ON NON-COERCIVE VARIATIONAL INEQUALITIES

SILKE GLAS AND KARSTEN URBAN

Abstract. We consider variational inequalities with different trial and test spaces and a possibly non-coercive bilinear form. Well-posedness is shown under general conditions that are e.g. valid for the space-time variational formulation of parabolic variational inequalities. Moreover, we prove an estimate for the error of a Petrov-Galerkin approximation in terms of the residual. For parabolic variational inequalities the arising estimate is independent of the final time.

1. Introduction

Let X, Y be two separable Hilbert spaces, $X \hookrightarrow Y$ dense, w.l.o.g. $\|w\|_Y \leq \|w\|_X$, $w \in X$. For a given $f \in Y'$ (the dual space of Y), we consider variational inequalities

$$u \in K \cap X: \quad a(u, v-u) \geq f(v-u) \quad \forall\, v \in K, \tag{1.1}$$

where $a : X \times Y \to \mathbb{R}$ is a bounded bilinear form (i.e., $\exists\, \gamma_a < \infty$ such that $a(v,w) \leq \gamma_a \|v\|_X \, \|w\|_Y$) and $K \subset Y$ is a closed convex set with $K \cap X \neq \emptyset$.[a] In order that (1.1) makes sense, one has to require that $X \subseteq Y$[b] since the test function $v-u$ needs to be in Y for $u \in X$ and $v \in K \subset Y$. A lot of research has been done in the case that $Y = X$ and $a(\cdot,\cdot)$ being coercive or at least satisfying a Gårding inequality, see e.g. [6, 10, 11], just to mention a few. This includes well-posedness (existence, uniqueness, stability) as well as numerical methods. Much less is known for the above mentioned general case.

One source of motivation for the present paper are time-dependent parabolic problems which fit into the described framework. The above references do contain interesting and important results also for the parabolic case. However, to the best of our knowledge, the following three issues have not been addressed so far: (1) Even though existence and uniqueness results for parabolic variational inequalities are known, we are not aware of corresponding results for general non-coercive bilinear forms $a(\cdot,\cdot)$. For variational *equalities* with a bounded bilinear form, it is well-known that Nečas condition (see Definition 2.2 below) is necessary *and* sufficient for the well-posedness of the considered problem. For (1.1) however, it is often assumed that $a : X \times X \to \mathbb{R}$ (i.e., $Y = X$) is coercive, which is significantly stronger

Date: July 2, 2014.

2010 *Mathematics Subject Classification.* 35K85,49J40,65M15.

Key words and phrases. Variational inequalities, parabolic problems, error estimates.

K.U. been supported by the Deutsche Forschungsgemeinschaft (DFG) under GrK 1100. Both authors have been supported by the DFG under SPP1324. The authors are grateful to Wolfgang Arendt and Mladjan Radic for helpful discussions and comments. We thank the anonymous referees for many helpful remarks that have significantly improved the paper.

[a]For $K = Y$, the variational inequality reduces to an equality.

[b]If nothing else is stated, we consider the case $X \subsetneq Y$. The special case $Y = X$ is only described for comparison purpose.

than Nečas condition. We want to relax the coercivity condition. (2) Stability results that allow to bound an unknown Petrov-Galerkin approximation error by a computable residual are also not known in the above described general case. (3) Space-time variational formulations of parabolic problems. The motivation for this comes from recent results in Reduced Basis Methods (RBM), where it has been shown that a space-time analysis can significantly improve error estimates for a RBM, [17, 18]. We are interested in the extension to variational inequalities.

Let us briefly describe for a simple example why we are interested in obtaining error/residual relations. Let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form and consider the variational problem of finding $u \in X$ such that $a(u, v) = \langle f, v\rangle$ for all $v \in Y$. Let $X_N \subset X$, $Y_N \subset Y$ be (finite-dimensional) subspaces and denote by $u_N \in X_N$ the Petrov-Galerkin approximation, i.e., $a(u_N, v_N) = \langle f, v_N\rangle$ for all $v_N \in Y_N$ (where we assume that Y_N is appropriately chosen in order to guarantee a stable solution). If $a(\cdot, \cdot)$ is inf-sup stable with constant $\beta_a > 0$, we get $\beta_a\|u - u_N\|_X \leq \sup_{v\in Y} \frac{a(u-u_N,v)}{\|v\|_Y} = \sup_{v\in Y} \frac{\langle f,v\rangle - a(u_N,v)}{\|v\|_Y} = \|r_N\|_{Y'}$ with the residual defined as $r_N(v) := \langle f, v\rangle - a(u_N, v)$, $v \in Y$. If (like in the case of Reduced Basis Methods, RBM) the dual norm of the residual, i.e., $\|r_N\|_{Y'}$, is efficiently computable (at least approximately), one obtains an estimate for the error $e_N := u - u_N$, as long as β_a, or a lower bound for it, is computable. Without going into the details, we remark that this is true for RBM, see e.g. [14]. However, we think that the investigation of (1.1) is interesting on its own. We like to understand which kind of additional properties are required in the more general framework to derive well-posedness and error/residual estimates as known in the coercive case.

The remainder of this paper is organized as follows. In Section 2, we consider well-posedness of variational inequalities. After collecting some preliminaries in §2.1, we consider a well-known technique for non-coercive problems [11], namely to regularize $a(\cdot, \cdot)$ by a coercive form $a^\varepsilon(\cdot, \cdot)$. This already leads to some required properties in terms of the relation of the spaces X and Y that are analyzed in §2.2. It turns out that in order to prove well-posedness in the general framework in §2.4, we need to pose some additional requirements on $a(\cdot, \cdot)$ which are collected in §2.3. Section 3 is devoted to the derivation of error/residual estimates for which we consider a well-known saddle-point formulation of (1.1). We generalize estimates from [6]. Next, we detail our general findings for the particular case of space-time variational formulations of parabolic variational inequalities in Section 4. The paper ends with conclusions in Section 5 and a summary of all results in Table 5.1.

We briefly summarize the –at least up to the best of our knowledge– new contributions of this paper:

- We show existence and uniqueness for a variational inequality (1.1) under milder conditions than what is known from the literature, see Theorems 2.15 and 2.16.
- Error vs. residual estimates are derived that are new and also cover a setting with less restrictive assumptions, see Theorem 3.12.
- A new error estimate for the space-time variational formulation of parabolic variational inequalities is derived. This improves on corresponding results e.g. in [8, §5] that involve the true error.

2. Well-Posedness

We consider well-posedness of (1.1), i.e., existence, uniqueness and stability.

2.1. **Preliminaries.** The following result is well-known in the coercive case and sets the benchmark for our analysis.

Theorem 2.1 (Coercive case, e.g. [5, Satz 7.1], [10])**.** *Let $a : X \times X \to \mathbb{R}$ be bounded and coercive, i.e., $a(v,v) \geq \alpha_a \|v\|_X^2$, $v \in X$. Then,* (1.1) *(for $Y = X$) admits a unique solution $u \in K \cap X$ for all $f \in X'$ and*

$$\|u\|_X \leq \frac{1}{\alpha_a}\|f\|_{X'} + \Big(\frac{\gamma_a}{\alpha_a} + 1\Big)\,\mathrm{dist}_{\|\cdot\|_X}(0,K), \tag{2.1}$$

where $\mathrm{dist}_{\|\cdot\|_X}(v,K) := \inf_{k\in K}\|v-k\|_X$*, $v \in X$.* □

Since we aim at relaxing the assumptions on the bilinear form $a(\cdot,\cdot)$ (mainly coercivity and allowing $Y \neq X$), it pays off to consider variational *equalities* in order to understand what kind of result might be achievable for *inequalities*.

Definition 2.2. We say that the bilinear form $a : X \times Y \to \mathbb{R}$ satisfies a *Nečas condition* on $U \subseteq Y$, if there exists a $\beta_a > 0$ such that

$$\sup_{w\in U}\frac{a(v,w)}{\|w\|_Y} \geq \beta_a\|v\|_X \;\forall\, v \in X \cap U, \qquad \sup_{v\in X\cap U} a(v,w) > 0 \;\forall\, 0 \neq w \in U.$$

Theorem 2.3 (Variational *equality*, see e.g. [12, Thm. 3.3], [13, Thm. 2])**.** *Let $a : X \times Y \to \mathbb{R}$ be bounded. Then, the variational* equality *$u \in X$: $a(u,v) = f(v)$ $\forall\, v \in Y$ is well-posed (i.e., admits a unique solution $u \in X$ for all $f \in Y'$, which depends continuously on f)* if and only *if $a(\cdot,\cdot)$ satisfies a Nečas condition on Y.* □

The above result sets the benchmark in the sense that we cannot hope to derive well-posedness of (1.1) under milder assumptions than a Nečas condition. We have not been able to derive results using only Nečas condition, but need additional conditions concerning the relation between the two Hilbert spaces X and Y.

2.2. **Regularization.** It is a standard technique in the analysis of non-coercive problems to define a regularized bilinear form $a^\varepsilon(\cdot,\cdot)$ that is coercive and then to consider the limit as $\varepsilon \to 0$. In order to do so, let $|\cdot|_X$ be a seminorm on X induced by some inner product $(\!(\cdot,\cdot)\!)_X$ on X, i.e., $(\!(v,v)\!)_X = |v|_X^2$, $(\!(v,w)\!)_X \leq |v|_X\,|w|_X$ for $v, w \in X$.[c] Then, we assume that the norms on X and Y are related as

$$\|v\|_X^2 = |v|_X^2 + \|v\|_Y^2, \quad v \in X. \tag{2.2}$$

Moreover, for $\varepsilon > 0$, we define a (well-known, [11]) "coercive (or elliptic) regularization" $a^\varepsilon(v,w) := \varepsilon(\!(v,w)\!)_X + a(v,w)$, $v, w \in X$, as well as the norm

$$\|v\|_\varepsilon^2 := \varepsilon|v|_X^2 + \|v\|_Y^2, \quad v \in X. \tag{2.3}$$

In particular we have that $\|\cdot\|_\varepsilon$ equals $\|\cdot\|_X$ for $\varepsilon = 1$ and $\|v\|_\varepsilon \to \|v\|_Y$ as $\varepsilon \to 0$, $v \in X$. The idea is to consider a coercive regularized problem (see Lemma 2.6 below) and then to derive well-posedness of (1.1) by considering the limit $\varepsilon \to 0+$. The following properties are easily shown.

Lemma 2.4. *Let $a : X \times Y \to \mathbb{R}$ be bounded and $0 \leq \varepsilon \leq 1$. Then, for all $v, w \in X$:*

(i) *$a^\varepsilon(v,w) \leq \gamma_a^+\,\|v\|_X\,\|w\|_\varepsilon$ with $\gamma_a^+ := \sqrt{2}\max\{1,\gamma_a\}$.*

(ii) *With γ_a^+ as in* (i) *it holds that $a^\varepsilon(v,w) \leq \frac{\gamma_a^+}{\sqrt{\varepsilon}}\|v\|_\varepsilon\,\|w\|_\varepsilon$.*

If, in addition $a(v,v) \geq \alpha_{a,Y}\|v\|_Y^2$ holds for all $v \in X$, we have

[c] Note that it is *not* required that $|v|_X = 0$ implies $v = 0$ in X.

