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Numerical Finance – Sheet 5

(Exercise Class May 28, 2014)

Exercise 1: Binomial Method

a) Consider the value $V(t,S)$ of an option in the Black-Scholes model. $\Delta(t,S) := \frac{\partial V(t,S)}{\partial S}$ is the so called *delta* hedge parameter. Show that one can approximate the *delta* by the hedge parameter

$$\Delta_{Binom}(t,S) := \frac{V^{(u)} - V^{(d)}}{uS - dS} \text{ with } V^{(u)} := V(t+\Delta t, uS),\ V^{(d)} := V(t+\Delta t, dS)$$

of the binomial model where for fixed S the error $|\Delta(S,t) - \Delta_{Binom}(S,t)|$ is of order $\mathcal{O}(\Delta t)$.

b) Show that $V_{0,M} \to V(0,S_0)$ by proving

(i) If we denote by $\mathbb{E}[V_T]$ the expectation of the payoff in the binomial model at maturity, show that

$$\mathbb{E}[V_T] = \sum_{j=\alpha}^{M} \binom{M}{j} p^j (1-p)^{M-j} (S_0 u^j d^{M-j} - K),$$

where

$$\alpha = \left\lceil -\frac{\log(\frac{S_0}{K}) + M\log(d)}{\log(u) - \log(d)} \right\rceil.$$

(ii) Show that for $\widetilde{p} := pue^{-r\Delta t}$

$$V_{0,M} = e^{-rT}\mathbb{E}[V_T] = S_0 B_{M,\widetilde{p}}(\alpha) - e^{-rT} K B_{M,p}(\alpha)$$

where $B_{M,p}(j) := \mathbb{P}[X \geq j]$ for a (M,p)-binomially distributed random variable X.

(iii) Use the central limit theorem and (ii) to show that

$$\lim_{M\to\infty} V_{0,M} = S_0 F(d_1) + e^{-rT} K F(d_2)$$

where

$$d_1 = \frac{\log(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\log(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

c) Derive complexity and storage estimates for the binomial method with respect to the number of time steps M.

Hint:
Using power series expansions for e^x and $\sqrt{1-x}$, one can show that

$$u = 1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + \mathcal{O}((\Delta t)^{3/2}) = e^{\sigma\sqrt{\Delta t}} + \mathcal{O}((\Delta t)^{3/2}),$$
$$d = 1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + \mathcal{O}((\Delta t)^{3/2}) = e^{-\sigma\sqrt{\Delta t}} + \mathcal{O}((\Delta t)^{3/2}).$$

Programming Exercise 1: Binomial Tree Algorithm **(8 Points)**

Implement Algorithm 5.2.1 to find the fair price of

a) a European put with $S_0 = 20$, $K = 25$, $r = 0.02$, $\sigma = 0.4$ and maturity $T = 1.5$,

b) a European digital option given by the payoff function $g(S) = \mathbb{1}_{\{S>K\}}$ with $S_0 = 5$, $K = 7$, $r = 0.06$, $\sigma = 0.3$ and maturity $T = 1$.

Which convergence rates do you observe with respect to the number of time steps M?

Hint: The exact value C_t^E of a European Call can be obtained by the Black-Scholes formula

$$C_t^E = S_t\Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t)),$$

$$\text{where} \qquad d_1(t) = \frac{\ln(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(t) = d_1(t) - \sigma\sqrt{T-t}.$$

The price P_t^E can then be determined through the Put-Call Parity

$$C_t^E + Ke^{-r(T-t)} = P_t^E + S_t.$$

The price $C_t^{E,\text{dig}}$ of a European digital option is given by $C_t^{E,\text{dig}} = e^{-rt}\Phi(d_2(t))$.

Programming Exercise 2: Implied Volatilities **(10 Points)**

An option price $V = V(0, S_0)$ depends on the parameters S_0, K, T, r and σ. Of these, only σ is not observable at the market. However, one can use the observed option price V_{traded} to infer the so-called *implied volatility* σ by solving $V(\sigma) = V_{traded}$ for σ. Usually, one then finds that σ is *not* constant, as assumed in the Black-Scholes model. For this reason, one often looks at the *implied volatility surface*, i.e. the implied volatility as a function of strike and maturity: $\sigma = \sigma(K, T)$.

On May 15th, 2014, Google Inc. (GOOG) stocks were traded at 526.88\$. At the same time, European Call options on Google were priced as in Table 1. The 1-year US Treasury Bills rate was $r = 0.11\%$.

	Maturity T				
Strike K	0.00555	0.0889	0.17222	0.3388	0.75
450	78.80	78.2	80.4	82.9	89.7
480	48.80	49.8	53.3	57.7	66.4
500	28.80	32.7	37.4	43.0	53.1
510	20.30	24.8	30.4	36.6	47.2
520	12.20	18.1	24.3	30.8	41.5
530	6.50	12.7	19.0	25.5	36.6
550	1.30	5.5	10.9	17.0	27.3
570	0.35	1.95	5.8	10.8	20.2
600	0.25	0.45	2.0	5.1	12.4

Table 1: Call option prices on Google

Use this data to construct an empirical volatility surface for the Google Call option by computing the implied volatility at the different data points. In order to do so,

- use Newton's method to find the implied volatilities, where you
- compute the option prices $V(\sigma)$ using a binomial tree,
- and the derivative $\frac{\partial}{\partial\sigma}V(\sigma)$ w.r.t σ using central differences, i.e. $\frac{\partial}{\partial\sigma}V(\sigma) \approx \frac{V(\sigma+h)-V(\sigma-h)}{2h}$.

Plot the empirical volatilities $\sigma(K,T)$ in a 3D-Plot. What do you see?

Hints:

- Of course, the Newton method is very sensitive w.r.t. initial values. Using $\sigma^{(0)} = 0.7$ should work for all given data points. (As an alternative to Newton, *Brent's method* is often used in practice.)
- On the homepage you find the data points in a text file, where the first line contains the strike values, the second the maturities and the rest of the file the data for V_{traded} as in Table 1.
- Gnuplot can plot 3D data given in the following format (note the blank line between different x-values):

```
x1 y1 z(x1,y1)
...
x1 yn z(x1,yn)

x2 y1 z(x2,y1)
...
x2 yn z(x2,yn)
```

  Use `splot ''data.txt'' w lp`. (*w lp* stands for *with linespoints*).