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Sheet 3

Due May 12, 2016.

Exercise 1 (Consistency and Increment Function)

Consider the IVP

$$\begin{aligned} y'(t) &= f(t, y(t)), \qquad t \in I := [0, T] \subset \mathbb{R}, \\ y(0) &= y_0. \end{aligned} \tag{1}$$

Let $h > 0$ be the step-size. Every one-step method for determination of the grid-function$\{y_m\}_{m=1}^N$ is of the form

$$y_{m+1} = y_m + h\Phi(t_m, y_m; h), \qquad m = 0, \ldots, N-1,$$

where Φ is the so called *increment function*.

(a) Determine the increment function for the explicit and implicit Euler method.

(b) Let $y \in C^1(I)$ be the exact solution of the IVP (1). Show: A one-step method is consistent if and only if

$$\lim_{h \to 0} \Phi(t, y(t); h) = f(t, y(t)).$$

Use this result to justify why the explicit and implicit Euler method are consistent.

(c) Let $f \in C^2(I \times \mathbb{R})$. Consider the increment function

$$\Phi(t, y; h) = \alpha_1 f(t, y) + \alpha_2 f(t + \beta_1 h, y + \beta_2 h f(t, y)).$$

Determine $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that the order of convergence equals 2. Is it possible to achieve higher order?

(d) Let $f \in C^3(I \times \mathbb{R})$. Consider the increment function

$$\Phi(t, y; h) = \frac{1}{6} \cdot (k_1 + 4 \cdot k_2 + k_3),$$

with

$$\begin{aligned} k_1 &= f(t, y), \\ k_2 &= f\left(t + \frac{h}{2}, y + \frac{h}{2} k_1\right), \\ k_3 &= f(t + h, y + h(-k_1 + 2k_2)). \end{aligned}$$

Show that the resulting one-step method is of consistency order $p = 3$. Which convergence order has the method?

Exercise 2 (Integral Equation III and the classical RKM)

Consider the initial value problem (IVP) (see Exercise 1 on Sheet 1):

$$
\begin{aligned}
y'(t) &= f(t, y(t)), \qquad t \in I := [0, T] \subset \mathbb{R}, \\
y(0) &= y_0
\end{aligned}
$$

with $f \in C^4(I \times \mathbb{R})$. On Sheet 1, Exercise 1, we have already shown, that

$$
y \text{ is soution of the IVP } \Leftrightarrow y(t) = Ky(t) := y_0 + \int_0^t f(s, y(s))\, ds. \tag{2}
$$

We discretize I equidistantly in N subintervals with stepsize $h := \frac{T}{N} > 0$, e.g.

$$
0 = t_0 < t_1 < \ldots < t_N = T
$$

with $t_k = k \cdot h$ for $k = 0, 1, \ldots, N$. With (2) we obtain therefore

$$
\frac{y(t_{k+1}) - y(t_k)}{h} = \frac{1}{h} \int_{t_k}^{t_{k+1}} f(s, y(s))\, ds \quad \text{with} \quad y(t_0) = y_0. \tag{3}
$$

(a) Derive the classical Runge-Kutta method by using (3) and

- using a quadrature formula (which one?) on $\int_{t_k}^{t_{k+1}} f(s, y(s))\, ds$,
- approximation of the function values of y by taylor formula,
- elimination of the derivatives of y.

(b) Consider the IVP

$$
\begin{aligned}
y'(t) &= y(t), \qquad t \in I := [0, T], \\
y(0) &= \alpha,
\end{aligned}
$$

with the exact solution $y(t) = \alpha \cdot \exp t$, $0 \le t \le T$. Formulate for this IVP the classical Runge-Kutta method and discuss your observations.

Exercise 3 (Explicit RKM)

Consider for $I := [\mathtt{t0}, \mathtt{tN}] \subset \mathbb{R}$ the IVP

$$
\begin{aligned}
y'(t) &= f(t, y(t)), \qquad t \in I, \\
y(\mathtt{t0}) &= \mathtt{y0}.
\end{aligned}
$$

(a) Consider the classical Runge-Kutta method with the Butcher-tableau:

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

Which consistency order has the resulting explicit Runge-Kutta method and prove your statement.