(iii) $a^\varepsilon(v,v) \geq \min\{1, \alpha_{a,Y}\}\|v\|_\varepsilon^2$. (iv) $a^\varepsilon(v,v) \geq \min\{\alpha_{a,Y}, \varepsilon\}\|v\|_X^2$.

Proof. Let $v, w \in X$, then the boundedness of $a(\cdot,\cdot)$ implies the estimate $a^\varepsilon(v,w) \leq \varepsilon|v|_X\,|w|_X + \gamma_a\|v\|_X\|w\|_Y \leq \|v\|_X(\varepsilon|w|_X + \gamma_a\|w\|_Y) \leq \gamma_a^+\|v\|_X\|w\|_\varepsilon$ by Remark 2.5 below with $a = |w|_X$, $b = \|w\|_Y$, $\gamma = \gamma_a$, which proves (i). As for (ii), we note that by $\varepsilon \leq 1$ we have $\|v\|_X^2 \leq \frac{1}{\varepsilon}(\varepsilon|v|_X^2 + \|v\|_Y^2) = \frac{1}{\varepsilon}\|v\|_\varepsilon^2$, so that (ii) follows from (i). The statement (iii) follows immediately by $a^\varepsilon(v,v) = \varepsilon|v|_X^2 + a(v,v) \geq \varepsilon|v|_X^2 + \alpha_{a,Y}\|v\|_Y^2 \geq \min\{1,\alpha_{a,Y}\}\|v\|_\varepsilon^2$ for $v \in X$. Finally, (iv) follows from (iii) and (2.3). □

Remark 2.5. *Let $a, b, \varepsilon, \gamma \geq 0$. Then, we have for all $0 \leq \varepsilon \leq 1$ that $\varepsilon a + \gamma b \leq \tilde{\gamma}\sqrt{\varepsilon a^2 + b^2}$ with $\tilde{\gamma} := \sqrt{2}\max\{1,\gamma\}$. In fact, by $\varepsilon \leq 1$ we get $(\varepsilon a + \gamma b)^2 = \varepsilon^2 a^2 + 2\varepsilon\gamma ab + \gamma^2 b^2 \leq 2(\varepsilon^2 a^2 + \gamma^2 b^2) \leq 2\max\{1,\gamma^2\}(\varepsilon a^2 + b^2)$.*

Note that the condition $a(v,v) \geq \alpha_{a,Y}\|v\|_Y^2$, $v \in X$, is milder than coercivity due to the weaker Y-norm on the right-hand side. The relationship to Nečas condition is at least not obvious. Now, the following statement is readily seen.

Lemma 2.6. *Let $a(\cdot,\cdot)$ be bounded and assume that $a(v,v) \geq \alpha_{a,Y}\|v\|_Y^2$ holds for all $v \in X$. Then, the regularized variational inequality*

$$u^\varepsilon \in K \cap X: \qquad a^\varepsilon(u^\varepsilon, v - u^\varepsilon) \geq f(v - u^\varepsilon) \quad \forall\, v \in K \cap X \tag{2.4}$$

has a unique solution for all fixed $\varepsilon > 0$.

Proof. By Lemma 2.4 (ii), $a^\varepsilon(\cdot,\cdot)$ is bounded on $(X, \|\cdot\|_\varepsilon)$ and Lemma 2.4 (iii) guarantees that $a^\varepsilon(\cdot,\cdot)$ is also coercive on $(X, \|\cdot\|_\varepsilon)$ for all $\varepsilon > 0$, which implies well-posedness by Theorem 2.1. □

Theorem 2.1 also yields a stability estimate (2.1) for $\|u^\varepsilon\|_\varepsilon$ with α_a, γ_a replaced by

$$\alpha_a^- := \min\{1, \alpha_{a,Y}\}, \qquad \gamma_\varepsilon := \varepsilon^{-1/2}\gamma_a^+, \quad \gamma_a^+ := \sqrt{2}\,\max\{1,\gamma_a\}.$$

However, since $\gamma_\varepsilon \nearrow \infty$ as $\varepsilon \to 0$ the stability estimate (2.1) is meaningless in the limit $\varepsilon \to 0$. Fortunately, a refined analysis leads to a useful estimate as follows.

Proposition 2.7. *Let $a : X \times Y \to \mathbb{R}$ be bounded and $a(v,v) \geq \alpha_{a,Y}\,\|v\|_Y^2$ for all $v \in X$. With $\|f\|_{\varepsilon'} := \sup_{v\in X}\frac{f(v)}{\|v\|_\varepsilon}$, the unique solution $u^\varepsilon \in X$ of (2.4) satisfies*

$$\|u^\varepsilon\|_\varepsilon \leq \frac{1}{\alpha_a^-}\|f\|_{\varepsilon'} + \Big(\frac{\gamma_a^+}{\alpha_a^-} + 1\Big)\mathrm{dist}_{\|\cdot\|_X}(0,K). \tag{2.5}$$

Proof. By Lemma 2.4 (iii), we get for any $v \in K \cap X$ that

$$\begin{aligned} \alpha_a^-\|v - u^\varepsilon\|_\varepsilon^2 &\leq a^\varepsilon(v - u^\varepsilon, v - u^\varepsilon) = a^\varepsilon(v, v - u^\varepsilon) - a(u^\varepsilon, v - u^\varepsilon) \\ &\leq a^\varepsilon(v, v - u^\varepsilon) - f(v - u^\varepsilon), \end{aligned}$$

where the last step follows from (2.4). Then, by Lemma 2.4 (i) we get $a^\varepsilon(v, v - u^\varepsilon) - f(v - u^\varepsilon) \leq \gamma_a^+\|v - u^\varepsilon\|_\varepsilon(\|v\|_X + \|f\|_{\varepsilon'})$. Since v was chosen arbitrarily, the triangle inequality proves the claim. □

Corollary 2.8. *Under the assumptions of Proposition 2.7 we have that*

$$\|u^\varepsilon\|_\varepsilon \leq \frac{1}{\alpha_a^-}\|f\|_{X'} + \Big(\frac{\gamma_a^+}{\alpha_a^-} + 1\Big)\mathrm{dist}_{\|\cdot\|_X}(0,K). \tag{2.6}$$

Proof. Since $f \in Y' \subset X'$, we easily get $\|f\|_{\varepsilon'} = \sup_{v\in X}\frac{f(v)}{\|v\|_\varepsilon} \leq \sup_{v\in X}\frac{f(v)}{\|v\|_Y} \leq \sup_{v\in Y}\frac{f(v)}{\|v\|_Y} = \|f\|_{Y'}$, and the rest follows from Proposition 2.7. □

Note that the right-hand side of (2.6) does *not* depend on ε, which will be relevant later. We cannot immediately take the limit as $\varepsilon \to 0$ in (2.6) since both the norm $\|\cdot\|_\varepsilon$ and u^ε depend on ε. It will turn out later that we need to replace the inf-sup condition in Definition 2.2 by exchanging $\|\cdot\|_Y$ by $\|\cdot\|_\varepsilon$. As we shall see now, the latter one can already be ensured by Nečas condition.

Lemma 2.9. *Let X, Y be Hilbert spaces with X being densely and continuously embedded into Y and $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form that satisfies an inf-sup condition $\beta_a := \inf_{v\in X} \sup_{w\in Y} \frac{a(v,w)}{\|v\|_X\, \|w\|_Y} > 0$. Then, for $\beta_{a,\varepsilon} := \inf_{v\in V} \sup_{x\in X} \frac{a(v,x)}{\|v\|_X\, \|x\|_\varepsilon}$ we have that $\lim_{\varepsilon\to 0} \beta_{a,\varepsilon} = \beta_a$.*

Proof. For $v \in X$ we define $J_\varepsilon(v) := \sup_{x\in X} \frac{a(v,x)}{\|v\|_X\, \|x\|_\varepsilon}$, i.e., $\beta_{a,\varepsilon} = \inf_{v\in V} J_\varepsilon(v)$. It is readily seen that $0 \le J_\varepsilon(v) \le \gamma_a$. Hence, there exists a minimizing sequence $(v_k)_{k\in\mathbb{N}} \subset X$, i.e., $\lim_{k\to\infty} J_\varepsilon(v_k) = \inf_{v\in X} J_\varepsilon(v) = \beta_{a,\varepsilon} \ge 0$. Since Y is closed, the inf-sup condition implies for any such $v_k \in X$ the existence of a supremizer $y_k \in Y$ with $a(v_k, y_k) \ge \beta_a \|v_k\|_X\, \|y_k\|_Y$. Now choose

$$0 < \delta < \frac{\beta_a}{\gamma_a}$$

fixed. Since $X \hookrightarrow Y$ dense, there exists a $x_{k,\delta} \in X$ such that $\|x_{k,\delta} - y_k\|_Y \le \delta\, \|y_k\|_Y$, which implies that $\|x_{k,\delta}\|_Y \le (1 + \delta)\|y_k\|_Y$. Then, we obtain

$$\begin{aligned} a(v_k, x_{k,\delta}) = a(v_k, y_k) - a(v_k, y_k - x_{k,\delta}) &\ge \beta_a \|v_k\|_X\, \|y_k\|_Y - \gamma_a\, \|v_k\|_X\, \|y_k - x_{k,\delta}\|_Y \\ &\ge (\beta_a - \delta\gamma_a)\, \|v_k\|_X\, \|y_k\|_Y \end{aligned}$$

and $\beta_a - \delta\gamma_a > 0$ due to the choice of δ. Next, we estimate $\|x_{k,\delta}\|_\varepsilon^2 = \varepsilon |x_{k,\delta}|_X^2 + \|x_{k,\delta}\|_Y^2 \le \varepsilon |x_{k,\delta}|_X^2 + (1 + \delta)^2 \|y_k\|_Y^2$, so that we get

$$J_\varepsilon(v_k) \ge \frac{a(v_k, x_{k,\delta})}{\|v_k\|_X\, \|x_{k,\delta}\|_\varepsilon} \ge \frac{(\beta_a - \delta\gamma_a)\|y_k\|_Y}{(\varepsilon |x_{k,\delta}|_X^2 + (1 + \delta)^2 \|y_k\|_Y^2)^{1/2}} =: b_{\varepsilon,k}.$$

