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Numerik von gewöhnlichen Differenzialgleichungen
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Sheet 8

Due June 09, 2016.

Exercise 1 (Stability regions for multi-step methods)

Consider the linear k-step method

$$\sum_{r=0}^{k} \alpha_r y_{j+r} = h \sum_{r=0}^{k} \beta_r f_{j+r}$$

with the corresponding polynomials

$$\rho(z) = \sum_{r=0}^{k} \alpha_r z^r \quad \sigma(z) = \sum_{r=0}^{k} \beta_r z^r.$$

Then the polynomial with respect to stability of the k-step method is given by

$$\pi(z, h\lambda) = \rho(z) - h\lambda\sigma(z)$$

and the roots of π are denoted by $z_i(h\lambda)$. The set

$$R = \{h\lambda \ : \ |z_j(h\lambda)| < 1 \ , \ j = 0, \dots, k\}$$

is called stability region of the k-step method. For the boundary of R it can be shown that

$$\partial R \subseteq \widehat{R} := \{\widehat{h} \in \mathbb{C} \ : \ \widehat{h} = \rho(\exp(\mathrm{i}\phi))/\sigma(\exp(\mathrm{i}\phi)), \ 0 \le \phi \le 2\pi\}.$$

Plot the set $\widehat{R}$ (with Matlab) for the following k-step-methods:

- k-step Adams-Bashforth-method $y_{i+1} = y_i + h\sum_{r=0}^{k} \alpha_r f_r$:

k	α_k			
1	1			
2	3/2	-1/2		
3	23/12	-16/12	5/12	
4	55/24	-59/24	37/24	-9/24

- k-step Adams-Moulton-method $y_{i+1} = y_i + h\sum_{r=0}^{k} \alpha_r f_r$:

k	α_k				
1	1/2	1/2			
2	5/12	8/12	-1/12		
3	9/24	19/24	-5/24	1/24	
4	251/720	646/720	-264/720	106/720	-19/720

Exercise 2 (Another Stiff Problem)

Consider the following IVP $y'(t) = Ay(t)$, $y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where

$$A = \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix}.$$

(a) Determine the exact solution.

(b) Consider the scheme $y_k = f_1(h,k)v_1 + f_2(h,k)v_2$ and formulate it explicitly for the explicit Euler-method, where v_1 and v_2 are the Eigenvectors of A. What are the requirements for the step-size?

(c) Consider the scheme $y_k = f_1(h,k)v_1 + f_2(h,k)v_2$ and formulate it explicitly for the implicit Euler-method, where v_1 and v_2 are again the Eigenvectors of A. Why is there no restriction for the step-size?

Exercise 3 (Consistency ⇒ Convergence for MSM?)

Consider the following IVP

$$y'(t) = 0, \qquad y(0) = 0.1. \tag{1}$$

We want to apply the following explicit linear multi-step method

$$y_{\ell+2} + 4y_{\ell+1} - 5y_\ell = h(4f(t_{\ell+1}, y_{\ell+1}) + 2f(t_\ell, y_\ell)).$$

(a) Determine the order of consistency of the above multi-step method.

(b) Solve the above IVP (1) with the multi-step method in MATLAB (with arbitrary h) on $I = [0, 50]$. What do you observe and why? For the first initial values, use the exact solution of (1) or any suitable method of your choice.

Exercise 4 (Double-Pendulum)

The equation for a double-pendulum is given by

$$\begin{aligned} (m_1+m_2)\ell_1\ddot{\varphi}_1 + m_2\ell_2\ddot{\varphi}_2\cos(\varphi_1-\varphi_2) + m_2\ell_2\dot{\varphi}_1^2\sin(\varphi_1-\varphi_2) + g(m_1+m_2)\sin(\varphi_2) &= 0, \\ m_2\ell_2\ddot{\varphi}_2 + m_2\ell_1\ddot{\varphi}_1\cos(\varphi_1-\varphi_2) - m_2\ell_1\dot{\varphi}_1^2\sin(\varphi_1-\varphi_2) + gm_2\sin(\varphi_2) &= 0. \end{aligned}$$

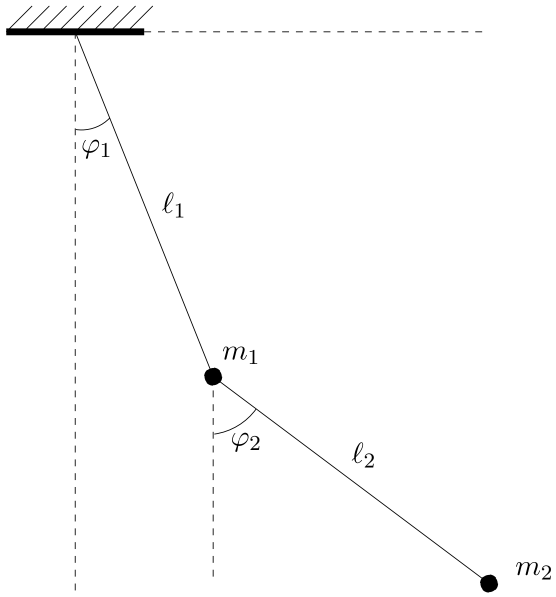
where $g = 9.81m/s^2$ the constant of gravitation.

(a) Rewrite the above equation into an equation of first order.

(b) Solve the equation by using a one-step- as well as a multi-step-method (of your choice!) with an appropriate step-size for $t \in [0, 20]$.

(c) Plot the solution and build an animation of the movement of the pendulum for the following configuration: $m_1 = 2$, $m_2 = 1$, $\ell_1 = 1$, $\ell_2 = 1.732051$ and

- $\varphi_1(0) = -\frac{\pi}{2}$, $\varphi_2(0) = \pi$, $\dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0$,
- $\varphi_1(0) = -\frac{\pi}{2}$, $\varphi_2(0) = \pi + 10^{-2}$, $\dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0$.



Hint: You can use the template on the homepage.