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Numerik von gewöhnlichen Differenzialgleichungen
SoSe 2016

Sheet 4

Due May 12, 2016.

Exercise 1 (Embedded RKM - RK7(8))

Consider the following embedded Runge-Kutta method:

0													
$\frac{2}{27}$	$\frac{2}{27}$												
$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{12}$											
$\frac{1}{6}$	$\frac{1}{24}$	0	$\frac{1}{8}$										
$\frac{5}{12}$	$\frac{5}{12}$	0	$-\frac{25}{16}$	$\frac{25}{16}$									
$\frac{1}{2}$	$\frac{1}{20}$	0	0	$\frac{1}{4}$	$\frac{1}{5}$								
$\frac{5}{6}$	$-\frac{25}{108}$	0	0	$\frac{125}{108}$	$-\frac{65}{27}$	$\frac{125}{54}$							
$\frac{1}{6}$	$\frac{31}{300}$	0	0	0	$\frac{61}{225}$	$-\frac{2}{9}$	$\frac{13}{900}$						
$\frac{2}{3}$	2	0	0	$-\frac{53}{6}$	$\frac{704}{45}$	$-\frac{107}{9}$	$\frac{67}{90}$	3					
$\frac{1}{3}$	$-\frac{91}{108}$	0	0	$\frac{23}{108}$	$-\frac{976}{135}$	$\frac{311}{54}$	$-\frac{19}{60}$	$\frac{17}{6}$	$-\frac{1}{12}$				
1	$\frac{2383}{4100}$	0	0	$-\frac{341}{164}$	$\frac{4496}{1025}$	$-\frac{301}{82}$	$\frac{2133}{4100}$	$\frac{45}{82}$	$\frac{45}{164}$	$\frac{18}{41}$			
0	$\frac{3}{205}$	0	0	0	0	$-\frac{6}{41}$	$-\frac{3}{205}$	$-\frac{3}{41}$	$\frac{3}{41}$	$\frac{6}{41}$	0		
1	$-\frac{1777}{4100}$	0	0	$-\frac{341}{164}$	$\frac{4496}{1025}$	$-\frac{289}{82}$	$\frac{2193}{4100}$	$-\frac{51}{82}$	$\frac{33}{164}$	$\frac{12}{41}$	0	1	
γ	$\frac{41}{840}$	0	0	0	0	$\frac{34}{105}$	$\frac{9}{35}$	$\frac{9}{35}$	$\frac{9}{280}$	$\frac{9}{280}$	$\frac{41}{840}$	0	0
$\hat{\gamma}$	0	0	0	0	0	$\frac{34}{105}$	$\frac{9}{35}$	$\frac{9}{35}$	$\frac{9}{280}$	$\frac{9}{280}$	0	$\frac{41}{840}$	$\frac{41}{840}$

Where the consistency order with respect to γ is 7 and $\hat{\gamma}$ is 8. For a good and reasonable application of the above embedded RKM we consider the following problem:
The orbit $(x(t), y(t))$ of a satellite in the gravitation field of the earth, moon and the sun can be described by the following equations

$$\begin{aligned} x'' &= x + 2y' - \hat{\mu}\frac{x+\mu}{N_1} - \mu\frac{x-\hat{\mu}}{N_2}, \\ y'' &= y - 2x' - \hat{\mu}\frac{y}{N_1} - \mu\frac{y}{N_2} \end{aligned}$$

with the relative masses

$$\mu = \frac{m_M}{m_E + m_M} \quad \text{and} \quad \hat{\mu} = \frac{m_E}{m_e + m_M} = 1 - \mu,$$

where m_E is the mass of the earth and m_M of the moon. N_1 and N_2 are given by

$$N_1 = \left((x+\mu)^2 + y^2\right)^{\frac{3}{2}} \quad \text{and} \quad N_2 = \left((x-\hat{\mu})^2 + y^2\right)^{\frac{3}{2}}.$$

The movement of the satellite in $\mathbb{R}^2$ with coordinates $(x(t), y(t))$ is a coordinate system, which rotates around the centre of gravitation (origin). The earth is assumed fix in the point $(-\mu, 0)$ and the moon in $(\hat{\mu}, 0)$ respectively. For the following initial values

$$x(0) = 0.994, \quad x'(0) = 0, \quad y(0) = 0, \quad y'(0) = -2.0015851063790825$$

and for $\mu = 0.012277471$ we recieve the solution of a s called (four leaved) Arenstorf-Orbit with period $T = 17.0652165601579625588 9$ (months).

(a) Transform the system into a system of first order of the form

$$u' = f(t, u) \quad \text{with} \quad u(t) = (x(t), x'(t), y(t), y'(t))^T.$$

(b) Complete the function `[t,y]=RK78(f,tspan,y0,tol)`, which you can find on the homepage. `f` is a function handle describing the right-hand side of the ODE, `tspan` is the interval on which the ODE has to be solved, in our case `tspan=[0,T]` and `y0` is the initial value. `tol` is the desired tolerance. The output parameter are `t` and `y`, which contain the time discretization $t_0, t_1, t_2, \dots, t_N$ and the corresponding computed values $y_0, y_1, y_2, \dots, y_N$ respectively. Note, that we don't have an equidistant discretization. Apply the function on the above IVP. What do you observe?

(c) Apply the explicit Euler method on the above IVP with $N \in \{10, 10^2, 10^3, 10^4, 10^6\}$.

(d) Apply the `RK7(8)` method on the Davis-Skodje model, see Sheet 3, Exercise 3, Example (iv) with $\varepsilon \in \{10^{-2}, 10^{-3}, 10^{-4}\}$. Compare your results with Sheet 3, Exercise 3 and 4.

Exercise 2 (Adams-Bashforth)

Consider the IVP

$$\begin{aligned} y'(t) &= -2ty(t)^2, \qquad t \in I := [0, 1], \\ y(0) &= 1. \end{aligned}$$

We want to solve the IVP with the Adams-Bashforth-method

$$y_{m+4} - y_{m+3} = \frac{h}{24} \cdot \left(55 \cdot f(t_{m+3}, y_{m+3}) - 59 \cdot f(t_{m+2}, y_{m+2}) + 37 \cdot f(t_{m+1}, y_{m+1}) - 9 \cdot f(t_m, y_m)\right),$$

where h denotes the step size for an equidistant mesh.

(a) Determine the exact solution of the IVP.

(b) Write a function `[y,t]=adams_bashforth(f,tspan,y0,N,type)`, in which the initial values y_1, y_2, y_3

(i) are the exact values (`type = 1`).

(ii) are calculated with the classical Runge-Kutta-method of order 4 with step size `h=1/N` (`type = 2`).

(iii) are calculated with the explicit Euler-method with step size `h=1/N` (`type = 3`).

`f` is a function handle describing the right-hand side of the ODE, `tspan` is the time interval on which the ODE has to be solved, in oure case `tspan =[0,1]`, `y0` the initial value of the IVP and `N` the number of iterations.

(c) Create a plot with the solutions of Exercise 2(b) and discuss your results. Use and complete the file `aufgabe2.m`, which can be found on the homepage.

(d) Modify your code from Exercise 2(b) in such a way, that the explicit Euler-method uses the step sizes $H = h/2^i$ for $i = 0, 1, 2, \dots$ and plot the solutions. For which stepsize do we reach the same convergence order as for exact initial values?

Remark: Which values do you have to choose for obtaining y_1, y_2 and y_3?