Since $\lim_{\varepsilon\to 0} b_{\varepsilon,k} = \frac{\beta_a - \delta\gamma_a}{1+\delta}$ (which is independent of k) and $0 \le b_{\varepsilon,k} \le J_\varepsilon(v_k) \le \gamma_a < \infty$, the sequence $(b_{\varepsilon,k})_{k\in\mathbb{N}}$ is bounded and thus contains a convergent subsequence $(b_{\varepsilon,k_m})_{m\in\mathbb{N}}$. Since a subsequence of a minimizing sequence is also a minimizing sequence, we denote the subsequence again by $(b_{\varepsilon,k})_{k\in\mathbb{N}}$, whose limit $\lim_{k\to\infty} b_{\varepsilon,k}$ thus exists. Hence, we may exchange the two limits and obtain

$$\lim_{\varepsilon\to 0} \beta_{a,\varepsilon} = \lim_{\varepsilon\to 0} \lim_{k\to\infty} J_\varepsilon(v_k) \ge \lim_{\varepsilon\to 0} \lim_{k\to\infty} b_{\varepsilon,k} = \lim_{k\to\infty} \lim_{\varepsilon\to 0} b_{\varepsilon,k} = \frac{\beta_a - \delta\gamma_a}{1 + \delta}.$$

Since this holds for all $0 < \delta < \frac{\beta_a}{\gamma_a}$, we may consider the limit $\delta \to 0+$, which proves the claim. □

Remark 2.10. *We can also interpret the statement in Lemma 2.9 as follows: There exists a $\bar\varepsilon > 0$ such that $\beta_{a,\varepsilon} > 0$ for all $0 \le \varepsilon < \bar\varepsilon$. Note, that Lemma 2.9 in general does* not *imply the validity of an inf-sup condition for $a(\cdot,\cdot)$ with $\|\cdot\|_\varepsilon$ replaced by $\|\cdot\|_X$. In fact, this would only hold if $\bar\varepsilon > 1$, which is definitely not true in general.*

2.3. **Additional Assumptions.** As already mentioned above, we have not been able to prove well-posedness of (1.1) only under the conditions formulated so far. However, this does not seem to be astonishing since it was shown in [11] that assuming just $a(v, v) \geq 0$ for $v \in Y = X$ yields a possibly *empty* closed convex set of solutions. Existing well-posedness results are available for quite specific cases only. We will now introduce properties mainly on $a(\cdot, \cdot)$ which are valid e.g. for parabolic variational inequalities. The first piece is an additional seminorm on X denoted by $[\cdot]_X$ induced by a scalar product $[\cdot, \cdot]_X$ such that

$$[v, v]_X = [v]_X^2, \qquad [v, w]_X \leq [v]_X [w]_X, \quad v, w \in X, \tag{2.7}$$ [d]

as well as

$$\exists\, C > 0 : \quad [v]_X \leq C \|v\|_X,\ v \in X. \tag{2.8}$$

Next, we consider a stronger (but equivalent) norm on X, namely

$$|\!|\!| v |\!|\!|_X^2 := |v|_X^2 + [v]_X^2 + \|v\|_Y^2 =: |v|_X^2 + [\![v]\!]_X^2, \tag{2.9}$$

i.e., $[\![v]\!]_X^2 := [v]_X^2 + \|v\|_Y^2$. Later, we will also use dual (semi-)norms defined as

$$[\![f]\!]_{X'} := \sup_{v \in X} \frac{f(v)}{[\![v]\!]_X} \quad \text{and} \quad |\!|\!| f |\!|\!|_{X'} := \sup_{v \in X} \frac{f(v)}{|\!|\!| v |\!|\!|_X}.$$

Note, that (2.8) has the following consequence which we will use later, namely

$$|\!|\!| v |\!|\!|_X \leq \sqrt{1 + C^2}\, \|v\|_X, \quad w \in X. \tag{2.10}$$

In fact, we have $|\!|\!| v |\!|\!|_X^2 = |v|_X^2 + [v]_X^2 + \|v\|_Y^2 \leq |v|_X^2 + C^2 \|v\|_X^2 + \|v\|_Y^2 = (1 + C^2)\|v\|_X^2$, recalling (2.2). Similarly, we get

$$|\!|\!| v |\!|\!|_X \leq \varepsilon^{-1/2} \sqrt{1 + C^2}\, \|v\|_\varepsilon, \quad w \in X,\ 0 < \varepsilon < 1. \tag{2.11}$$

Since trivially $\|v\|_X \leq |\!|\!| v |\!|\!|_X = (\|v\|_X^2 + [\![v]\!]_X^2)^{1/2}$, the norms $\|\cdot\|_X$ and $|\!|\!| \cdot |\!|\!|_X$ are equivalent on X. One might think that we could simply replace $\|\cdot\|_X$ by the equivalent norm $|\!|\!| \cdot |\!|\!|_X$. This, however, would lead to error/residual estimates which may be far from being sharp. The precise interplay between $\|\cdot\|_X$ and $|\!|\!| \cdot |\!|\!|_X$ turns out to be crucial. As we shall see below, $[\cdot]_X$ will play a significant role in our analysis, keeping footnote d in mind.

Definition 2.11. We call a bilinear form $a : X \times Y \to \mathbb{R}$, $X \subset Y$,

(a) *weakly coercive*, if there exists a constant $\alpha_{\mathrm{w}} > 0$ such that $a(v, v) \geq \alpha_{\mathrm{w}} [\![v]\!]_X^2$ for all $v \in X$;
(b) *symmetrically bounded*, if there exists a constant $\gamma_{\mathrm{s}} < \infty$ such that $a(v, w) \leq \gamma_{\mathrm{s}} [\![v]\!]_X |\!|\!| w |\!|\!|_X$ for all $v, w \in X$.

At a first glance (b) might be astonishing. However, as we shall see below in the case of parabolic variatiational inequalities, it is not realistic to hope that an estimate of the form $a(v, w) \leq \gamma \|v\|_Y \|w\|_X$ holds true, not even for $v, w \in X$. In fact, (b) will be verified by integration by parts so that the final time contribution will be $[v]_X$ leading to $[\![v]\!]_X$ instead of $\|v\|_Y$. This is also the reason for introducing the above stronger norms. We also remark that weak coercivity in general does not imply Nečas condition. In fact, if $a(\cdot, \cdot)$ is weakly coercive, we obtain for $v \in X \subset Y$

$$\sup_{w \in Y} \frac{a(v, w)}{\|w\|_Y} \geq \frac{a(v, v)}{\|v\|_Y} \geq \alpha_{\mathrm{w}} \frac{[\![v]\!]_X^2}{\|v\|_Y} \geq \alpha_{\mathrm{w}} [\![v]\!]_X,$$

[d]Again, we have to keep in mind that $[v]_X = 0$ does *not* imply that $v = 0$ in X.

but $[\![\cdot]\!]_X$ is not a norm on X and cannot be bounded from below by $\|\cdot\|_X$ (only from above by (2.10)). The next step is to modify $\|\cdot\|_\varepsilon$ as given in (2.3) by

$$\|\!|v|\!\|_\varepsilon^2 := \varepsilon|v|_X^2 + [v]_X^2 + \|v\|_Y^2 = \varepsilon|v|_X^2 + [\![v]\!]_X^2 = \|v\|_\varepsilon^2 + [v]_X^2. \tag{2.12}$$

Then, the analogue of Lemma 2.4 reads as follows:

Corollary 2.12. *Let $a : X \times Y \to \mathbb{R}$ be bounded, symmetrically bounded and $0 \le \varepsilon \le 1$. Then, we have for all $v, w \in X$:*

(i) *$a^\varepsilon(v, w) \le \gamma_{\mathrm{s}}^+ \|\!|v|\!\|_X \|\!|w|\!\|_\varepsilon$ with $\gamma_{\mathrm{s}}^+ := \sqrt{2}\max\{1, \gamma_{\mathrm{s}}\}$.*

(ii) *With γ_{s}^+ as in (i) it holds that $a^\varepsilon(v, w) \le \frac{\gamma_{\mathrm{s}}^+}{\sqrt{\varepsilon}} \|\!|v|\!\|_\varepsilon \|\!|w|\!\|_\varepsilon$.*

If in addition $a(\cdot, \cdot)$ is weakly coercive, we have

(iii) *$a^\varepsilon(v, v) \ge \min\{1, \alpha_{\mathrm{w}}\}\|\!|v|\!\|_\varepsilon^2$.* (iv) *$a^\varepsilon(v, v) \ge \min\{\alpha_{\mathrm{w}}, \varepsilon\}\|\!|v|\!\|_X^2$.* □

Proposition 2.13. *Let $a(\cdot, \cdot)$ be symmetrically bounded, then we have the estimate $a^\varepsilon(v, w) \le \gamma_{\mathrm{s}}^+ \|\!|v|\!\|_\varepsilon \|\!|w|\!\|_X$ for all $v, w \in X$.*

Proof. Let $v, w \in X$, then $a^\varepsilon(v, w) = \varepsilon(\!(v, w)\!)_X + a(v, w) \le \varepsilon|v|_X\, |w|_X + \gamma_{\mathrm{s}}[\![v]\!]_X \|\!|w|\!\|_X \le \|\!|w|\!\|_X(\varepsilon|v|_X + \gamma_{\mathrm{s}}[\![v]\!]_X) \le \sqrt{2}\max\{1, \gamma_{\mathrm{s}}\} \|\!|v|\!\|_\varepsilon \|\!|w|\!\|_X$, by Remark 2.5. □

We now consider the regularized variational inequality (2.4) on $(X, \|\!|\cdot|\!\|_\varepsilon)$. Using the same arguments as in Proposition 2.7 and Corollary 2.8 yields the following:

Corollary 2.14. *Let $a(\cdot, \cdot)$ be bounded, symmetrically bounded and weakly coercive. Then, the unique solution $u^\varepsilon \in X$ of (2.4) satisfies*

$$\|\!|u^\varepsilon|\!\|_\varepsilon \le \frac{1}{\alpha_{\mathrm{w}}^-}\|\!|f|\!\|_{X'} + \Big(\frac{\gamma_{\mathrm{s}}^+}{\alpha_{\mathrm{w}}^-} + 1\Big)\mathrm{dist}_{\|\!|\cdot|\!\|_X}(0, K), \tag{2.13}$$

where $\alpha_{\mathrm{w}}^- := \min\{1, \alpha_{\mathrm{w}}\}$ and γ_{s}^+ as in Corollary 2.12 (i).

2.4. **Well-posedness in the Non-Coercive Case.** Now, we are ready to consider well-posedness also in the non-coercive case.

2.4.1. *Existence.* We start by proving existence of a solution.

