

2D-FEM

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$

$$u = g \quad \text{auf } \Gamma = \partial\Omega$$

$$u := u_0 + u_g \quad u_0 \in H_0^1(\Omega) \quad , \quad u_g \in H^1(\Omega) \text{ Fortsetzung von } g$$

Schwache Formulierung:

$$\int_{\Omega} \nabla u_0 \cdot \nabla v \, d\vec{x} = \int_{\Omega} f v \, d\vec{x} - \int_{\Omega} \nabla u_g \cdot \nabla v \, d\vec{x} \quad \forall v \in H_0^1(\Omega)$$

Diskretes Problem:

$$X_n \subset H^1(\Omega) \quad , \quad X_{n,0} \subset H_0^1(\Omega)$$

$$X_{n,0} = \text{span} \{ \varphi_1, \dots, \varphi_n \} \quad , \quad X_n = \text{span} \{ \varphi_1, \dots, \varphi_{n+m} \}$$

$$\Rightarrow \sum_{j=1}^n u_j \underbrace{\int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i \, d\vec{x}}_{a(\varphi_i, \varphi_j)} = \underbrace{\int_{\Omega} f \varphi_i \, d\vec{x}}_{(f, \varphi_i)} - \sum_{j=1}^{n+m} u_{g,j} \underbrace{\int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i \, d\vec{x}}_{a(\varphi_i, \varphi_j)} \quad , \quad i=1, \dots, n$$

$$=: a_{ij} \quad \quad \quad =: f_j$$

Assemblierung der Steifigkeitsmatrix $A = (a_{ij})$

Wir gehen elementweise vor

Dreieck T_e hat Knoten $\vec{x}_{e1} \quad \vec{x}_{e2} \quad \vec{x}_{e3}$
 $(x_{e1}, y_{e1}) \quad (x_{e2}, y_{e2}) \quad (x_{e3}, y_{e3})$

Abbildung auf Referenzdreieck $\hat{T} = \{ \xi \in \mathbb{R}^2 : 0 \leq \xi_1 \leq 1, 0 \leq \xi_2 \leq 1 - \xi_1 \}$

$$\gamma: \hat{T} \rightarrow T_e : \xi \mapsto \vec{x}_{e1} + \xi_1 (\vec{x}_{e2} - \vec{x}_{e1}) + \xi_2 (\vec{x}_{e3} - \vec{x}_{e1})$$

$$= \begin{pmatrix} x_{e1} \\ y_{e1} \end{pmatrix} + \underbrace{\begin{pmatrix} x_{e2} - x_{e1} & x_{e3} - x_{e1} \\ y_{e2} - y_{e1} & y_{e3} - y_{e1} \end{pmatrix}}_{=: B_e} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

2D-FEM:

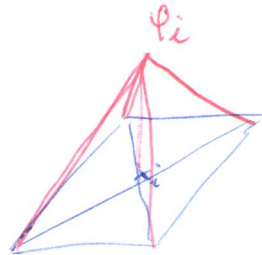
Assemblierung der Steifigkeitsmatrix: elementweise!

LGS:

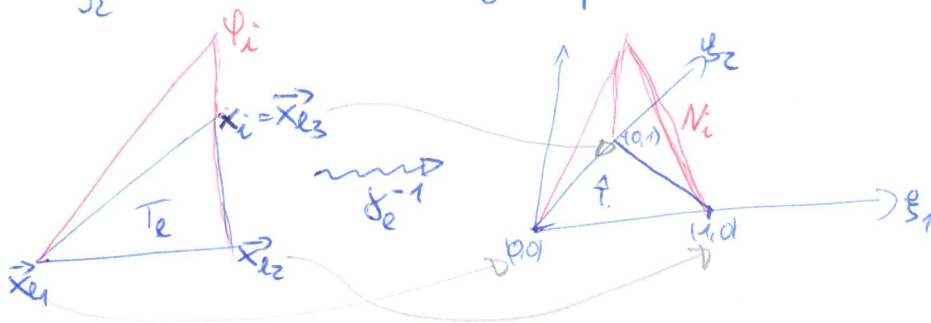
$$\sum_{j=1}^n u_j \int_{\Omega} \nabla \phi_i \nabla \phi_j \, d\vec{x} = \int_{\Omega} f \phi_i \, d\vec{x} - \sum_{j=1}^{n+m} u_{sj} \int_{\Omega} \nabla \phi_i \nabla \phi_j \, d\vec{x}$$

