Numerical Finance – Sheet 7
(Exercise Class June 11, 2014)

Exercise 1: Higher Order Schemes

Derive the following higher order Taylor scheme:

\[ Y_{n+1} = Y_n + \left[ a - \frac{1}{2} bb' \right] \Delta_n + b \Delta W_n + \frac{1}{2} \left[ bb'(\Delta W_n)^2 \right] + \frac{1}{2} \left[ aa' + b b' a'' \right] \Delta_n^2 + \frac{1}{2} \left[ a' b + b' a + \frac{1}{2} b b'' \right] \Delta_n \Delta W_n. \] (1)

Consider only the terms that are not already in the Milstein scheme. Use the fact that one can replace the integral

\[ I_{t_0, t} := \int_{t_0}^t (W_s - W_t) ds \]

by its conditional expectation

\[ \mathbb{E} [I_{t_0, t} \mid W_{t_0}, W_t - W_{t_0}] = \frac{1}{2} (t - t_0) (W_t - W_{t_0}). \]

Exercise 2: Multidimensional Schemes

Consider a \( d \)-dimensional Itô process \( X(t) = (X_1(t), \ldots, X_d(t))^T \), driven by an \( m \)-dimensional Brownian motion \( W(t) = (W_1(t), \ldots, W_m(t))^T \), i.e. \( X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW(s) \) with \( a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \), \( b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m} \).

Use the multidimensional Itô formula to derive the appropriate Euler and Milstein schemes.

Hint:

- Consider each component \( X_i \) separately, proceed as in Section 6.7 and keep in mind that we neglect almost all double integrals.
- Integrals \( I_{t_0, t}^{[k, l]} := \int_{t_0}^t \int_{s}^t dW_k(z) dW_l(s) \) with \( k \neq l \) can be approximated by

\[ I_{t_0, t}^{[k, l]} \approx \frac{1}{2} \left[ (W_k(t) - W_k(t_0))(W_l(t) - W_l(t_0)) - V_{kl} \right], \]

where \( V_{kl} \) is a random variable with \( V_{kl} = -V_{lk} \) for \( l > k \) and \( V_{kl} = \pm \Delta_n \) with probability \( \frac{1}{2} \) for \( l < k \).
- Multidimensional Itô Formula: For a \( d \)-dim. Itô process \( X, f : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) with appropriate partial derivatives, \( \Sigma := b \cdot b^T \) and \( Y(t) := f(t, X(t)) \), it holds with \( b_i \), the \( i \)-th row of \( b \)

\[ Y(t) - Y(t_0) = \int_{t_0}^t \left[ \frac{\partial f}{\partial t}(s, X(s)) + \sum_{i=1}^d \frac{\partial f}{\partial x_i}(s, X(s)) a_i(s, X(s)) + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial f}{\partial x_i \partial x_j}(s, X(s)) \Sigma_{i,j}(s, X(s)) \right] ds \\
+ \int_{t_0}^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(s, X(s)) b_i(s, X(s)) dW_i(s). \]
**Programming Exercise 1: Higher Order Schemes**

Consider (again) the European Put from Programming Exercise 1, Sheet 5 ($S_0 = 20$, $K = 25$, $r = 0.02$, $\sigma = 0.4$, $T = 1.5$). For the three methods

- Euler-Maruyama,
- Milstein,
- the scheme from (1),

compute

(i) the strong errors,

(ii) the weak errors w.r.t. the option payoff function,

(iii) the error w.r.t the Black-Scholes option price.

Compare the convergence rates of the errors w.r.t the time discretization, using e.g. $M = 10^5$ Monte-Carlo simulations. What do you see? What is the strong (weak) convergence order of the Taylor scheme (1)?

*Compare your results for the error of the option price with those obtained for significantly more MC simulations runs, e.g. $M = 10^6, M = 10^7$ (note that this may take some time!). What do you see? What does this imply for the relation between Monte-Carlo and SDE discretization error?

**Programming Exercise 2: Heston Model**

As the assumption of constant volatility in the Black-Scholes framework is often not consistent with market option prices, many models use local or stochastic volatility functions. One example for a stochastic volatility model is the Heston model, which models the volatility as a mean reverting square-root diffusion process and in its simplest form looks as follows:

\[
\begin{align*}
    dS(t) &= rS(t)dt + \sqrt{V(t)}S(t)dW_1(t), \\
    dV(t) &= \alpha(\theta - V(t))dt + \sqrt{V(t)}\sigma dW_2(t).
\end{align*}
\]

Use your results from Exercise 2 to compute the price of a European call with parameters $T = 1$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, $\alpha = 1.2$, $\theta = 0.04$ and initial values $S_0 = 100$, $V_0 = 0.04$ in this model,

a) with the Euler scheme,

b) with Milstein.

**Hint:** $W_1$ and $W_2$ are assumed to be independent. This simplifies the multidimensional schemes significantly!