Theorem 2.15 (Existence)**.** *Let $a : X \times Y \to \mathbb{R}$ be bounded, symmetrically bounded, weakly coercive and satisfy a Nečas condition on Y for $X \hookrightarrow Y$ dense. Then, for given $f \in Y'$, the unique solution u^ε of (2.4) converges to $u \in X$ as $\varepsilon \to 0$ which solves (1.1).*

Proof. The proof follows partly some ideas and lines from [11, §7.4] (there for the specific case of parabolic variational inequalities). By Corollary 2.14 we have that $\|\!|u^\varepsilon|\!\|_\varepsilon \le \kappa_1$ with κ_1 independent of ε. Next, we use Lemma 2.9 with ε replaced by δ and fix some $0 < \delta < \bar{\delta}$. Since X is closed, there exists a supremizer $w \in X$ with

$$\begin{aligned} \beta_{a,\delta}\, \|w\|_\delta \|u^\varepsilon\|_X &\le a(u^\varepsilon, w) \le \gamma_{\mathrm{s}}\, [\![u^\varepsilon]\!]_X\, \|\!|w|\!\|_X \le \gamma_{\mathrm{s}} \|\!|u^\varepsilon|\!\|_\varepsilon\, \|\!|w|\!\|_X \le \gamma_{\mathrm{s}}\kappa_1 \|\!|w|\!\|_X \\ &\le \gamma_{\mathrm{s}}\kappa_1\, \delta^{-1/2}\sqrt{1 + C^2}\, \|w\|_\delta =: \kappa_2(\delta)\|w\|_\delta,^{\mathrm{e}} \end{aligned}$$

where we have used the symmetric boundedness, (2.12), (2.13) and (2.11). Thus, we get $\|u^\varepsilon\|_X \le \frac{\kappa_2(\delta)}{\beta_{a,\delta}}$ and this bound is independent of ε. Hence, $(u^\varepsilon)_{\varepsilon>0}$ is bounded in X and we can extract a subsequence $(u^\eta)_{\eta>0}$ that is weakly convergent, i.e.,

[e] It is precisely this part of the proof that requires the symmetric boundedness in order to obtain an estimate versus $\|\!|u^\varepsilon|\!\|_\varepsilon$ which was previously shown to be bounded independently of ε. In [10, 11] this step was done by different techniques for certain special cases.

$u^\eta \rightharpoonup u$ in X as $\eta \to 0$. Since $K \cap X$ is closed convex, it is weakly closed, hence $u \in K \cap X$. We finally need to show that u solves (1.1). From (2.4), we get $a^\varepsilon(u^\varepsilon, v) - f(v - u^\varepsilon) \geq a^\varepsilon(u^\varepsilon, u^\varepsilon) = \varepsilon |u^\varepsilon|_X^2 + a(u^\varepsilon, u^\varepsilon) \geq a(u^\varepsilon, u^\varepsilon)$. Since –possibly again by considering subsequences– it holds that $u^\varepsilon \rightharpoonup u$ in X as $\varepsilon \to 0$, thus we have $a^\varepsilon(u^\varepsilon, v) \to a(u, v)$ for all $v \in X$ as $\varepsilon \to 0$ and hence $a(u, v) - f(v - u) \geq \liminf_{\varepsilon \to 0} a(u^\varepsilon, u^\varepsilon) \geq a(u, u)$, which is equivalent to (1.1). □

2.4.2. *Uniqueness.* Concerning uniqueness: Let $u_1, u_2 \in X$ be two solutions of (1.1), then $\alpha_{\mathrm{w}} [\![u_1 - u_2]\!]_X^2 \leq a(u_1 - u_2, u_1 - u_2) = a(u_1, u_1 - u_2) + a(u_2, u_2 - u_1) \leq f(u_1 - u_2) + f(u_2 - u_1) = 0$, where we have used the properties of $a(\cdot, \cdot)$ and (1.1). This implies that u_1 and u_2 coincide with respect to $[\![\cdot]\!]_X$ – which does *not* imply uniqueness in X which is equipped with a stronger norm.

2.4.3. *Stability.* In order to derive a stability estimate, one could now use (2.13) and perform the limit $\varepsilon \to 0$. This leads to

$$[\![u]\!]_X \leq \frac{1}{\alpha_{\mathrm{w}}^-} \|f\|_{X'} + \Big(\frac{\gamma_{\mathrm{s}}^+}{\alpha_{\mathrm{w}}^-} + 1 \Big) \mathrm{dist}_{\|\cdot\|_X}(0, K).$$

The main drawback is the control only in $[\![\cdot]\!]_X$ which is weaker than $\|\cdot\|_X$ or even $|\!|\!| \cdot |\!|\!|_X$. This deficiency can be overcome using the inf-sup stability of $a(\cdot, \cdot)$.

Theorem 2.16 (Stability)**.** *Let $u \in K$ solve* (1.1)*. If $a : X \times Y \to \mathbb{R}$ is bounded and satisfies a Nečas condition on Y, we have*

$$\|u\|_X \leq \frac{1}{\beta_a} \|f\|_{Y'} + \Big(\frac{\gamma_a}{\beta_a} + 1 \Big) \mathrm{dist}_{\|\cdot\|_X}(0, K). \tag{2.14}$$

Proof. Let $\phi \in K$ be arbitrary. Then, we use Nečas condition, (1.1) and the boundedness of $a(\cdot, \cdot)$ to obtain

$$\begin{aligned} \beta_a \|u - \phi\|_X &\leq \sup_{v \in Y} \frac{a(\phi - u, v)}{\|v\|_Y} = \sup_{v \in Y} \frac{a(\phi - u, v - u)}{\|v - u\|_Y} \\ &= \sup_{v \in Y} \frac{a(\phi, v - u) - a(u, v - u)}{\|v - u\|_Y} \leq \sup_{v \in Y} \frac{a(\phi, v - u) - f(v - u)}{\|v - u\|_Y} \\ &\leq \sup_{v \in Y} \frac{(\gamma_a \|\phi\|_X + \|f\|_{Y'}) \|v - u\|_Y}{\|v - u\|_Y} = \gamma_a \|\phi\|_X + \|f\|_{Y'}. \end{aligned}$$

Using triangle inequality and taking the infimum over $\phi \in K$ proves the result. □

The equivalence of $\|\cdot\|_X$ and $|\!|\!| \cdot |\!|\!|_X$ yields a stability result also for $|\!|\!| \cdot |\!|\!|_X$.

3. Error Versus Residual Estimates

Now we come to the derivation of estimates of the Petrov-Galerkin error versus the residual. It turns out that the key for this derivation lies in the consideration of the well-known saddle-point formulation of (1.1).

3.1. **Saddle-point Formulation.** It is well-known that under certain conditions the problem (1.1) is equivalent to a saddle-point inequality. This amounts a (natural) condition, namely that the convex set K has a certain representation.

Definition 3.1. We say that a convex set $K \subset Y$ is of *dual cone form* if there exists a Hilbert space W (with norm $\|\cdot\|_W$ induced by an inner product $(\cdot, \cdot)_W$), a continuous bilinear form $b : Y \times W \to \mathbb{R}$, a convex cone $M \subset W$ (the so-called *dual cone*) and some $g \in W'$ such that $K = \{v \in Y : b(v, q) \leq g(q),\, q \in M\}$. □

We now consider the saddle-point inequality: Find $(u, p) \in X \times M$ such that

$$a(u, v) + b(v, p) = f(v) \qquad \forall\, v \in Y, \tag{3.1a}$$

$$b(u, p - q) \quad \leq \; g(p - q) \qquad \forall\, q \in M. \tag{3.1b}$$

The proof of the following well-known result can e.g. be found in [7, Lemma A4].

Lemma 3.2. *Let K be of dual cone form and assume that $a : X \times Y \to \mathbb{R}$ is bounded and satisfies a Nečas condition on $Ker(B) := \{v \in Y : b(v, q) = 0\, \forall\, q \in W\}$ for $X \hookrightarrow Y$ dense. If the bilinear form $b(\cdot, \cdot)$ is* inf-sup stable*, i.e., there exists a $\beta_b > 0$ such that $\sup_{v \in Y} \frac{b(v,q)}{\|v\|_Y} \geq \beta_b \|q\|_W \quad \forall\, q \in W$, then, the variational inequality* (1.1) *and the saddle-point inequality* (3.1) *are equivalent in the following sense:*

- (i) *If u is a solution of* (1.1)*, then there exists a unique $p \in M$ such that (u, p) is a solution of* (3.1)*.*
- (ii) *If (u, p) is a solution of* (3.1)*, then u solves* (1.1)*.*

One advantage of (3.1) for the numerical treatment is that one does not need to construct a conforming discretization of the convex set K, i.e., some finite-dimensional $K_N \subset K$. Often, it is easier to construct $M_N \subset M$ or to solve (3.1) by a primal-dual active set strategy (i.e., no need for a conforming discretization of the space M), [6]. Moreover, as we shall see below, the saddle-point form allows us to derive a-posteriori error estimates.

3.2. **(Petrov-)Galerkin Methods.** Now, we consider approximations to the solution u (resp. u and p) in finite-dimensional spaces $X_N \subset X$ and $M_N \subset M$, the dimension of both somehow related to $N \in \mathbb{N}$. Moreover, we will consider a finite-dimensional $K_N \subset K$. The discrete problem corresponding to (1.1) then reads

$$u_N \in K_N \cap X_N : \quad a(u_N, v_N - u_N) \geq f(v_N - u_N) \quad \forall\, v_N \in K_N. \tag{3.2}$$

We denote the *error* by $e_N := u - u_N$ and the *residual* $R_N \in Y'$ by

$$R_N(v) := f(v - u_N) - a(u_N, v - u_N), \quad v \in Y. \tag{3.3}$$

With $K_N = \{v_N \in X_N : b(v_N, q_N) \leq g(q_N),\, q_N \in M_N\}$, the discrete saddle-point problem corresponding to (3.1) reads: Find $(u_N, p_N) \in X_N \times M_N$ such that

$$a(u_N, v_N) + b(v_N, p_N) = f(v_N) \qquad \forall\, v_N \in X_N, \tag{3.4a}$$

$$b(u_N, p_N - q_N) \quad \leq \; g(p_N - q_N) \qquad \forall\, q_N \in M_N. \tag{3.4b}$$

For $v \in X$, the *residual* $r_N \in X'$ of the *equation* (3.4a) is denoted by

$$r_N(v) := f(v) - a(u_N, v) - b(v, p_N) = a(e_N, v) + b(v, \delta_N), \tag{3.5}$$

with the *dual error* $\delta_N := p - p_N$. The *inequality residual* of (3.4b) is $s_N \in W'$

$$s_N(q) := b(u_N, q) - g(q), \qquad q \in W. \tag{3.6}$$

For the following, we basically follow [6]. We define the Riesz representators $\hat{r}_N \in X, \hat{s}_N \in W, \sigma \in W$ of residuals and the inequality functional, respectively, by

$$(v, \hat{r}_N)_X = r_N(v),\; v \in X, \qquad (q, \hat{s}_N)_W = s_N(q),\; q \in W,$$
$$(\sigma, q)_W = b(u, q) - g(q),\; q \in W.$$

Additionally, we need a projection $\pi : W \to M$ which is assumed to be orthogonal with respect to some scalar product $\langle \cdot, \cdot \rangle_\pi$ on W. Furthermore, we define an induced norm $\|\eta\|_\pi := \sqrt{\langle \eta, \eta \rangle_\pi}$, which is assumed to be equivalent to the norm $\|\cdot\|_W$, i.e., there exist constants c_π, C_π, such that $0 < c_\pi < C_\pi$ and $c_\pi \|q\|_W \le \|q\|_\pi \le C_\pi \|q\|_W$ for all $q \in W$. We assume that π has the following properties:

$$(q - \pi(q), \eta)_W \le 0, \qquad q \in W, \eta \in M, \tag{3.7a}$$

$$\pi(\sigma) = 0, \tag{3.7b}$$

$$\langle q, \sigma \rangle_\pi \le 0, \qquad q \in M. \tag{3.7c}$$

3.3. **The Coercive Case.** We recall some known estimates for the case $Y = X$ and the bilinear form $a(\cdot,\cdot)$ being coercive. One can prove an error/residual estimate.

