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Sheet 6

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Exercise 1 (Supplement Sheet 3 - Implicit Euler Method)

Let $f \in C(I \times \mathbb{R}^n)$ be a global Lipschitz continuous with respect to the second argument, e.g.

$$\|f(t, y_1) - f(t, y_2)\|_2 \leq L\|y_1 - y_2\|_2, \qquad \forall (t, y_1), (t, y_2) \in I \times \mathbb{R}^n$$

and consider the fixpoint iteration

$$y_{k+1}^{(l+1)} = y_k + hf(t_{k+1}, y_{k+1}^{(l)}), \qquad l = 0, 1, 2, \dots$$

for the computation of the unknown values y_{k+1} needed for the implicit Euler method.

(a) Show, that the fixpoint iteration converges, if the step-size $h > 0$ fulfills the condition $h < 1/L$.

(b) Select a function $f \in C(I \times \mathbb{R}^n)$, which is global Lipschitz continuous with respect to the second argument, s.t. the resulting fixpoint iteration diverges for $h = 1/L$.
Hint: Remember Sheet 3, Exercise 3. Combine your results of Sheet 3 with this exercise.

Exercise 2 (Supplement Sheet 3 - Implicit RKM)

Let $f \in C(I \times \mathbb{R}^n)$ be a global Lipschitz continuous with respect to the second argument, e.g.

$$\|f(t, y_1) - f(t, y_2)\|_\infty \leq L\|y_1 - y_2\|_\infty, \qquad \forall (t, y_1), (t, y_2) \in I \times \mathbb{R}^n$$

and consider the fixpoint iteration

$$k_j^{(l+1)} = f\left(t_k + \alpha_j h, y_k + h\sum_{i=1}^{m} \beta_{ji} k_i^{(l)}\right), \qquad j = 1, \dots, m, \quad l = 0, 1, 2, \dots$$

for the computation of the stages k_j needed for the implicit RKM. Show that the fixpoint iteration converges for every initial value $k_1^{(0)}, \dots, k_m^{(0)}$ if the step-size $h > 0$ fulfills the condition

$$q := h\ L \max_{j=1,\dots,m} \sum_{i=1}^{m} |\beta_{ji}| < 1.$$

Exercise 3 (Inherent Instability)

Consider the IVP

$$\begin{aligned} y'(t) &= f(t, y(t)), \qquad t \in [0, T] \subset \mathbb{R} \\ y(0) &= y_0 \end{aligned} \tag{1}$$

and the disturbed problem

$$\begin{aligned} \hat{y}'(t) &= f(t, \hat{y}(t)) + \delta(t), \qquad t \in [0, T] \subset \mathbb{R} \\ \hat{y}(0) &= y_0 + \delta_0, \end{aligned}$$

where $\delta_0 \in \mathbb{R}$ und $\delta \in C([0,T])$. The IVP (1) is called *Ljapunov-stable* on $[0,T]$, if for every disturbation $(\delta_0, \delta(t))$, which satisfies for $\varepsilon > 0$ (small enough)

$$|\delta_0| < \varepsilon, \qquad |\delta(t)| < \varepsilon \qquad \forall t \in [0,T],$$

implies

$$\exists C > 0 \text{ (independend of } \varepsilon\text{), such that } |y(t) - \hat{y}(t)| < C\varepsilon, \qquad \forall t \in [0,T].$$

Note, that the Ljapunov-stability is a property of the cauchy-problem. In the following we want to consider a special class of ODE's, whose analytical solution is known. The IVP, we want to investigate, reads for $\lambda \in \mathbb{R}$ and $f \in C^1([0,T])$:

$$\begin{aligned} y'(t) &= \lambda(y(t) - f(t)) + f'(t), \qquad t \in [0,T] \\ y(0) &= y_0. \end{aligned} \tag{2}$$

(a) Determine the analytical solution of (2) for the initial value $y_0 = f(0)$

(b) Determine the analytical solution of (2) for the disturbed initial value $\hat{y}_0 = f(0) + \delta_0$ with $\delta_0 \in \mathbb{R}$. Discuss your results.

