, $n \in \mathcal{Q}$, such that

$$\partial\mathcal{U}^{[j]} = \{a\} \cup \Gamma_+ \cup \Gamma_- \cup \bigcup_{n\in\mathcal{Q}} C_n.$$

In particular, Γ_+ is a negative and Γ_- is a positive separatrix. Furthermore, for every $n \in \mathcal{Q}$ the orbit C_n is a positive and negative separatrix.

Proof

The proof of this Theorem works in the same way as the proof of Theorem 5.35 with obvious changes. Section 5.2 ensures that sector components $\mathcal{U}^{[1]}$ and $\mathcal{U}^{[2]}$ of $\mathcal{U}$ have the same properties as the global sectors of an equilibrium with order at least 3, cf. Proposition 5.14, 5.16 and 5.18. In particular, Theorem 5.20 ensures that both sector components are simply connected.

□

Example 5.37
In this example we want to visualize the results of Theorem 5.35. Define the polynomial $F : \mathbb{C} \to \mathbb{C}$ by $F(x) := x^2(x-1)^2$, $F \in \mathcal{O}(\mathbb{C})$. This is a polynomial of degree 4 having two zeros $a_1 := 0$ and $a_2 := 1$, both of order $m := 2$. By Theorem 4.24, there exists a minimal FSD in both equilibria. In addition, both equilibria have two elliptic sectors and thus two (global) sector components. But in this example, the area between these two equilibria appears in some sense not intuitive.

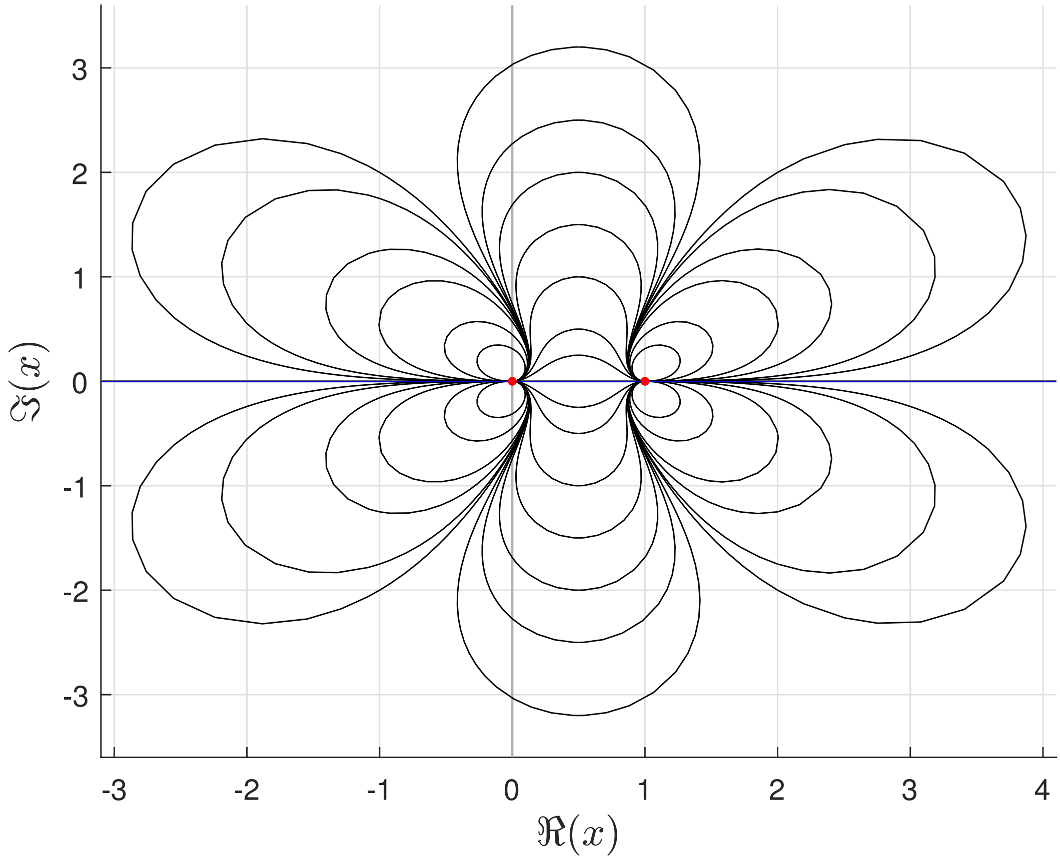

Figure 5.2: Phase portrait of system (4.1) with $F(x) = x^2(x-1)^2$

One can calculate that all rays of the FSD lie on the $\Re$-axis, i.e. we have

$$\mathcal{E}(F, a_1) = \mathcal{E}(F, a_2) = \{0, \pi\}\,.$$

Moreover, these blue rays are also orbits of (4.1). It appears that there are three elliptical sectors each. Upon closer inspection, however, it becomes apparent that the orbits „between" the two equilibria are heteroclinic. They tend to a_1 and a_2 in the definite direction given by the middle blue connecting line between a_1 and a_2. In particular, one can show

that the following holds true: For all $x \in \{z \in \mathbb{C} : \Re(z) < 0, \Im(z) \neq 0\}$ there exists a $T \in \mathbb{R}$ such that $\Phi(T, x) \in \{z \in \mathbb{C} : \Re(z) > 0, \Im(z) \neq 0\}$. By symmetry of the orbits, the same holds true for the right side: For all $x \in \{z \in \mathbb{C} : \Re(z) > 1, \Im(z) \neq 0\}$ there exists a $T \in \mathbb{R}$ such that $\Phi(T, x) \in \{z \in \mathbb{C} : \Re(z) < 1, \Im(z) \neq 0\}$.
In this example, the region between the two equilibria is a „global parabolic region" with exclusively heteroclinic orbits. In particular, except of the two equilibria and the boundary separatrices there should only be either homoclinic (as part of one of the four sector components), or heteroclinic (as part of the „global parabolic region") orbits.