Inequality formulation. Define the residual of the *equality* as

$$\varrho_N(v) := f(v) - a(u_N, v), \quad v \in Y.$$

Note, that $R_N(v) = \varrho_N(v) - \varrho_N(u_N)$ with R_N as in (3.3). Moreover, in general $\varrho_N(u_N) \ne 0$ and $\varrho_N(v) \not\to 0$ as $N \to \infty$.

Proposition 3.3. *Assume that $a : X \times X \to \mathbb{R}$ is coercive with constant $\alpha_a > 0$. Let $K_N \subset K$ and denote by $u_N \in K_N$ the unique solution of* (3.2). *Then we have $\|e_N\|_X \le d_1 + (d_1^2 + d_2)^{1/2}$, where $d_1 := \frac{1}{2\alpha_a}\|R_N\|_{X'}$ and $d_2 := |\varrho_N(u_N)|$.*

Proof. Using $v = u_N$ in (1.1) yields $a(u, u - u_N) \le f(u - u_N)$. Hence, we get

$$\begin{aligned}\alpha_a \|e_N\|_X^2 &\le a(u - u_N, u - u_N) = a(u, u - u_N) - a(u_N, u - u_N)\\ &\le f(u - u_N) - a(u_N, u - u_N) = R_N(u) = \varrho_N(u) - \varrho_N(u_N)\\ &= R_N(u - u_N) + \varrho_N(u_N) \le \|R_N\|_{X'}\|e_N\|_X + |\varrho_N(u_N)|.\end{aligned}$$

Hence, we get the estimate $\|e_N\|_X^2 - 2d_1\|e_N\|_X - d_2 \le 0$ with $d_1, d_2 \ge 0$. Thus, we can estimate $\|e_N\|_X$ by the largest root of the quadratic equation. $\square$

Proposition 3.3 and (2.14) show that the variational inequality is stable and yields an error/residual relation in the conforming case, i.e., if $K_N \subset K$. This estimate, however, might not be optimal since d_2 might not be small or not tend to zero as $N \to \infty$. One can also derive an estimate of the form $\|e_N\|_X \le C\|R_N\|_{X'}^{1/2}$, which is also not sufficient.

Saddle-point formulation. Now, we consider on the saddle-point formulation (3.1), (3.4). We recall from [6] the primal/dual error relation which we will need later on.

Lemma 3.4 (Primal/Dual Error Relation, coercive case)**.** *In addition to the assumptions of Lemma 3.2 assume that $b(\cdot,\cdot)$ is inf-sup stable for $Y = X$. Then, we obtain the following primal/dual error relation for $\delta_N := p - p_N$ and $e_N := u - u_N$*

$$\|\delta_N\|_W \le \frac{1}{\beta_b}(\|r_N\|_{X'} + \gamma_a\|e_N\|_X). \tag{3.8}$$

Proof. Due to the inf-sup stability on $Y = X$ and the closedness of X, there exists a $v \in X$, $v \ne 0$ such that we get $\beta_b\|v\|_X\|\delta_N\|_W \le b(v, \delta_N) = r_N(v) - a(e_N, v) \le \|v\|_X\|r_N\|_{X'} + \gamma_a\|v\|_X\|e_N\|_X$. The estimate follows directly. $\square$

The following estimate has been derived in [6, Proposition 4.2]. We report the proof since we will need to modify it later in the case where $a(\cdot,\cdot)$ is not coercive.

Proposition 3.5 (A-posteriori Error Bound, coercive case)**.** *Let K be of dual cone form and assume that $a : X \times X \to \mathbb{R}$ ($Y = X$) is bounded and coercive. If $b(\cdot,\cdot)$ is* inf-sup stable *on $X \times W$ and* (3.7) *holds. Then we get the error bounds*

$$\|e_N\|_X \le \Delta_N^u := c_1 + (c_1^2 + c_2)^{1/2}, \quad \|\delta_N\|_W \le \Delta_N^p := \frac{1}{\beta_b}(\|r_N\|_{X'} + \gamma_a \Delta_N^u), \tag{3.9}$$

$$c_1 := \frac{1}{2\alpha_a}\left(\|r_N\|_{X'} + \frac{\gamma_a}{\beta_b}\|\pi(\hat{s}_N)\|_W\right),\ c_2 := \frac{1}{\alpha_a}\left(\frac{\|\pi(\hat{s}_N)\|_W \|r_N\|_{X'}}{\beta_b} + (p_N, \pi(\hat{s}_N))_W\right).$$

Proof. First, we want to prove (3.9). Using coercivity, (3.1b) and (3.7a) and the definition of the residual yields

$$\begin{aligned}
\alpha_a \|e_N\|_X^2 &\le a(e_N, e_N) = r_N(e_N) - b(e_N, \delta_N) \\
&\le \|r_N\|_{X'}\|e_N\|_X - b(u, \delta_N) + b(u_N, \delta_N) \\
&\le \|r_N\|_{X'}\|e_N\|_X - g(\delta_N) + g(\delta_N) + s_N(\delta_N) \\
&= \|r_N\|_{X'}\|e_N\|_X + s_N(\delta_N) = \|r_N\|_{X'}\|e_N\|_X + (p, \hat{s}_N)_W \\
&= \|r_N\|_{X'}\|e_N\|_X + (p, \pi(\hat{s}_N))_W + (p, \hat{s}_N - \pi(\hat{s}_N))_W \\
&\le \|r_N\|_{X'}\|e_N\|_X + (p, \pi(\hat{s}_N))_W \\
&= \|r_N\|_{X'}\|e_N\|_X + (p - p_N, \pi(\hat{s}_N))_W + (p_N, \pi(\hat{s}_N))_W \\
&= \|r_N\|_{X'}\|e_N\|_X + (\delta_N, \pi(\hat{s}_N))_W + (p_N, \pi(\hat{s}_N))_W. \qquad (3.10)
\end{aligned}$$

Next, we continue by the Cauchy-Schwarz inequality

$$\alpha_a \|e_N\|_X^2 \le \|r_N\|_{X'}\|e_N\|_X + \|\delta_N\|_W \|\pi(\hat{s}_N)\|_W + (p_N, \pi(\hat{s}_N))_W. \tag{3.11}$$

Inserting (3.8) yields: $\|e_N\|_X^2 - 2c_1\|e_N\|_X - c_2 \le 0$. Note that c_1 and c_2 are positive and we can bound $\|e_N\|_X$ again by the largest root of the quadratic equation, which yields the upper a-posteriori error bound Δ_N^u for the primal solution. Inserting (3.9) into (3.8) proves also the second inequality of the claim. ☐

Remark 3.6. *It should be noted that the error bounds Δ_N^u and Δ_N^p both tend to zero as $N \to \infty$. In fact, we have for any $q \in W$ that $(q, \hat{s}_N)_W = s_N(q) = b(u_N, q) - g(q) \to b(u, q) - g(q) = (\sigma, q)_W \le 0$ as $N \to \infty$ since $u \in K$. Since π is continuous, we get $(q, \pi(\hat{s}_N))_W \to (\pi(\sigma), q)_W = 0$ as $N \to \infty$ since $\pi(\sigma) = 0$. Hence, $\pi(\hat{s}_N) \to 0$ as $N \to \infty$ and r_N and δ_N trivially tend to zero.*

3.4. **Beyond Coercivity.** Now, we derive error estimates for the case where $a(\cdot,\cdot)$ is *not* required to be coercive. One might hope to derive an error/residual relation without weak coercivity and symmetric boundedness (see Definition 2.11) following the proof of Proposition 3.5 with $\|\cdot\|_X$ replaced by $\|\cdot\|_Y$. In fact, if $a(\cdot,\cdot)$ would be coercive on the larger space Y, i.e., $a(v,v) \ge \alpha_{a,Y}\|v\|_Y^2$, $v \in Y$, we get

$$\alpha_{a,Y}\|e_N\|_Y^2 \le [\![r_N]\!]_{X'}\,\|e_N\|_Y + (\delta_N, \pi(\hat{s}_N))_W + (p_N, \pi(\hat{s}_N))_W, \tag{3.12}$$

where the residuals $r_N \in Y'$ and $s_N \in W'$ are defined by (3.5) and (3.6), respectively. However, the subsequent argument involving the primal-dual error relation fails since following the lines of the proof of Lemma 3.4 leads

$$\beta_b\|v\|_Y\,\|\delta_N\|_W \le r_N(v) - a(e_N, v) \le \|r_N\|_{Y'}\|v\|_Y + \gamma_a\|e_N\|_X\,\|v\|_Y,$$

i.e., the error e_N appears in the stronger $\|\cdot\|_X$-norm on the right-hand side as opposed to the left-hand side of (3.12). Even if we would pose an inf-sup condition for $b(\cdot,\cdot)$ with $\|\cdot\|_Y$ replaced by the stronger norm $[\cdot]_X$, we would get $[v]_X$ on the left-hand side. In both cases we are stuck here. A way out is to use the symmetric boundedness, which we have already needed for the well-posedness.

Proposition 3.7. *Let $a : X \times Y \to \mathbb{R}$ be bounded, symmetrically bounded, weakly coercive and satisfying a Nečas condition on Y for $X \hookrightarrow Y$ dense. Then we have for $K_N \subset K$ the estimate $[\![e_N]\!]_X \le d_1 + (d_1^2 + d_2)^{1/2}$, where $d_1 := \frac{1}{2\alpha_{\rm w}} [\![R_N]\!]_{X'}$ and $d_2 := |\varrho_N(u_N)|$ for the error $e_N := u - u_N$ and the residual R_N defined in* (3.3).