(b) Write a function `y = runge_kutta_exp(f,tspan,y0,N)`, which runs the classical Runge-Kutta method with the Butcher-tableau given in (a). The time interval `tspan=[t0,tN]` shall be divided in N equidistent subintervals. `f` is a function handle.

(c) Apply the method on the following examples and vary $\texttt{N} \in \{10, 50, 100, 500, 1000\}$

(i) $u'(t) = -\lambda u(t)$, $u(0) = 1$, for $\lambda \in \{-10, -1, 1, 10\}$ and $t \in [0, 1]$.

(ii) $u'(t) = -t^2 u(t)^2$, $u(0) = 1$, for $t \in [0, 1]$.

(iii) $u'(t) = -2tu(t)^2$, $u(0) = 1$, for $t \in [0, 1]$.

(iv) Davis-Skodje problem:

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} -u(t) \\ -\frac{1}{\varepsilon}v(t) + \frac{u(t)}{\varepsilon(1+u(t))} - \frac{u(t)}{(1+u(t))^2} \end{pmatrix}$$

with the initial values

$$u(0) = 1 \qquad \text{and} \qquad v(0) = 1$$

for $\varepsilon \in \{1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-6}\}$ and $t \in [0, 4]$.

(d) Apply further the explicit, implicit Euler method as well as the Crank-Nicolson method on the above examples with the same discretization. Discuss and explain your observations.

Exercise 4 (Implicit RKM)

Consider for $I := [\texttt{t0}, \texttt{tN}] \subset \mathbb{R}$ the IVP

$$\begin{aligned} y'(t) &= f(t, y(t)), \qquad t \in I, \\ y(\texttt{t0}) &= \texttt{y0}. \end{aligned}$$

(a) Consider two implicit Runge-Kutta methods, which are characterized by the following Butcher-tableaus:

$$\begin{array}{c|cc} \alpha_1 & \beta_{11} & \frac{3-2\sqrt{3}}{12} \\ \frac{3+\sqrt{3}}{6} & \frac{3+2\sqrt{3}}{12} & \frac{1}{4} \\ \hline & \frac{1}{2} & \gamma_2 \end{array} \qquad\qquad \begin{array}{c|ccc} \frac{5-\sqrt{15}}{10} & \frac{5}{36} & \frac{10-3\sqrt{15}}{45} & \frac{25-6\sqrt{15}}{180} \\ \frac{1}{2} & \frac{10+3\sqrt{15}}{72} & \frac{2}{9} & \frac{10-3\sqrt{15}}{72} \\ \frac{5+\sqrt{15}}{10} & \frac{25+6\sqrt{15}}{180} & \frac{10+3\sqrt{15}}{45} & \frac{5}{36} \\ \hline & \frac{5}{18} & \frac{4}{9} & \frac{5}{18} \end{array}$$

Determine the coefficients α_1, β_{11} and γ_2 of the left Butcher-tableau, such that the resulting implicit Runge-Kutta method is of consistency order 4.

(b) Write a function `y = runge_kutta_imp(f,tspan,y0,N,alpha,gamma,A)`, with `tspan=[t0,tN]` which runs the implicit Runge-Kutta method with the Butcher-tableaus (described by `alpha,gamma,A`) given in (a). The time interval `tspan=[t0,tN]` shall be again divided in N equidistent subintervals. `f` is a function handle. Further, for solving the implicit system, use

(i) fixpoint iteration,

(ii) Newton method.

Compare and discuss the two methods.

(c) Apply the method on the Examples (i)-(iv) from Exercise 3 again with $\texttt{N} \in \{10, 50, 100, 500, 1000\}$.

Additional remarks to Exercise 3 and 4

Create the plots in a meaningful manner! Make them as neat as possible! Plots without a title, without detailed description of the x-and $y-$ axis or without a legend (if necessary) will not be accepted!

The following keywords could be useful:

- `doc figure`
- `doc plot`
- `doc subplot`
- `title`
- `xlabel,ylabel`
- `legend`
- `axis`