Prof. Dr. Karsten Urban
Mazen Ali

Ulm University
Institute for Numerical Mathematics
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Numerical Finance – Sheet 1

(due 23.04.2018)

Exercise 1: Congruential Generators

a) Show that

- Congruence modulo M on $\mathbb{Z}$ defines an equivalence relation $\sim$, i.e., is symmetric, reflexive and transitive.
- $(a+b) \mod M \equiv \big[(a \mod M) + (b \mod M)\big] \mod M$, $a, b \in \mathbb{Z}$.
- $(a \cdot b) \mod M \equiv \big[(a \mod M) \cdot (b \mod M)\big] \mod M$, $a, b \in \mathbb{Z}$.

b) Let $(y_n)_{n\in\mathbb{N}} \subset \mathbb{Z}_M$ be a sequence of pseudo random numbers (PRNs) generated by a linear congruential generators, i.e.

$$y_{n+1} = (ay_n + b) \mod M.$$

Usually, one is interested in uniformly distributed PRNs on $[0,1]$, so that usually the fractions $u_n = \frac{y_n}{M} \in [0,1]$ are considered.

- Show that $(u_n)_{n\in\mathbb{N}}$ fulfills the recurrence $u_{n+1} = \big(au_n + \frac{b}{M}\big) \mod 1$, where $z \mod 1 := z - \lfloor z \rfloor$.
- Why is it not a good idea to use that equation directly?

c) The so-called *Fibonacci sequence* is given by

$$y_{n+1} = (y_{n-1} + y_n) \mod M.$$

It is one of the examples for bad PRGs. One reason is the following: A reasonable requirement for a generator is that $y_{n-1} < y_{n+1} < y_n$ for about one sixth of the time (as all orderings of the numbers y_{n-1}, y_n, y_{n+1} should be equally probable). Show that this ordering never occurs for the Fibonacci sequence.

d) One (once) very popular generator, implemented by IBM in 1970, is the *RANDU generator*, a linear congruential generator with $a = 2^{16} + 3 = 65539$, $b = 0$, y_0 odd and $M = 2^{31} = 2147483648$.
Show that for $u_n := \frac{y_n}{M} \in [0,1)$, $u_{n+2} - 6u_{n+1} + 9u_n$ is an integer. What does this imply for the distribution of triples (u_n, u_{n+1}, u_{n+2}) in the unit cube?

Hint: First show that $y_{n+2} = 6y_{n+1} - 9y_n + c \cdot 2^{31}$ for some $c \in \mathbb{N}$.

e) Numbers of the form $M_n = 2^n - 1$ are called *Mersenne numbers.*

- What are the first 4 Mersenne prime numbers?
- Is M_{11} a prime number?

Programming Exercise 1: Linear Congruential Generators **(10 Points)**

There are many different implementations of linear congruential generators. We want to compare the following two examples:

- RANDU: See Exercise 1(d).
- UNIX rand(): standard Unix random number generator.

  $a = 1103515245$, $b = 12345$ and $M = 2^{31}$.

Implement a linear congruential generator. For both examples, using for example $y_0 = 1$,

a) simulate 30000 uniformly distributed 1-dimensional pseudo-random numbers on $[0, 1]$ and plot a histogram.

b) simulate 10000 uniformly distributed 3-dimensional pseudo-random vectors on $[0, 1]^3$ and visualize these samples in a 3D plot.

Compare the performance of the generators. Which one would you prefer?

Hints:

- In C/C++, use `long long int` to avoid floating point exceptions. Usage:

  ```
  long long int M = 2147483648LL;
  ```

- GNUPLOT can plot histograms with the following script:

  ```
  n = 50 # number of intervals
  width = 1./n
  bin(x,width) = width*floor(x/width) + width/2.0
  plot  "data.txt" using (bin($1,width)):(1.0) smooth freq with boxes title "MyData"
  ```

- Obtain 3-dimensional vectors by setting $u_1 = \left(\frac{y_1}{M}, \frac{y_2}{M}, \frac{y_3}{M}\right)^T$, $u_2 = \left(\frac{y_4}{M}, \frac{y_5}{M}, \frac{y_6}{M}\right)^T$, etc.
- 3D vectors can be plotted with GNUPLOT using `splot "file"`, where the file is of the form

  ```
  u11 u12 u13
  u21 u22 u23
  ...
  ```

  Be sure that the terminal type is wxt (`set terminal wxt`), so that you can rotate the plot.

Programming Exercise 2: χ^2-Test **(10 Points)**

One possibility to verify if a sequence of independent and identically distributed random variables $t_1, \ldots, t_n$ follows a certain distribution is the χ^2-test: We know that

$$\chi^2_{(n)} \xrightarrow{d} \chi^2_m \qquad \text{as } n \to \infty,$$

so that

$$\lim_{n\to\infty} \mathbb{P}[\chi^2_{(n)} > \chi_{m,1-\alpha}] = \alpha, \tag{1}$$

where is $\chi_{m,1-\alpha}$ the $(1-\alpha)$-quantile of the χ^2-distribution with m degrees of freedom.

a) Show that

$$\chi^2_{(n)}(x_1, ..., x_n) = \sum_{i=0}^{m} \frac{B_i^2}{E_i} - n.$$

b) Implement the computation of the test statistic $\chi^2_{(n)}$ for uniformly distributed random variables. Test whether the sequence of random numbers generated by the RANDU-algorithm is accepted by the test or not.

Hint: Recall that, knowing (1), the hypothesis that $t_1, \ldots, t_n$ are iid is rejected (at significance level α) if $\chi^2_{(n)} > \chi_{m,1-\alpha}$. You can find tabulated values of the χ^2-distribution on our homepage.