Proof. Note that (1.1) and (3.2) are well-posed. Using the weak coercivity, the proof follows the lines of the proof of Proposition 3.3, i.e.,

$$\begin{aligned}
\alpha_{\rm w} [\![e_N]\!]_X^2 &\le a(u - u_N, u - u_N) = a(u, u - u_N) - a(u_N, u - u_N) \\
&\le f(u - u_N) - a(u_N, u - u_N) = R_N(u) = \varrho_N(u) - \varrho_N(u_N) \\
&= R_N(u - u_N) + \varrho_N(u_N) \le [\![R_N]\!]_{X'} [\![e_N]\!]_X + |\varrho_N(u_N)|
\end{aligned}$$

and then the usual arguments prove the claim. □

Now, we turn to the saddle-point inequality. The first step is to derive an error/residual-relation for (3.1). We start by fixing some notation. The equality and inequality residuals are defined as in (3.5) and primal respectively dual errors are again abbreviated by $e_N := u - u_N$, $\delta_N := p - p_N$. A first attempt could be to try to generalize the primal/dual error relation in Lemma 3.4.

Remark 3.8. *Let $a : X \times Y \to \mathbb{R}$ be bounded, symmetrically bounded, weakly coercive and satisfy a Nečas condition. Let $b : Y \times W \to \mathbb{R}$ be bounded and*

$$\exists\, \beta_{b,|\!|\!|\cdot|\!|\!|_X} > 0 : \qquad \inf_{q \in W} \sup_{v \in X} \frac{b(v,q)}{|\!|\!| v |\!|\!|_X \|q\|_W} \ge \beta_{b,|\!|\!|\cdot|\!|\!|_X}. \tag{3.13}$$

Then, $\|\delta_N\|_W \le \beta_{b,|\!|\!|\cdot|\!|\!|_X}^{-1} (|\!|\!| r_N |\!|\!|_{X'} + \gamma_{\rm s} [\![e_N]\!]_X)$.

Proof. As in the proof of Lemma 3.4, we have for $0 \ne v \in X$ the estimate $\beta_{b,|\!|\!|\cdot|\!|\!|_X} |\!|\!| v |\!|\!|_X \|\delta_N\|_W \le b(v, \delta_N) = r_N(v) - a(e_N, v)$. Now, use the symmetric boundedness of $a(\cdot,\cdot)$ and get $\beta_{b,|\!|\!|\cdot|\!|\!|_X} |\!|\!| v |\!|\!|_X \|\delta_N\|_W \le |\!|\!| r_N |\!|\!|_{X'} |\!|\!| v |\!|\!|_X + \gamma_{\rm s} [\![e_N]\!]_X |\!|\!| v |\!|\!|_X$. □

Remark 3.9. *As already noted in Remark 2.10, the inf-sup condition* (3.13) *is generally* not *satisfied by our previous assumptions even if we would use $\|\cdot\|_X$ instead of $|\!|\!|\cdot|\!|\!|_X$. Moreover, it is by no means clear if such a condition holds at all. If it would hold, however, we would get an estimate analogous to Proposition 3.5.*

As already said, an inf-sup condition as (3.13) might not be realistic. Thus, we now aim at replacing the primal/dual error relation. In order to do so, we need some relationship between $b(\cdot,\cdot)$ describing the convex set (in terms of the Hilbert space W –or the convex cone $M \subset W$– the bilinear form is defined on) and the space X. We will show later how to verify this at least in the space-time framework.

Definition 3.10. The convex set K is called *X-compatible* if there exists a linear mapping $D : M \to X$ such that (1) $b(Dp, q) = (p, q)_W$ for $p, q \in M$; (2) There exists a $C_D < \infty$[f] such that $|\!|\!| Dp |\!|\!|_X \le C_D \|p\|_W$ for all $p \in M$.

Lemma 3.11. *Let K be X-compatible and let $a : X \times Y \to \mathbb{R}$ be symmetrically bounded. Then, $\|\delta_N\|_W \le C_D (|\!|\!| r_N |\!|\!|_{X'} + \gamma_{\rm s} [\![e_N]\!]_X)$.*

[f]In fact, C_D can be chosen as the operator norm with respect to $|\!|\!|\cdot|\!|\!|_X$, i.e., $C_D = \|D\|_{\mathcal{L}(M,X)}$.

Proof. Since K is X-compatible, we have for $w \in M$ that $(w, \delta_N)_W = b(Dw, \delta_N) = r_N(Dw) - a(e_N, Dw) \le |||r_N|||_{X'}\, |||Dw|||_X + \gamma_s |||Dw|||_X [\![e_N]\!]_X$, which proves the claim by choosing $w = \delta_N = p - p_N \in M$. $\square$

Theorem 3.12. *Let $a : X \times Y \to \mathbb{R}$ be bounded, symmetrically bounded, weakly coercive and satisfy a Nečas condition on Ker(B) for $X \hookrightarrow Y$ dense. Let $b : Y \times W \to \mathbb{R}$ be bounded and inf-sup stable. If K is X-compatible, the following error/residual estimates hold $[\![e_N]\!]_X \le \Delta_N^u := c_1 + (c_1^2 + c_2)^{1/2}$, where $c_1 := \frac{1}{2\alpha_w}([\![r_N]\!]_{X'} + \gamma_s C_D \|\pi(\hat{s}_N)\|_W)$ and $c_2 := \frac{1}{\alpha_w}(C_D |||r_N|||_{X'} \|\pi(\hat{s}_N)\|_W + (p_N, \pi(\hat{s}_N))_W)$. Moreover, we get $\|\delta_N\|_W \le \Delta_N^p := C_D(|||r_N|||_{X'} + \gamma_s \Delta_N^u)$.*

Proof. First, note that the continuous problems (1.1), (3.1) and the discrete ones (3.2), (3.4) are well-posed and equivalent in the sense of Lemma 3.2. In order to prove the claimed estimate, we follow until (3.10) the lines of the proof of Proposition 3.5. Hence, we arrive at $\alpha_w [\![e_N]\!]_X^2 \le [\![r_N]\!]_{X'} [\![e_N]\!]_X + (\delta_N, \pi(\hat{s}_N))_W + (p_N, \pi(\hat{s}_N))_W$. We use Lemma 3.11 and get $(\delta_N, \pi(\hat{s}_N))_W \le \|\delta_N\|_W \|\pi(\hat{s}_N)\|_W \le C_D \|\pi(\hat{s}_N)\|_W (|||r_N|||_{X'} + \gamma_s [\![e_N]\!]_X)$, i.e. $\alpha_w [\![e_N]\!]_X^2 \le ([\![r_N]\!]_{X'} + \gamma_s C_D \|\pi(\hat{s}_N)\|_W) [\![e_N]\!]_X + C_D \|\pi(\hat{s}_N)\|_W\, |||r_N|||_{X'} + (p_N, \pi(\hat{s}_N))_W$. Again, we get an estimate of the form $x^2 - 2c_1 x - c_2 \le 0$. As $c_1, c_2, x \ge 0$, it follows that $x \le c_1 + \sqrt{c_1^2 + c_2}$, which proves the claim. The estimate for $\|\delta_N\|_W$ follows from Lemma 3.11. $\square$

Remark 3.13. *(a) As in Remark 3.6 we obtain that Δ_N^u, $\Delta_N^p \to 0$ as $N \to \infty$.*
(b) If a stronger inf-sup condition (3.13) *holds, we can derive an error/residual estimate similar to Theorem 3.12 (just replacing M_D by $\beta_{b,|||\cdot|||_X}^{-1}$) without the requirement of X-compatibility of the convex set K.*

4. Space-Time Formulation of Parabolic Variational Inequalities

Now we apply the general theory to the space-time variational formulation of parabolic variational inequalities (PVIs). Let us first describe the framework.

4.1. Space-Time Variational Formulation.

We recall space-time formulations of parabolic initial value problems and then generalize to variational inequalities.

Spaces. Let $V \hookrightarrow H \hookrightarrow V'$ be a Gelfand triple of Hilbert spaces and $I := (0, T)$, $T > 0$. The spaces V, H and V' arise from the spatial variational formulation of a parabolic problem (e.g. $H = L_2(\Omega)$, $V = H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^d$). We denote by

$$\varrho := \sup_{\phi \in V} \frac{\|\phi\|_H}{\|\phi\|_V} \tag{4.1}$$

the embedding constant of V in H. For the space-time variational formulation, we require the notion of *Bochner spaces* (see [4, §5.9.2]) for any normed space U, i.e., $L_2(I; U) := \{v : I \to U \text{ strongly measurable: } \|v\|_{L_2(I;U)}^2 := \int_I \|v(t)\|_U^2 < \infty\}$, and choose from here on

$$X := \{v \in L_2(I; V) : \dot{v} \in L_2(I; V'), v(0) = 0\}, \quad Y := L_2(I; V), \text{ i.e., } X \hookrightarrow Y \text{ dense.}$$

Note that $X \hookrightarrow C(\bar{I}; H)$ so that $v(0)$ and $v(T)$ are well-defined in H. Then, set

$$\|v\|_Y := \|v\|_{L_2(I;V)}, \quad [\![v]\!]_X^2 := \|v\|_Y^2 + \|v(T)\|_H^2, \quad |||v|||_X^2 := [\![v]\!]_X^2 + \|\dot{v}\|_{Y'}^2, \quad v \in X,$$

and we keep these norms.[g] The norm in X, even though equivalent to the standard graph norm, allows for a control of the state at the final time, [18].

[g]Note that $\|\dot{v}\|_{Y'}$ plays the role of $|v|_X$ in the previous section.

Forms. Now, we detail the variational formulation. To this end, let $c : V \times V \to \mathbb{R}$ be the bilinear form corresponding to the weak form in space. We start by a parabolic initial value problem (PIVP) that reads for given $f(t) \in V'$, $t \in I$ a.e.,

$$\langle \dot{u}(t), v(t)\rangle_{V' \times V} + c(u(t), v(t)) = \langle f(t), v(t)\rangle_{V' \times V} \;\; \forall\, v(t) \in V, t \in I \text{ a.e.,} \tag{4.2a}$$

$$u(0) = 0 \text{ in } H. \tag{4.2b}$$

Next, we define space-time bilinear forms

$$[u, v] := \int_I \langle u(t), v(t)\rangle_{V' \times V} dt, \qquad \mathcal{C}[u, v] := \int_I c(u(t), v(t))\, dt$$

and we finally obtain the variational formulation

$$u \in X : \quad a(u, v) = f(v) \;\forall\, v \in Y, \tag{4.3}$$

where $a(u, v) := [\dot{u}, v] + \mathcal{C}[u, v]$ as well as $f(v) := [f, v]$.