$$A = (a_{ij}) \quad , \quad a_{ij} = \int_{\Omega} \nabla \phi_i \nabla \phi_j \, d\vec{x}$$

$$\phi_i(\vec{x}_j) = \delta_{ij}$$



$$a_{ij} = \int_{\Omega} \nabla \phi_i \nabla \phi_j \, d\vec{x} = \sum_{T: x_i, x_j \in T} \int_T \nabla \phi_i \nabla \phi_j \, d\vec{x}$$



$$\gamma_e: \hat{T} \rightarrow T_e$$

$$\xi \mapsto \vec{x}_{e1} + (\vec{x}_{e2} - \vec{x}_{e1}, \vec{x}_{e3} - \vec{x}_{e1}) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_{e1} \\ y_{e1} \end{pmatrix} + \underbrace{\begin{pmatrix} x_{e2} - x_{e1} & x_{e3} - x_{e1} \\ y_{e2} - y_{e1} & y_{e3} - y_{e1} \end{pmatrix}}_J \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

Es gilt:

$$\bullet \quad |T_e| = \int_{T_e} 1 \, d\vec{x} = |\det J| \int_{\hat{T}} 1 \, d\vec{x} = \frac{1}{2} |\det J|$$

$$\bullet \quad \vec{x} \in T_e: \quad v(\vec{x}) = v(\vec{x}_{e1} + J_e \cdot \xi) \quad , \quad \xi \in \hat{T}$$

$$\nabla_{\xi} v(\vec{x}_{e1} + J_e \xi) = J_e^T \nabla_{\vec{x}} v(\vec{x})$$

$$\Rightarrow \nabla_{\vec{x}} v(\vec{x}) = J_e^{-T} \nabla_{\xi} v(\vec{x}_{e1} + J_e \xi)$$

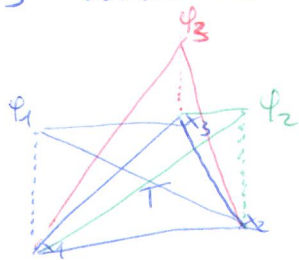
wobei:

$$J_e^{-1} = \frac{1}{|\det J_e|} \begin{pmatrix} y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{pmatrix} \Rightarrow J_e^{-T} = \frac{1}{|\det J_e|} \begin{pmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{pmatrix}$$

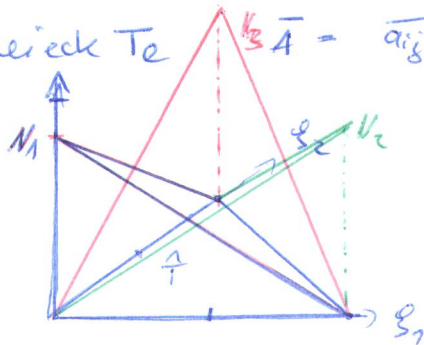
3x3 Untermatrix

für das Dreieck T_e

$$K_3 A = \bar{a}_{ij}, \quad i, j = 1 \dots 3$$



$$J_e^{-1}$$



$$N_1 = 1 - \xi_1 - \xi_2$$

$$N_2 = \xi_1$$

$$N_3 = \xi_2$$

$$\begin{aligned} \bar{a}_{11} &= \int_T \nabla_x \phi_1(\vec{x}) \nabla_x \phi_1(\vec{x}) d\vec{x} = |\det J_e| \int_{\bar{T}} J_e^{-T} \nabla_{\xi} N_1(\xi) J_e^{-T} \nabla_{\xi} N_1(\xi) d\bar{\xi} \\ &= \frac{1}{|\det J_e|} \int_{\bar{T}} \left[\begin{pmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right]^2 d\bar{\xi} \\ &= \frac{1}{2|\det J_e|} \left[\begin{pmatrix} (y_1 - y_3)(y_2 - y_1) \\ (x_3 - x_1)(x_1 - x_2) \end{pmatrix} \right]^2 = \frac{1}{2|\det J_e|} \left[(y_3 - y_1)^2 (y_2 - y_1)^2 + (x_3 - x_1)^2 (x_2 - x_1)^2 \right] \end{aligned}$$