6 Conclusion and Outlook

In summary, it can be stated that Planar Analytic Dynamical Systems provide a rich quantitative and qualitative theory. First, existence and uniqueness of solutions can be guaranteed for any dimension of the differential equation, regardless of whether the time variable is real or complex, cf. Theorems 3.2 and 3.8.
For two-dimensional analytic differential equations, the phase portrait can be described both locally and globally based on the equilibria of the system. The analysis, geometry, and topology of the phase space can all be described in detail. In particular, we are able to show the local existence of a minimal finite sectorial decomposition that exclusively consists of elliptic sectors, cf. Theorem 4.24. But this special structure cannot be easily transferred to the global case, more precisely, we have seen examples where parts of a (local) elliptic sector are not part of the corresponding global sector, cf. Examples 4.28 and 5.37. Nevertheless, global sectors and sector components can be viewed as a global point of view on elliptic sectors, cf. Remarks 5.2 and 5.19.
Furthermore, many geometric structures have been excluded for holomorphic vector fields, including hyperbolic sectors, saddle points and limit cycles, cf. Theorems 4.3, 4.24 and 4.34. The latter result ensures a more specific description of the limit sets of bounded orbits as a generalization of the Theorem of Poincare-Bendixson, cf. Corollary 4.35. Another crucial result is the existence of a countable separatrix configuration on the boundary of global neighborhoods and global sectors, cf. Theorems 5.31, 5.32 and 5.35. Based on these findings, many further research questions and unsolved problems arise. Some of these are presented in the following.

Throughout this work, it was often pointed out how the qualitative analysis of the phase space can be even more precise and which areas of the phase space have not yet been investigated,. These include, in particular, the parabolic areas between elliptic sectors (cf. Remark 5.19), the areas outside the maximum extent of influence of equilibria (cf. Remark 5.33 a)), and the global structure of the boundary of nodes and foci near nodes and foci on such boundaries (cf. Remark 5.11). Specifically, one can extend and proof my conjectures and ideas formulated in Remark 5.11.

The question also arises as to how separatrices can be characterized. We are able to show that certain orbits are separatrices, but is this already the complete picture? When considering a larger section of the phase portrait of the function in Example 4.28, it can be conjectured that this is not the case. Hence, to gain an even better understanding of phase portraits of holomorphic vector fields, a more precise characterization of separatrices will be necessary.

One theory that attempts to do this is the theory of so-called Newton flows with real- and complex-valued time, cf. [21], [22] and [23]. There is a direct connection between the qualitative properties of the phase space of a flow and its corresponding Newton flow. Moreover, considering a dynamical system on the Poincare Sphere also leads to noteworthy and crucial results on the position of separatrices, cf. [9, Chapter 3.10], [24, Theorem 3.6] and [25].

All in all, these theories use the existence and uniqueness in complex time, cf. Theorem 3.8. Hence, the question arises as to how the two phase portraits of a dynamical system, one with complex-valued time and one with real-valued time, are related. What happens when a complex-valued function is restricted to real time? Does this restriction coincide with the (unique) solution of the real-valued dynamical system? Can certain properties of complex-valued solutions of a dynamical system provide information about the position of separatrices in the phase portrait with real time? Additional literature on these questions includes [24, Chapter 5], [26] and [27]. Furthermore, many conjectures and approaches can be found in [28].

Given the importance of the meromorphic vector fields in the theories and papers discussed, it is now the logical next step to reformulate the existing results in this thesis for meromorphic (instead of holomorphic) vector fields. Some first results and attempts can be found in [2] and [3]. The greatest difficulty in doing so may be that poles generate punctured, and therefore not simply connected, sets on which the investigated vector field is holomorphic. Note that this was always a fundamental condition for the result in this work. This difficulty is not solved yet, see for example the conjecture formulated after the proof of [2, Theorem 3.3].

Finally, there is one more important application field for this thesis: The location of zeros of the holomorphic Riemann ξ-function significantly influences the position of separatrices in the phase space generated by this function, viewed as a vector field. For this reason,

many authors come to the following conclusion: If we can characterize the position of separatrices of the ξ-flow sufficiently well, statements can be made about the position of zeros and thus about the truth of the Riemann hypothesis.
Moreover, understanding the structure of the separatrices for the complex-valued Newton flow of the Riemann ξ-function might yield insight into the location of the ξ-zeros via topology and geometry of the solution manifold, cf. Remark 3.5 and [24, Chapter 5]. Additional numerical results about the zeros of the Riemann ξ- and ζ-function can be found in [29], [30]. Furthermore, in [27] many interesting and consequential results on the position of these zeros are formulated, but not proved rigorously.

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Ehrenwörtliche Erklärung

Ich, Nicolas Kainz, erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbständig verfasst habe. Die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Außerdem erkläre ich, dass ich die in der Satzung der Universität Ulm zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung festgelegten Grundsätze eingehalten habe.

Ich bin mir bewusst, dass eine unwahre Erklärung rechtliche Folgen haben wird.

Ulm, den 19. März 2023

Nicolas Kainz