Theorem 4.1 (Well-posedness of space-time equation, e.g. [16, Theorem 5.1, Appendix A], [3, Ch. XVIII, §3])**.** *Let $c : V \times V \to \mathbb{R}$ be a bounded bilinear form satisfying a Gårding inequality, i.e. there exist $\alpha_c > 0$ and $\lambda_c \geq 0$ such that $c(\phi, \phi) + \lambda_c \|\phi\|_H^2 \geq \alpha_c \|\phi\|_V^2$ for all $\phi \in V$. Then, problem* (4.3) *is well-posed.* □

4.2. **Parabolic Variational Inequalities.** Given a closed convex subset $K \subset Y (= L_2(I; V))$, i.e. $K(t) \subseteq V$ for $t \in I$ a.e., consider the parabolic variational inequality, which reads: Find $u \in H^1(I; H) \cap C(\bar{I}; V)$ such that $u(t) \in K(t)$ and

$$(\dot{u}(t), v(t) - u(t))_H + c(u(t), v(t) - u(t)) \geq \langle f(t), v(t)\rangle_{V' \times V} \;\; \forall\, v(t) \in K(t), \tag{4.4a}$$

$$u(0) = 0 \text{ in } H, \tag{4.4b}$$

for all $t \in I$. According to [9], such a function u is called a *strong solution* (also due to the stronger regularity assumption as compared to (4.3), which also allows to use $(\cdot, \cdot)_H$ in the first term in (4.4a) instead of $\langle \cdot, \cdot \rangle_{V' \times V}$). It is also investigated there when a strong solution exists. The space-time variational formulation now reads: Find $u \in X \cap K$ with $u(t) \in K(t)$ for all $t \in I$ a.e. and

$$a(u, v - u) \geq f(v - u) \,\forall\, v \in K, \tag{4.5}$$

with $a(\cdot, \cdot)$ and $f(\cdot)$ given as in the previous section. It should be noted that (4.5) does *not* correspond to the weak formulation in [9], since there integration by parts with respect to time is performed. As a byproduct of our analysis we obtain also new well-posedness results for (4.5). To this end, we verify that the space-time non-coercive form $a(\cdot, \cdot)$ is bounded, symmetrically bounded and weakly coercive.

Proposition 4.2. *If the bilinear form $c(\cdot, \cdot)$ is bounded and satisfies a Gårding inequality, such that $\alpha_c - \lambda_c \varrho^2 > 0$, then the bilinear form $a(\cdot, \cdot)$ is bounded, symmetrically bounded and weakly coercive.*

Proof. First we show that $a(\cdot, \cdot)$ is bounded. Let $v \in X$, $w \in Y$, then

$$a(v, w) = [\dot{v}, w] + \mathcal{C}[v, w] \leq \|\dot{v}\|_{Y'} \|w\|_Y + \gamma_c \|v\|_Y \|w\|_Y \leq \max\{1, \gamma_c\}\, \|v\|_X \,\|w\|_Y,$$

by Cauchy-Schwarz inequality, which proves the boundedness. For the weak coercivity, we apply the Gårding inequality for some $v \in X$ (recall $v(0) = 0$)

$$
\begin{aligned}
a(v, v) = [\dot{v}, v] + \mathcal{C}[v, v] &= \frac{1}{2}\|v(T)\|_H^2 + \int_0^T c(v(t), v(t))\, dt \\
&\geq \frac{1}{2}\|v(T)\|_H^2 + \int_0^T (\alpha_c \|v(t)\|_V^2 - \lambda_c \|v(t)\|_H^2)\, dt \\
&\geq \frac{1}{2}\|v(T)\|_H^2 + (\alpha_c - \lambda_c \varrho^2)\|v\|_Y^2 \geq \min\{\frac{1}{2}, \alpha_c - \lambda_c \varrho^2\} [\![v]\!]_X^2,
\end{aligned}
$$

i.e., weak coercivity with constant $\alpha_{\mathrm{w}} := \min\{\frac{1}{2}, \alpha_c - \lambda_c \varrho^2\}^{1/2}$. Finally, integration by parts and recalling that $v(0) = 0$ in H for $v \in X$ yields for $v, w \in X$ that

$$
\begin{aligned}
a(v, w) &= (v(T), w(T))_H - (v(0), w(0))_H - [v, \dot{w}] + \mathcal{C}[v, w] \\
&\leq \|v(T)\|_H \|w(T)\|_H + \|v\|_Y \|\dot{w}\|_{Y'} + \gamma_c \|v\|_Y \|w\|_Y \leq \gamma_s [\![v]\!]_X \|w\|_X,
\end{aligned}
$$

with $\gamma_{\mathrm{s}} := \max\{1, \gamma_c\}$. □

As a direct consequence of Theorem 2.15, we get:

Corollary 4.3. *If the assumptions of Proposition* (4.2) *hold, the space-time variational inequality* (4.5) *has a solution which is unique with respect to* $[\cdot]_X$.

Saddle-point Inequality. We can also formulate a space-time saddle-point inequality, which we will use for deriving an error bound. Let us assume that $K(t)$, $t \in I$ a.e., can be represented as $K(t) = \{v(t) \in V : \tilde{b}(v(t), q(t)) \leq g(t; q(t))\, \forall\, q(t) \in \tilde{M}\}$, where $\tilde{M} \subset \tilde{W}$ is the dual cone, $\tilde{b} : V \times \tilde{W} \to \mathbb{R}$ is the bilinear form in space only and $g(t) \in \tilde{W}'$ is given.[h] Based upon these, we need to define W and $b : Y \times W \to \mathbb{R}$. The precise definition of W and $b(\cdot, \cdot)$ is influenced by the question how the convex space-time set K arises from $K(t)$, i.e., its temporal evolution or how the dual cone $M \subset W$ arises from $\tilde{M}$. Then, given $g \in W'$, the saddle-point inequality reads as in (3.1) with $a(\cdot, \cdot)$ and $f(\cdot)$ defined as before.

4.3. **Modeling of the Cone.** We will now discuss different possibilities for the choice of M, W and $b(\cdot, \cdot)$. For each choice, we need to

- prove that $b(\cdot, \cdot)$ is inf-sup stable, so that Lemma 3.2 ensures that the saddle-point inequality is equivalent to the variational inequality. This ensures well-posedness;
- specify an operator $D : M \to X$ from Definition 3.10 in order to ensure that K is X-compatible and to obtain an error estimate by Theorem 3.12;
- detail the projector π in order to derive the precise form of an error estimate.

Let us start by thinking of an obstacle or a barrier. In this setting, it makes sense to require that a solution $u \in X$ of a PVI satisfies the cone condition for (at least for almost) all times, i.e.,

$$
\tilde{b}(v(t), \tilde{q}) \leq \tilde{g}(t; \tilde{q}) \quad \forall\, \tilde{q} \in \tilde{M}\ \forall\, t \in I \text{ a.e.} \tag{4.6}
$$

In some applications, a condition for 'almost' all $t \in I$ might not be sufficient.

4.3.1. *Straightforward approaches.* We start by some straightforward approaches.

[h]From here on, we use tildes to indicate quantities in space (only).

Stationary Cones. In case of a stationary cone, i.e., $g(t) \equiv g \in \tilde{W}'$, one might try to set $M := \tilde{M}$, $W := \tilde{W}$ and now $b^{\text{stat}}(v,q) := \int_0^T \tilde{b}(v(t),q)\,dt$ for $v \in Y$, $q \in W$. This, however, does *not* yield that the cone-condition holds for (almost) all times since $b^{\text{stat}}(v,q) = \tilde{b}(\bar{v},q)$ with the temporal average $\bar{v} := \int_0^T v(t)\,dt$. Thus, this would only ensure that the temporal average is in K which might not be what we want (depending on the problem at hand, of course).

Average Condition. Another possibility is, that we define the space $W := L_2(I;\tilde{W})$ and the bilinear form as $b(v,q) := \int_0^T \tilde{b}(v(t),q(t))dt$. Given $g \in W' = L_2(I;\tilde{W}')$, the cone condition reads $\int_0^T \tilde{b}(v(t),q(t))\,dt \le \int_0^T \langle g(t),q(t)\rangle_{\tilde{W}'\times\tilde{W}}dt$, which can be seen as a cone condition 'on average'. From an application point of view, this might again be insufficient (e.g. a company should not be bankrupt just in average). On the other hand, we can show that $b(\cdot,\cdot)$ is inf-sup stable:

Proposition 4.4. *Let $\tilde{b} : V \times \tilde{W} \to \mathbb{R}$ be bounded and inf-sup stable, i.e.,*

$$\tilde{b}(\eta,\mu) \le \gamma_{\tilde{b}} \|\eta\|_V \|\mu\|_{\tilde{W}}, \qquad \inf_{\mu\in\tilde{W}} \sup_{\eta\in V} \frac{\tilde{b}(\eta,\mu)}{\|\eta\|_V\|\mu\|_{\tilde{W}}} \ge \beta_{\tilde{b}} > 0. \tag{4.7}$$

Then, $b : Y \times W \to \mathbb{R}$ defined for $W := L_2(I;\tilde{W})$ by $b(y,q) := \int_0^T \tilde{b}(y(t),q(t))\,dt$ is inf-sup stable on $Y \times W$ with constant $\beta_b \ge \beta_{\tilde{b}}$.

Proof. Define the operator $\tilde{B} : V \to \tilde{W}'$ by $\langle \tilde{B}\eta,\mu\rangle_{\tilde{W}'\times\tilde{W}} := \tilde{b}(\eta,\mu)$ for $\eta \in V$, $\mu \in \tilde{W}$. By (4.7), the operator $\tilde{B}$ is boundedly invertible with $\|\tilde{B}\eta\|_{\tilde{W}'} \le \gamma_{\tilde{b}}\,\|\eta\|_V$ and $\|\tilde{B}^{-1}\hat{\mu}\|_V \le \beta_{\tilde{b}}{}^{-1}\|\hat{\mu}\|_{\tilde{W}'}$. Let $q \in W$ be given, i.e., $q(t) \in \tilde{W}$ for almost all $t \in I$. For such a $t \in I$, define $\hat{q}(t) \in \tilde{W}'$ as the Riesz representation, i.e., $\langle \hat{q}(t),\eta\rangle_{\tilde{W}'\times\tilde{W}} = (q(t),\eta)_{\tilde{W}}$, for $\eta \in \tilde{W}$ (where $(\cdot,\cdot)_{\tilde{W}}$ is the inner product in $\tilde{W}$) in particular $\|\hat{q}(t)\|_{\tilde{W}'} = \|q(t)\|_{\tilde{W}}$. Then, set $y_q(t) := \tilde{B}^{-1}\hat{q}(t) \in V$. This implies that $\tilde{b}(y_q(t),q(t)) = \langle \hat{q}(t),q(t)\rangle_{\tilde{W}'\times\tilde{W}} = \|q(t)\|^2_{\tilde{W}}$. Since

$$\|y_q\|_Y^2 = \int_0^T \|y_q(t)\|_V^2\,dt \le \beta_{\tilde{b}}{}^{-2}\int_0^T \|\hat{q}(t)\|^2_{\tilde{W}'}dt = \beta_{\tilde{b}}{}^{-2}\int_0^T \|q(t)\|^2_{\tilde{W}}dt = \beta_{\tilde{b}}{}^{-2}\|q\|_W^2$$

we get $\|y_q\|_Y \le \beta_{\tilde{b}}^{-1}\|q\|_W$ and in particular that $y_q \in Y$. Finally,

$$b(y_q,q) = \int_0^T \tilde{b}(y_q(t),q(t))dt = \int_0^T \|q(t)\|^2_{\tilde{W}}dt = \|q\|_W^2 \ge \beta_{\tilde{b}}\|q\|_W\|y_q\|_Y,$$

which proves the claim. □

So far, we have not specified the dual cone M based upon the dual cone $\tilde{M}$ in space only. This also relates to the X-compatibility of K. We have not been able to construct an operator $D : M \to X$ as in Definition 3.10. The reason is that X requires regularity in time which is not provided by the norm in W with the above choice. Hence, the next try is to equip W (and M) with some temporal regularity.