$$\bar{a}_{22} = \int_T \nabla_x \phi_2(\vec{x}) \nabla_x \phi_2(\vec{x}) d\vec{x} = \frac{1}{2|\det J_e|} \left[\begin{pmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]^2 = \frac{1}{2|\det J_e|} \left[(y_3 - y_1)^2 (y_2 - y_1)^2 \right]$$

$$\bar{a}_{33} = \int_T \nabla_x \phi_3(\vec{x}) \nabla_x \phi_3(\vec{x}) d\vec{x} = \frac{1}{2|\det J_e|} \left[\begin{pmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^2 = \frac{1}{2|\det J_e|} \left[(x_3 - x_1)^2 (x_2 - x_1)^2 \right]$$

$$\begin{aligned} \bar{a}_{12} &= \int_T \nabla_x \phi_1(\vec{x}) \nabla_x \phi_2(\vec{x}) d\vec{x} = \frac{1}{2|\det J_e|} \left[\begin{pmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &= \frac{1}{2|\det J_e|} \left[\begin{pmatrix} (y_1 - y_3)(y_2 - y_1) \\ (x_3 - x_1)(x_1 - x_2) \end{pmatrix} \cdot \begin{pmatrix} y_3 - y_1 \\ x_1 - x_3 \end{pmatrix} \right] \\ &= \frac{1}{2|\det J_e|} \left[- (y_3 - y_1)^2 (y_2 - y_1) + (x_3 - x_1)^2 (x_2 - x_1) \right] \end{aligned}$$

$$\begin{aligned} \overline{a_{13}} &= \int_T \varphi_1(\vec{x}) \varphi_3(\vec{x}) dx = \frac{1}{2|\det J|} \left[\begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] \\ &= \frac{1}{2|\det J|} \left[\begin{pmatrix} y_1 - y_3 \\ x_3 - x_1 \end{pmatrix} \cdot \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} \right] \\ &= \frac{1}{2|\det J|} \left[+ (y_3 - y_1)(y_2 - y_1)^2 - (x_3 - x_1)(x_2 - x_1)^2 \right] \end{aligned}$$

$$\begin{aligned} \overline{a_{23}} &= \int_T \varphi_2(\vec{x}) \varphi_3(\vec{x}) dx = \frac{1}{2|\det J|} \left[\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] \\ &= \frac{1}{2|\det J|} \left[\begin{pmatrix} y_3 - y_1 \\ x_1 - x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} \right] \\ &= \frac{1}{2|\det J|} \left[- (y_3 - y_1)(y_2 - y_1) - (x_3 - x_1)(x_2 - x_1) \right] \end{aligned}$$

Elegante Variante (Furker, Rostkowsky, Albrecht 2008)

$$\varphi_j(x, y) = \det \begin{pmatrix} 1 & x & y \\ 1 & x_{j+1} & y_{j+1} \\ 1 & x_{j+2} & y_{j+2} \end{pmatrix} / \det \begin{pmatrix} 1 & x_j & y_j \\ 1 & x_{j+1} & y_{j+1} \\ 1 & x_{j+2} & y_{j+2} \end{pmatrix}$$

$$\text{und } \nabla \varphi_j(x, y) = \frac{1}{2|T|} \begin{pmatrix} y_{j+1} - y_{j+2} \\ x_{j+2} - x_{j+1} \end{pmatrix}$$

Indizes modulo 3

$$\Rightarrow A_{jk} = \int_T \nabla \varphi_j \cdot \nabla \varphi_k = \frac{|T|}{(2|T|)^2} (y_{j+1} - y_{j+2} \quad x_{j+2} - x_{j+1}) \begin{pmatrix} y_{k+1} - y_{k+2} \\ x_{k+2} - x_{k+1} \end{pmatrix}$$

in Matrixschreibweise:

$$A = \frac{|T|}{2} \cdot G G^T \quad \text{wobei} \quad G = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$