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Numerik von gewöhnlichen Differenzialgleichungen
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Sheet 2

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Exercise 1 (Integral Equation II)

Consider the initial value problem (IVP) (see Exercise 1 on Sheet 1):

$$\begin{aligned} y'(t) &= f(t, y(t)), \qquad t \in I := [0, T], \\ y(0) &= y_0 \end{aligned}$$

with a function $f^2 \in C(I \times \mathbb{R})$. On Sheet 1, Exercies 1, we have already shown, that

$$y \text{ is soution of the IVP } \Leftrightarrow \; y(t) = Ky(t) := y_0 + \int_0^t f(s, y(s))\, ds. \tag{1}$$

We discretize I equidistantly in N subintervals with stepsize $h := \frac{T}{N} > 0$, e.g.

$$0 = t_0 < t_1 < \ldots < t_N = T$$

with $t_k = k \cdot h$ for $k = 0, 1, \ldots, N$. With (1) we obtain therefore

$$\frac{y(t_{k+1}) - y(t_k)}{h} = \frac{1}{h} \int_{t_k}^{t_{k+1}} f(s, y(s))\, ds \quad \text{with} \quad y(t_0) = y_0.$$

The idea is now, to apply a quadrature rule on $\int_{t_k}^{t_{k+1}} f(s, y(s))\, ds$.

(a) Show, that by applying the left rectangle rule, we obtain the explicit Euler method.

(b) Show, that by applying the right rectangle rule, we obtain the implicit Euler method.

(c) Which method do we obtain, if we apply the center- rule? Which one with the trapezoidal rule?

(d) Discuss and derive with this idea (geometrically) the order of consistency as well as convergence for the above methods.

Exercise 2 (Euler)

Show the following statements:

(a) Show, that the Euler method solves the following IVP

$$y'(t) = \frac{t+1}{y(t)}, \qquad y(0) = 1,$$

in an exact manner. Can this result be formulated more generally?

(b) Consider the following linear first order ODE of the form

$$y'(t) = a(t)y(t) + b(t)$$

with given functions $a(t), b(t)$. Show, that the implicit Euler method will lead to a linear (explicit) system of equations for the unknown y_{k+1}.

(c) Consider the IVP

$$y'(t) = -\frac{1}{y(t)}\sqrt{1 - y(t)^2}, \qquad y(0) = 1,$$

with the exact nontrivial solution $y(t) = \sqrt{1 - t^2}$, $t \in [0, 1]$. Why does the explicit Euler method computes the solution $y_n = 1$ for all n and for arbitrary step size?

Exercise 3 (A little bit of programming...)

(a) We want to compute the solution for the following IVP

$$y'(t) = -200ty(t)^2, \qquad y(t_0) = \frac{1}{901} \qquad t_0 := -3 \leq t \leq 3,$$

with the explicit Euler method $y_n = y_{n-1} + hf(t_{n-1}, y_{n-1})$ for $n = 1, \ldots, N := 4/h$, with step size $h = 2^{-i}$, $i = 5, \ldots, 10$. The exact solution is given by $y(t) = (1 + 100t^2)^{-1}$. Compare the numerical solution with the exact solution for $t = 1$ in a logarithmic plot, i.e., plot the error $|y_h(1) - y(1)|$ in dependence of h.

(b) Repeat the above computation with the trpezoidal rule $y_n = y_{n-1} + \frac{h}{2}\left(f(t_{n-1}, y_{n-1}) + f(t_n, y_n)\right)$ for $n = 1, \ldots, N := 6/h$ and h as above. Discuss the „convergence rate"

$$|y(3) - y_N| = \mathcal{O}(h^p).$$

How does this method behave for the coarse step size $h = 2^{-4}$. Explain this effect.

(c) Derive and determine the convergence for the following method. We consider the numerical solution gained with the trapezoidal rule y_N^i with respect to the step size h_i and we define the approximation

$$\tilde{y}_N^i := \frac{1}{3}\left\{4y_N^i - y_N^{i-1}\right\}, \qquad i = 2, \ldots, 8.$$

Try to explain the method, its results and effects.