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Numerik von gewöhnlichen Differenzialgleichungen
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Sheet 9

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Introduction Finite Element Method:

Consider the Boundary-Value-Problem (BVP)

$$-u''(x) = f(x), \qquad x \in (-1,1), \tag{1}$$
$$u(-1) = u(1) = 0, \tag{2}$$

with the classical solution $u \in C^2((-1,1)) \cap C([-1,1])$. To derive the so called *weak formulation* of the BVP, we proceed as follows: Multiplying equation (1) with any test-function $v \in X$ with the property $v(-1) = v(1) = 0$ (X is a suitable space. Choosing the space $X = H_0^1((-1,1)))$ and integrating the equation, one gets

$$-\int_{-1}^{1} u''(x)v(x)\,dx = \int_{-1}^{1} f(x)v(x)\,dx, \qquad \forall v \in X.$$

Integration by parts and the use of $[u'(x)v(x)]_{-1}^{1} = 0$ because of the property $v(-1) = v(1) = 0$ yields

$$\int_{-1}^{1} u'(x)v'(x)\,dx = \int_{-1}^{1} f(x)v(x)\,dx, \qquad \forall v \in X. \tag{3}$$

Equation (3) is the so called weak formulation of the BVP. In order to obtain a discrete problem, we choose u and v from a suitable finite dimensional subspace $X_h := \operatorname{span}\{\varphi_1, \dots, \varphi_N\}$ of X. Therefore any element of X_h can be formulated as linear combination of the basis functions $\varphi_1, \dots, \varphi_N$ of X_h, e.g.

$$u_h(x) = \sum_{j=1}^{N} u_j \varphi_j(x).$$

Due to the linearity of the problem it is sufficient to use the basis functions $\varphi_1, \dots, \varphi_N$ as test functions instead of any $v_h \in X_h$. Applying these ideas to equation (3) we get:

$$\int_{-1}^{1} u_h'(x)\varphi_i'(x)\,dx = \int_{-1}^{1} f(x)\varphi_i(x)\,dx, \qquad i = 1, \dots, N$$
$$\iff \sum_{j=1}^{N} u_j \int_{-1}^{1} \varphi_j'(x)\varphi_i'(x)\,dx = \int_{-1}^{1} f(x)\varphi_i(x)\,dx, \qquad i = 1, \dots, N.$$

Thus we obtain a system of linear equations of the form $\mathbf{A}\mathbf{u_h} = \mathbf{f}$, where the so called *stiffness-matrix* $\mathbf{A} = (a_{i,j})_{i,j=1}^{N} \in \mathbb{R}^{N \times N}$ is given by

$$a_{ij} = \int_{-1}^{1} \varphi_j'(x)\varphi_i'(x)\,dx, \qquad 1 \le i, j \le N$$

and the right hand side $\mathbf{f} = (f_i)_{i=1}^{N} \in \mathbb{R}^N$ is given by

$$f_i = \int_{-1}^{1} f(x)\varphi_i(x)\,dx, \qquad 1 \le i \le N.$$

Solving this system of linear equations yields the vector $\mathbf{u_h} = (u_1, \dots, u_N)^T \in \mathbb{R}^N$, which is representing (together with the basis functions of X_h) the solution of the BVP.

Exercise 1 (FDM vs. FEM)

Consider the BVP

$$\begin{aligned} -u''(x) &= f(x), \qquad x \in \Omega := (a,b), \\ u(a) &= u(b) = 0. \end{aligned} \tag{4}$$

The aim of this task is to study the some distinctions between the FDM/FEM and the effect of using different basis functions for the finite dimensional subspace X_h, which fullfills the condition $X_h \subset X$, used in the FEM. For the FEM we consider the following basis functions for the subspace X_h:

- *Hat-functions:* We discretize the interval $[a,b]$ in N subintervals, which are not necessarily equidistant. Then for the resulting mesh $a = x_0 < x_1 < \ldots < x_N = x_{N+1} = b$ we define the hat-functions

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in (x_i, x_{i+1}], \\ 0, & \text{else}, \end{cases} \quad , \qquad i = 1, \ldots, N.$$

  For simplicity we will choose an equidistant mesh, e.g. $x_{i+1} - x_i = h$ for all $i = 1, \ldots, N$ and $h > 0$.

- *Trigonometric-functions:*

$$\varphi_k(x) = \sin\left(k\pi \frac{x-a}{b-a}\right), \qquad k = 1, \ldots, N.$$

(a) Determine the entries of the stiffness-matrix of the FEM using

  (i) hat-functions as basis functions for the discrete space X_h,

  (ii) trigonometric-functions as basis functions for the discrete space X_h.

Consider now the BVP (4) with $a = -1$ and $b = 1$, e.g.

$$\begin{aligned} -u''(x) &= f(x), \qquad x \in \Omega := (-1,1), \\ u(-1) &= u(1) = 0. \end{aligned} \tag{5}$$

(b) Determine the exact solution of the BVP (4) using

  (i) $f(x) = 1$ for $x \in [-1,1]$,

  (ii) $f(x) = |x|$ for $x \in [-1,1]$.

(c) Write a function `uh = fem_hat(f,tspan,N)`, which computes the FEM-solution of this BVP, using the hat-functions as basis functions for the discrete space X_h. `f` is a function handle describing the right-hand side of the BVP, `tspan` is the interval on which the BVP has to be solved, in our case `tspan=[-1,1]`, and `N` is the number of basis functions.
For a given function f compute the right-hand side of the weak formulation by a quadrature formula. Use the Matlab-function `quad` for this task.

(d) Consider again the BVP (5). Write a function `uh = fem_sin(f,tspan,N)`, which computes the FEM-solution of this BVP, using the trigonometric-functions as basis functions for the discrete space X_h. `f` is a function handle describing the right-hand side of the BVP, `tspan` is the interval on which the BVP has to be solved, in our case `tspan=[-1,1]`, and `N` is the number of basis functions.
For a given function f compute the right-hand side of the weak formulation by a quadrature formula. Use a Gauss-quadrature formula for this task. Use the file `gauss.m`, which can be found on the homepage.

(e) Test your functions `fem_hat.m` and `fem_sin.m` using the BVP (5) with the right-hand-side

  (i) $f(x) = 1$ for $x \in [-1,1]$,

(ii) $f(x) = |x|$ for $x \in [-1, 1]$,

for $N \in \{10 \cdot i : i = 1, \ldots, 10\}$ and plot your solutions together with the exact solution. Furthermore create a plot, which shows the L_2-error between the exact solution u determined in (b) and the numerical solution u_h. To do this use a suitable quadrature-formula (Matlab-function `quad` for hat-functions and gauss-quadrature-formula for the trigonometric-functions).

(f) It is known that for the 1D-case the stiffness-matrix of the FEM using hat-functions and the matrix produced by the FDM, are almost (up to a factor h) the same for the BVP (5). Modify your code from Exercise 2(e) in such a way, that the plot additionally shows the FDM-solution. Discuss your results. **Note:** Pay attention to the right-hand-side!