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Numerik von gewöhnlichen Differenzialgleichungen
SoSe 2016

Sheet 1

Due April 21, 2016.

Hinweise:

a) Im SLC zur Veranstaltung "Numerik von gewöhnlichen Differenzialgleichungen" anmelden.

b) Theorie-Aufgaben werden von den Studenten in den jeweiligen Übungen vorgerechnet und vorgestellt! Die Matlab-Aufgaben werden ebenfalls in der jeweiligen Übung vorgestellt. Es wird Anwesenheit vorausgesetzt!

c) Zulassungskriterien zur Prüfung: 50% der Übungspunkte müssen erreicht und zudem muss mindestens 1 mal vorgerechnet werden.

d) Die Abgabe der Lösungen der Matlab-Aufgaben erfolgt per Email (rechtzeitig vor der Besprechung, d.h. mindestens einen Tag vorher! Mittwochs ab 18 Uhr werden keine Lösungen mehr akzeptiert!) an

   mladjan.radic@uni-ulm.de oder **stefan.hain@uni-ulm.de**.

   Der Betreff sollte lauten: "NumIVBlattx" (wobei x für die Nummer des jeweiligen Blattes steht). Die Lösungen müssen als Anhang an die Email versendet werden. Für jede Programmieraufgabe ist ein zip-file "AufgabeMy" zu erstellen (wobei y für die Nummer der Aufgabe steht), das die nötigen .m-files enthält.

Exercise 1 (Integral Equation)

Consider the initial value problem (IVP)

$$\begin{aligned} y'(t) &= f(t, y(t)), \qquad t \in [0, T], \\ y(0) &= y_0 \end{aligned}$$

with a global Liptschitz continuous function $f \in C([0, T] \times \mathbb{R})$ with respect to the second argument, e.g.:

$$\exists L > 0 : \quad |f(t, y_1) - f(t, y_2)| \leq L\,|y_1 - y_2|, \qquad \forall (t, y_1),\ (t, y_2) \in [0, T] \times \mathbb{R}.$$

(a) Show: y is a solution of the IVP if and only if

$$y(t) = Ky(t) := y_0 + \int_0^t f(s, y(s))\, ds. \tag{1}$$

(b) The right hand side of (1) defines an integral operator

$$K : C([0, T]) \to C([0, T]) \qquad (\text{actually } K : C([0, T]) \to C^1([0, T]))$$

on the Banach space $C([0, T])$ equipped with the maximum norm

$$\|f\|_\infty := \max_{t \in [0, T]} |f(t)|.$$

Show: For $T < 1/L$ the integral operator has a fixpoint $y \in C([0, 1])$ (actually $y \in C^1([0, T])$), which is a solution of the IVP.

(c) Show: As a byproduct of exercise (b) we obtain a uniform convergence in $C([0,T])$ of the successive approximation

$$y^k(t) = y_0 + \int_0^t f\left(s, y^{k-1}(s)\right)\, ds, \qquad 0 \le t \le T, \quad k = 1, 2, \ldots$$

for example with the initial function $y^0 \equiv y_0$. Derive further an a-priori and an a-posteriori error estimation.

Exercise 2 (Direction Field)

It's well known that it's advantageous to consider a Direction Field for getting a better feeling of ordinary differential equations (= ODE). For an ODE

$$y'(t) = f(t, y(t)) \qquad \text{with} \qquad f : I \times D \to \mathbb{R}, \quad I, D \subset \mathbb{R}$$

a Direction Field is a Vector Field, which assigns every point $(t, y(t))$ a normed vector with the direction $(1, y'(t)) = (1, f(t, y(t)))$.

(a) Write a Matlab-function `plotVectorfield(f,t,y)`, which creates a plot showing a Direction Field of an ODE $y' = f(t, y(t))$. `f` is a function handle and `t` and `y` are vectors, which contains the grid for drawing the Direction Field. Use the Matlab-Function `quiver` for the plot.

(b) Create a plot with the Direction Field for the following ODEs:

- $y'(t) = 0.5\, y(t),\ t \in [0, 5]$,
- $y'(t) = y(t)^2 + t^2,\ t \in [0, 2]$,
- $y'(t) = (5 - 5y(t))\, y(t),\ t \in [0, 2]$.

Use the file `mainVectorfield.m`, which can be found on the homepage.

Exercise 3 (Euler Method)

Consider an ODE or a system of ODEs of first order $y' = f(t, y)$. We are interested in a solution y of the ODE in the time inverval $I = [t_a, t_e]$. We discretize I equidistantly in N subintervals, i.e.,

$$t_a = t_0 < t_1 < \ldots < t_N = t_b$$

with $t_k = t_a + (t_b - t_a) \cdot \frac{k}{N}$ for $k = 0, \ldots, N$.

(a) Show, that the Euler scheme can be derived from the following idea:

- Create a tangent T_k at time-step t_k to the function y.
- Evaluate T_k at the time-step t_{k+1}.
- The value $T_k(t_{k+1})$ is an approximation of $y(t_{k+1})$.

Illustrate this derivation with a short sketch. How can this method be improved?

(b) Write a function `y = euler_exp1(f,y0,t0,tN,N)`, which runs the Euler Scheme for the initial value problem (IVP)

$$y'(t) = f(t, y(t)),$$
$$y(t0) = y0.$$

The time interval $[t0, tN]$ shall be divided in N equidistent subintervals. `f` is a function handle.

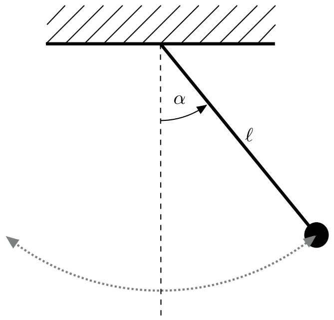
(c) Complete the script of the previous exercise, such that the solution of the corresponding IVP is plotted for various suitable initial values $y0$.

Exercise 4 (Pendulum)

We consider a pendulum with the length ℓ. $\alpha(t)$ denotes the angle of deflection depending on the time t. The corresponding ODE is then given by

$$\alpha''(t) + \frac{g}{\ell}\sin\left(\alpha(t)\right) = 0, \tag{2}$$

where $g = 9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}$ denotes the acceleration of gravity.



(a) Rewrite the ODE of second order into a system of first order.

(b) For an autonomous ODE, i.e., the right hand side depends only on y (and therefore not on t), we can draw the vector field $f(y)$. Draw the vector field f for $\alpha \in [-\pi/4, \pi/4]$ and $\alpha' \in [-1, 1]$. Discuss the corresponding vector field.

(c) Apply the above written function `euler_expl` with 1000 time-steps. Solve the ODE additionally with the function `ode45` (see matlab help for further informations) for $\alpha(0) = \pi/5$. Plot both solutions for $t \in [0, 20]$ in one figure. Explain the resulting observations.
Hints: `ode45` does not necessarily use equidistant time-steps. The vector with the corresponding time discretization will be returned. If we want to compare both solutions, we have to evaluate the solution computed with `ode45` on the equidistant grid from the Euler scheme. The function `interp1` can be useful.

(d) Compute the position of the pendulum from both solutions (`euler_expl` and `ode45`) with respect to t and create an animation for both solutions, which illustrates the swing of the pendulum.

(e) Compute the solutions of the ODE for $\alpha(0) \in \{\pi/5, \pi/4, \pi/3, \pi/2, \pi\}$. Create a plot for the curve $\alpha'(t)$ over α. Discuss and explain the corresponding figure.