Problems such as (2), which identify themselves by a high sensitivity to their initial conditions are called *inherently-instable*. Consider the concrete IVP

$$\begin{aligned} y'(t) &= 10\left(y(t) - \frac{t^2}{t^2+1}\right) + \frac{2t}{(t^2+1)^2}, \qquad t \in [0,3] \\ y(0) &= 0, \end{aligned} \tag{3}$$

which is of the form (2) and inherently-instable.

(c) Show, that the IVP (3) is Ljapunov-stable on $[0,3]$. To do this use the lemma of Gronwall.

(d) Solve the IVP (3) with the implicit RKM, treated on sheet 3, with the Adams-Bashforth-Method (initial values calculated with the RKM) and the embedded RKM, both treated on sheet 4. Discuss your results.

Use for every method `N = 1000`, where `N` is the number of iterations. Plot all your solutions together with the analytical solutions $y(t) = \frac{t^2}{1+t^2}$, $t \in [0,3]$ in a common plot. Use the command `ylim([-1.5,1.5])`.

Exercise 4 (Predictor-Corrector Method)

We consider the following test problem

$$\begin{aligned} y'(t) &= \lambda y(t), \qquad t \in [0,15], \\ y(0) &= 1, \end{aligned}$$

with $\lambda = -1$ and $\lambda = 1$. We want to solve this problem with the following two predictor-corrector methods

(a) Milne:

$$\begin{aligned} y_{\ell+4}^{(0)} &= y_\ell + \frac{4}{3}h(2f_{\ell+3} - f_{\ell+2} + 2f_{\ell+1}) \\ y_{\ell+4}^{(\nu+1)} &= y_{\ell+2} + \frac{1}{3}h(f_{\ell+4}^{(\nu)} + 4f_{\ell+3} + f_{\ell+2}), \qquad \nu = 0, 1, \ldots. \end{aligned}$$

(b) Hamming:

$$\begin{aligned} y_{\ell+4}^{(0)} &= y_\ell + \frac{4}{3}h(2f_{\ell+3} - f_{\ell+2} + 2f_{\ell+1}) \\ y_{\ell+4}^{(\nu+1)} - \frac{9}{8}y_{\ell+3} + \frac{1}{8}y_{\ell+1} &= \frac{3}{8}h(f_{\ell+4}^{(\nu)} + 2f_{\ell+3} - f_{\ell+2}), \qquad \nu = 0, 1, \ldots, \end{aligned}$$

where $f_\ell = f(t_\ell, y_\ell)$ and $f_\ell^{(\nu)} = f(t_\ell, y_\ell^{(\nu)})$ respectively. Computing the first initial values for the multy-step method we use the classical Runge-Kutta method and as a criterion for termination for the corrector iteration we consider

$$\frac{|y_{\ell+4}^{(\nu+1)} - y_{\ell+4}^{(\nu)}|}{|y_{\ell+4}^{(\nu)}|} \leq 10^{-5}.$$

For the step-size h, we consider $h = 0.1$ and we compute in a tabular for $t = 0.1, 0.2, \ldots, 15$ the exact solution $y^*(t)$ and the numerical solution $y_h(t)$ and the error $e_h(t) = y^*(t) - y_h(t)$ and the number of iterations. Vary now h in a meaningful manner. Plot the corresponding error e_h also in a meaningful manner. May you verify the following statement (which you have already seen in the lecture):

Satz. *If f is regular enough and Lipschitz continuous with respect to y, then the order of consisteny of the predictor-corrector method is* $\min\{p, m_0 + p^*\}$*, where p^* and p are the order of consistency of the predictor and corrector respectively and m_0 is the number of iterations (or number of corrector steps respectively).*