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Numerical Finance – Sheet 6

(due 4.06.2018)

Exercise 1: Strong Consistency of the Euler Scheme

Show that the Euler scheme is strongly consistent with $c(\delta) \equiv 0$.

Hints:

- τ_{n+1}, τ_n are $\mathcal{A}_{\tau_n}$-measurable.
- ΔW_n is independent of $\mathcal{A}_{\tau_n}$.
- Lyapunov inequality: For $X \in L_1(\Omega, \mathcal{A}, \mathbb{P})$ it holds that $\mathbb{E}(|X|) \leq \sqrt{\mathbb{E}(|X|^2)}$.

Exercise 2: Weak Consistency of the Euler Scheme

Show that the Euler scheme is weakly consistent.

Hints:

- Chebyshev inequality: $\mathbb{P}(\{\omega : |X(\omega)|^2 \geq a\}) \leq \frac{1}{a}\mathbb{E}(|X^2|)$ for all $a > 0$.
- Inequality: $(a + b + c)^3 \leq 3(a^2 + b^2 + c^2)$
- $\mathbb{E}((\Delta W_n)^2) = \Delta_n$.

Exercise 3: Stability of the Euler Scheme

Show that the Euler scheme is numerically stable under the assumptions of Theorem 4.2.4(b) (the existence of unique pathwise strong solutions).

Programming Exercise 1: Euler-Maruyama Error Analysis (12 Points)

Implement a function that computes the Euler-Maruyama approximation of a process

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t)$$

for a given Wiener process path $W(t)$, a maturity T, an initial value X_0 and functions $a(t, x)$, $b(t, x)$. Use this function to compute for the Geometric Brownian Motion

$$dX(t) = 2X(t)dt + X(t)dW(t)$$

a) the absolute error at the endpoint T,

b) the error of the entire process path.

Plot the error for different discretization levels. Which convergence rate do you expect? What do you observe?

Hints:

- Use the solution of the SDE to compare the approximation with the exact value.
- Use for example $N = 10000$ simulations for each error.
- You can pass the drift and diffusion coefficient functions as function pointers of the form

  ```
  double (*a)(const double t, const double x)
  double (*b)(const double t, const double x)
  ```

  to your Euler-Maruyama function.

Programming Exercise 2: Option Pricing with Euler-Maruyama (7 Points)

a) Use your function from programming exercise 1 to compute the option price of the European Put from Sheet 4, Prog.Ex. 2, with a Monte-Carlo simulation ($S_0 = 10$, $K = 12$, $r = 0.04$, $\sigma = 0.4$, $T = 2$).

b) Adapt your program such that it prices a European Up&Out Put, i.e. a European Put with payout function

$$V_T = (K - S_T)^+ \mathbb{1}_{\{S_t < H \ \forall t \in [0,T]\}}$$

that becomes worthless if $S_t > H$ for any $t \in [0, T]$. Options of this type are generally called *Barrier Options*. Compute the fair price of such an option with parameters as in (a) and $H = 13$.

Which convergence rates do you observe with respect to the time discretization?

Hints:

- The analytic solution of an Up&Out Put is given by

$$P_0 = Ke^{-rT}\left(\Phi(-d+\sigma\sqrt{T}) - \left(\frac{H}{S}\right)^{2\lambda-2}\Phi(-y+\sigma\sqrt{T})\right) - S_0\left(\Phi(-d) - \left(\frac{H}{S}\right)^{2\lambda}\Phi(-y)\right)$$

  with

$$\lambda = \frac{r+\frac{1}{2}\sigma^2}{\sigma^2}, \qquad y = \frac{\ln\left(\frac{H^2}{SK}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \qquad d = \frac{\ln\left(\frac{S}{K}\right) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

- For the above option, this formula yields the price $P_0 = 2.047849$.