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Numerical Finance – Sheet 7

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Exercise 1: Higher Order Schemes

Derive the following higher order Taylor scheme:

$$Y_{n+1} = Y_n + \left[a - \frac{1}{2}bb'\right]\Delta_n + b\Delta W_n + \frac{1}{2}bb'(\Delta W_n)^2 + \frac{1}{2}\left[aa' + \frac{1}{2}b^2 a''\right]\Delta_n^2 + \frac{1}{2}\left[a'b + b'a + \frac{1}{2}b''b^2\right]\Delta_n \, \Delta W_n. \quad (1)$$

Consider only the terms that are not already in the Milstein scheme. Use the fact that one can replace the integral

$$I_{t_0,t} := \int_{t_0}^{t} (W_s - W_t)ds$$

by its conditional expectation

$$\mathbb{E}\left[I_{t_0,t} \mid W_{t_0}, W_t - W_{t_0}\right] = \frac{1}{2}(t - t_0)\,(W_t - W_{t_0}).$$

Exercise 2: Multidimensional Schemes

Consider a d-dimensional Itô process $X(t) = (X_1(t), \dots, X_d(t))^T$, driven by an m-dimensional Brownian motion $W(t) = (W_1(t), \dots, W_m(t))^T$, i.e. $X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW(s)$ with $a : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$.

Use the multidimensional Itô formula to derive the appropriate Euler and Milstein schemes.

Hint:

- Consider each component X_i separately, proceed as in Section 6.7 and keep in mind that we neglect almost all double integrals.
- Integrals $I_{t_0,t}^{[k,l]} := \int_{t_0}^{t}\int_{t_0}^{s} dW_k(z)dW_l(s)$ with $k \neq l$ can be approximated by

$$I_{t_0,t}^{[k,l]} \approx \frac{1}{2}\left[(W_k(t) - W_k(t_0))(W_l(t) - W_l(t_0)) - V_{kl}\right],$$

where V_{kl} is a random variable with $V_{kl} = -V_{lk}$ for $l > k$ and $V_{kl} = \pm\Delta_n$ with probability $\frac{1}{2}$ for $l < k$.

- **Multidimensional Itô Formula**: For a d-dim. Itô process X, $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ with appropriate partial derivatives, $\Sigma := b \cdot b^T$ and $Y(t) := f(t, X(t))$, it holds with $b_{i,\cdot}$ the i-th row of b

$$Y(t) - Y(t_0) = \int_{t_0}^{t} \left[\frac{\partial f}{\partial t}(s, X(s)) + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(s, X(s)) a_i(s, X(s)) + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial f}{\partial x_i \partial x_j}(s, X(s)) \Sigma_{i,j}(s, X(s)) \right] ds$$
$$+ \int_{t_0}^{t} \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(s, X(s)) b_{i,\cdot}(s, X(s)) dW(s).$$

Programming Exercise 1: Higher Order Schemes (13 + 2 Points)

Consider (again) the European Put from Programming Exercise 2, Sheet 4. For the three methods

- Euler-Maruyama,
- Milstein,
- the scheme from (1),

compute

(i) the strong errors,

(ii) the weak errors w.r.t. the option payoff function,

(iii) the error w.r.t the Black-Scholes option price.

Compare the convergence rates of the errors w.r.t the time discretization, using e.g. $M = 10^5$ Monte-Carlo simulations. What do you see? What is the strong (weak) convergence order of the Taylor scheme (1)?

*Compare your results for the error of the option price with those obtained for significantly more MC simulations runs, e.g. $M = 10^6$, $M = 10^7$ (note that this may take some time!). What do you see? What does this imply for the relation between Monte-Carlo and SDE discretization error?

Programming Exercise 2: Heston Model (11 Points)

As the assumption of constant volatility in the Black-Scholes framework is often not consistent with market option prices, many models use *local* or *stochastic volatility* functions. One example for a stochastic volatility model is the *Heston model*, which models the volatility as a mean reverting square-root diffusion process and in its simplest form looks as follows:

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)dW_1(t),$$
$$dV(t) = \alpha(\theta - V(t))dt + \sqrt{V(t)}\sigma dW_2(t).$$

Compute the price of a European call with parameters $T = 1$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, $\alpha = 1.2$, $\theta = 0.04$ and initial values $S_0 = 100$, $V_0 = 0.04$ in this model,

a) with the Euler scheme,

b) with Milstein.

Hint: W_1 and W_2 are assumed to be independent. This simplifies the multidimensional schemes significantly!