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Numerical Finance – Sheet 3

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Exercise 1: Multivariate Normal Distribution

a) Let $X \in \mathbb{R}^n$ be a random vector with expectation vector $\mu_X \in \mathbb{R}^n$ and covariance matrix $\Sigma_X \in \mathbb{R}^{n\times n}$. Prove that for the expectation vector and the covariance matrix of a linear transformation $Z = AX \in \mathbb{R}^m$ with $A \in \mathbb{R}^{m\times n}$ it holds:

$$\mu_Z = A\mu_X \qquad \text{and} \qquad \Sigma_Z = A\Sigma_X A^T.$$

b) Derive an algorithm for the generation of multivariate normal random vectors $Z \sim \mathcal{N}(\mu, C)$ $(Z, \mu \in \mathbb{R}^n, C \in \mathbb{R}^{n\times n})$.

Exercise 2: Variance Reduction Techniques I (Control variates)

Consider a random variable Z with expected value $z = \mathbb{E}[Z]$ and variance $\mathrm{Var}[Z] = \sigma_Z^2$. The usual Monte-Carlo estimator for the z is the empirical mean

$$\widehat{z} := \frac{1}{N}\sum_{i=1}^{N} Z_i, \qquad Z_i \text{ independent realizations of } Z.$$

As the convergence of Monte-Carlo behaves like $\frac{\sigma_Z^2}{\sqrt{N}}$, the idea of so-called *variance reduction techniques* is to construct a different estimator with a lower variance.

One possibility is to consider a *control variate* W with known mean $\mathbb{E}[W] = w$, variance $\mathrm{Var}[W] = \sigma_W^2$, and N independent copies $W_1, \dots, W_N$ of W, where we assume that

- $\mathrm{Cov}(W_i, Z_i) = \mathrm{Cov}(W, Z) > 0$ for all $i = 1, \dots, N$.
- W_i, Z_j are independent for $i \neq j$.

Instead of $\widehat{z}$, one then uses the estimator $\widehat{z}_{CV}$ as approximation for z, where

$$\widehat{z}_{CV} := \widehat{z} + \alpha(\widehat{w} - w) \quad \text{with} \quad \widehat{w} := \frac{1}{N}\sum_{i=1}^{N} W_i.$$

a) Show that for all $\alpha \in \mathbb{R}$, $\mathbb{E}[\widehat{z}_{CV}] = z$, $\mathrm{Var}[Z + \alpha(W - w)] = \sigma_Z^2 + 2\alpha\mathrm{Cov}(W, Z) + \alpha^2\sigma_W^2$ and $\mathrm{Var}[\widehat{z}_{CV}] = \frac{1}{N}(\sigma_Z^2 + 2\alpha\mathrm{Cov}(W, Z) + \alpha^2\sigma_W^2)$.

b) Show that $\mathrm{Var}[\widehat{z}_{CV}]$ attains a global minimum $\frac{1}{N}\sigma_Z^2(1 - \rho^2)$ for $\alpha = -\frac{\mathrm{Cov}(W,Z)}{\sigma_W^2}$ where

$$\rho := \frac{\mathrm{Cov}(W, Z)}{\sqrt{\sigma_W^2\sigma_Z^2}}.$$

Exercise 3: Sparse Grids

The sequence of one-dimensional grids with $n_i = 2^i - 1$, $i = 1, 2, \ldots$ equidistant points $x_1, \ldots, x_{n_i}$ on $[a, b]$ forms a nested grid. We can use the (open) Newton Cotes formulas to construct a simple sparse grid. They are given by

$$\begin{aligned} n_i = 1: &\quad (b-a)f(x_1), \\ n_i = 3: &\quad \frac{b-a}{3}(2f(x_1) - f(x_2) + 2f(x_3)). \end{aligned}$$

Using these as one-dimensional quadrature formulas $Q^{(1)}$ and $Q^{(2)}$, compute the first two-dimensional Smolyak Quadrature formula $Q(1, 2)$ on $[0, 1]^2$. What does the grid look like?

Programming Exercise 1: Variance Reduction Techniques II (10 Points)

a) **Antithetic variables**
Antithetic variables use the fact that if $u \sim U[0, 1]$ then also $\tilde{u} := 1 - u \sim U[0, 1]$. Using $u_1, \tilde{u_1}, u_2, \tilde{u_2}, \ldots$ in a simulation might reduce the variance σ_F if $\text{Cov}\big(F(u), F(\tilde{u})\big) < 0$, as is the case for example for monotone functions F.
Compute the integral

$$\int_0^1 (x - b)^3 \mathrm{d}x$$

by Monte Carlo integration for $b = 0.55$ and $b = 0.5$ with and without the use of antithetic variables and compare the error and the convergence rates.

b) **Control variates**
Consider the estimator $Z := \mathbf{1}_{\{U_1^2 + U_2^2 \leq 1\}}$ of $\frac{\pi}{4}$ where U_1, U_2 are independent and uniformly distributed on $[0, 1]$. As a control variate, consider $W := \mathbf{1}_{\{U_1 + U_2 \geq \sqrt{2}\}}$ with $\mathbb{E}[W] = \frac{1}{2}(2 - \sqrt{2})^2$.

(i) Give a geometrical interpretation for Z and W. Are there even better choices for W?

(ii) Estimate π via Monte-Carlo simulation with and without the use of the control variate W. Compare the error and the convergence rates. As σ_Z^2, σ_W^2 and $\text{Cov}(W, Z)$ are not given, use their empirical estimators to get an approximation for α.

Programming Exercise 2: MC vs QMC (6+2* Points)

Compute the integral

$$I_3[f] = \int_{[0,1]^3} x_1^2 \, x_2^2 \, x_3^2 \; \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3,$$

using

a) Monte Carlo integration,

b) Quasi-Monte Carlo integration, using the Halton sequence.

c) Quasi-Monte Carlo integration, using Sobol numbers (a (t, s)-sequence). You can find a text file with three-dimensional Sobol numbers on the homepage.

*(It is often recommended to skip the first Sobol numbers, since they are not as evenly distributed as later ones. One (heuristic) rule is e.g. to skip the first 2^{n-1} numbers if one uses 2^n numbers in the simulation. Try this for the above example.)

Visually compare all methods by plotting their integration errors and their theoretical convergence rates.

Gnuplot hints:

- You can use logarithmic scaling for one or both axes: `set logscale xy`
- You can define functions in Gnuplot, e.g. `f(x) = 1./x` and plot them using e.g. `plot[1:100][0:1] f(x)`.
  (The square brackets define the x-range and the y-range of the plot.)
- You can plot several datasets in one plot: `plot "data1", "data2"`.
- If you have several columns in your data file (e.g. N MC-Error(N) QMC-Error(N)), you can tell Gnuplot which columns to use:
  `plot "data" using 1:2 with lines, "data" using 1:3 with lines`.
- You can label the graphs:
  `plot "data" using 1:2 title "MC", "data" using 1:3 title "QMC"`.
- You can label the x- and y-axes and set a title: `set xlabel "Sample Size"`, `set ylabel "Error"`, `set title "Plot title"`.
- You can save your plot in ps, pdf or other formats (use `help terminal` for more information):

  ```
  > set terminal pdf enhanced color
  > set output "myplot.pdf"
  > plot  .....
  > set output
  > set terminal wxt
  ```

  This creates a file "myplot.pdf" with the plot . If you don't change back to terminal `wxt`, subsequent plots will be added to this file.