More Regularity. In this regard, another possibility is to set

$$W := H^1_{\{0\}}(I;\tilde{W}) := \{q \in H^1(I;\tilde{W}) : q(0) = 0\}. \tag{4.8}$$

Note that $W \subset C(\bar{I};\tilde{W})$.[i] At a first glance, the left boundary condition might be surprising. However, keeping in mind that a parabolic variational inequality is equipped with an initial condition which lies in the convex set K, there is no need to ensure this by the variational formulation. Thus, the homogeneous initial condition is a natural choice. In order to prove inf-sup stability, we define: $\|q\|_W^2 := \|\dot{q}\|^2_{L_2(I;\tilde{W})} + \|q\|^2_{L_2(I;\tilde{W})} + \|q(T)\|^2_{\tilde{W}}$. Similar to the proof of Proposition 4.4, we can show the following result, whose proof we skip.

Proposition 4.5. *Let $\tilde{b} : V \times \tilde{W} \to \mathbb{R}$ be bounded and inf-sup stable as in* (4.7)*. Then, $b : Y \times W \to \mathbb{R}$ defined by $b(y,q) := \int_0^T \tilde{b}(y(t), \dot{q}(t) + q(t))\,dt$ is inf-sup stable with constant $\beta_b \geq 2^{-1/2}\beta_{\tilde{b}}$.*

As above, we obtain well-posedness, but we have not been able to show X-compatibility. A straightforward candidate would be $M := H_0^1(I;\tilde{M})$ (i.e., $M \subset C(\bar{I};\tilde{M})$) and $(Dp)(t) := \tilde{D}(p(t)) + \dot{p}(t))$ with $\tilde{D} : \tilde{M} \to V$ accordingly, which –however– is not bounded in X.

4.3.2. *Pointwise Conditions.* Given $K(t)$ as above, we now describe a possible choice of W, M and $b(\cdot,\cdot)$ that ensures $v(t) \in K(t)$ for all $t \in \bar{I}$ provided that $v \in X \cap K$. Moreover, this approach also allows to construct appropriate operators D and π. To this end, we assume that $\tilde{W}' \hookrightarrow \tilde{H} \hookrightarrow \tilde{W}$ is another Gelfand triple with a (pivot) Hilbert space $\tilde{H}$. Moreover, we require that $\tilde{M} \subset \tilde{H}$. Then, $g(t;q(t)) = \langle g(t), q(t)\rangle_{\tilde{W}'\times\tilde{W}}$ for $q(t) \in \tilde{W}$. Defining $H^1_{\{0\}}(I;\tilde{W}') := \{w \in H^1(I;\tilde{W}') : w(0) = 0 \text{ in } \tilde{H}\}$, we set $W := (H^1_{\{0\}}(I;\tilde{W}'))'$, $M := H^1(I;\tilde{M})$ and

$$b(y,q) := \int_0^T \tilde{b}(y(t), q(t))dt, \quad y \in Y, q \in W. \tag{4.9}$$

First, note that $M \subset W$ in the sense of a continuous embedding. In fact, let $q \in M$, i.e., $\|q\|^2_{H^1(I;\tilde{W})} = \int_0^T \{\|\dot{q}(t)\|^2_{\tilde{W}} + \|q(t)\|^2_{\tilde{W}}\}dt < \infty$ and $q(t) \in \tilde{M} \subset \tilde{H} \subset \tilde{W}$ for all $t \in \bar{I}$. Then, $\ell_q(v) := \int_0^T \{\langle \dot{q}(t), \dot{v}(t)\rangle_{\tilde{W}\times\tilde{W}'} + \langle q(t), v(t)\rangle_{\tilde{W}\times\tilde{W}'}\}dt$, $v \in H^1_{\{0\}}(I;\tilde{W}') = W'$, defines a linear mapping $\ell_q : W' \to \mathbb{R}$, which is bounded, i.e., $|\ell_q(v)| \leq C_q\|v\|_{W'}$, hence $\ell_q \in (H^1_{\{0\}}(I;\tilde{W}'))' = W$, which proves the embedding $M \subset W$. Next, we note for later reference that for $g \in W'$ and $q \in M$ we have

$$\langle g, q\rangle_{W'\times W} = \int_0^T \langle g(t), q(t)\rangle_{\tilde{W}'\times\tilde{W}}dt. \tag{4.10}$$

In fact, note that $W' \hookrightarrow L_2(I;\tilde{H}) \hookrightarrow W \cong H^{-1}(I;\tilde{W})$ so that the duality of W' and W is induced by $L_2(I;\tilde{H})$ in the sense that $\langle g, h\rangle_{W'\times W} = (g,h)_{L_2(I;\tilde{H})}$ if $g \in W'$ and $h \in L_2(I;\tilde{H})$. For $q \in M = H^1(I;\tilde{M}) \subset L_2(I;\tilde{H})$, we thus get $\langle g, q\rangle_{W'\times W} = (g,q)_{L_2(I;\tilde{M})}$ and the duality of $\tilde{W}'$ and $\tilde{W}$ is induced by $\tilde{H}$, which shows (4.10). This setting in fact realizes a pointwise condition for K:

Proposition 4.6. *Assume that $\tilde{W}' \hookrightarrow \tilde{H} \hookrightarrow \tilde{W}$ is a Gelfand triple with a Hilbert space $\tilde{H}$ such that $\tilde{M} \subset \tilde{H}$. Let $W := (H^1_{\{0\}}(I;\tilde{W}'))'$, $M := H^1(I;\tilde{M})$ and $b(\cdot,\cdot)$ be as in* (4.9)*. If $w \in X \cap K$, then $w(t) \in K(t)$ for all $t \in I$.*

---

[i]The following fact is well-known, see e.g. [1, Satz 8.24]: For any Hilbert space U and any $v \in H^1(I;U)$ there exists a unique $w \in C(\bar{I};U)$ with $v(t) = w(t)$ for $t \in I$ a.e. In the sequel, we will always identify $v \in H^1(I;U)$ with its continuous representative.

Proof. Let $t \in I$ and pick an arbitrary $\eta \in \tilde{M}$. Denoting by δ_t the Dirac distribution in time at t, we consider a sequence $(q_\delta)_{\delta>0} \subset M$ with $q_\delta \to \delta_t \otimes \eta$ as $\delta \to 0+$. Then, $b(w, q_\delta) = \int_0^T \tilde{b}(w(s), q_\delta(s))ds \to \tilde{b}(w(t), \eta)$ as $\delta \to 0+$. On the other hand, since $q_\delta \in M$, we get by (4.10) that $b(w, q_\delta) \geq \langle g, q_\delta\rangle_{W'\times W} = \int_0^T \langle g(s), q_\delta(s)\rangle_{\tilde{W}'\times\tilde{W}} ds \to \langle g(t), \eta\rangle_{\tilde{W}'\times\tilde{W}}$ as $\delta \to 0+$, i.e., $\tilde{b}(w(t), \eta) \geq \langle g(t), \eta\rangle_{\tilde{W}'\times\tilde{W}}$ for all $\eta \in \tilde{M}$, i.e., $w(t) \in K(t)$. □

Next, we need to prove inf-sup stability in order to show well-posedness.

Proposition 4.7. *Let the assumptions of Proposition 4.6 hold and assume* (4.7)*, i.e.,* $\tilde{b} : V \times \tilde{W} \to \mathbb{R}$ *is inf-sup stable with constant* $\beta_{\tilde{b}} > 0$*. Then,* $b : Y \times W \to \mathbb{R}$ *is inf-sup stable with constant* $\beta_b \geq \beta_{\tilde{b}}$*.*

Proof. Let $q \in W$ be given and denote by $\hat{q} \in W'$ its Riesz representation, i.e., $(\hat{q}, w)_{W'} = \langle q, w\rangle_{W\times W'}$ for all $w \in W'$, in particular $\langle q, \hat{q}\rangle_{W\times W'} = \|\hat{q}\|^2_{W'} = \|q\|^2_W$. Since $W' = H^1_{\{0\}}(I; \tilde{W}')$, we have $\hat{q}(t) \in \tilde{W}'$ for all $t \in \bar{I}$ and similar to the proof of Proposition 4.4 we define $y_q : \bar{I} \to V$ by $y_q(t) := \tilde{B}^{-1}(\hat{q}(t)) \in V$, $t \in \bar{I}$. Since

$$
\begin{aligned}
\|y_q\|^2_Y &= \int_0^T \|y_q(t)\|^2_V dt = \int_0^T \|\tilde{B}^{-1}(\hat{q}(t))\|^2_V dt \leq \beta_{\tilde{b}}^{-2} \int_0^T \|\hat{q}(t)\|^2_{\tilde{W}'} dt \\
&= \beta_{\tilde{b}}^{-2} \int_0^T \langle \hat{q}(t), q(t)\rangle_{\tilde{W}'\times W} dt = \beta_{\tilde{b}}^{-2} \langle \hat{q}, q\rangle_{W'\times W} = \beta_{\tilde{b}}^{-2} \|q\|^2_W,
\end{aligned}
$$

we get $y_q \in Y$. Finally,

$$
\begin{aligned}
b(y_q, q) &= \int_0^T \tilde{b}(y_q(t), q(t))dt = \int_0^T \langle \tilde{B} y_q(t), q(t)\rangle_{\tilde{W}'\times\tilde{W}} dt \\
&= \int_0^T \langle \hat{q}(t), q(t)\rangle_{\tilde{W}'\times\tilde{W}} dt = \langle \hat{q}, q\rangle_{W'\times W} = \|q\|^2_W \geq \beta_{\tilde{b}} \|q\|_W \|y_q\|_Y,
\end{aligned}
$$

which proves the claim. □

4.3.3. *X-compatibilty.* With $\tilde{R} : \tilde{W} \to \tilde{W}'$ we denote the Riesz operator in space $\langle \tilde{R}\tilde{q}, \tilde{w}\rangle_{\tilde{W}'\times\tilde{W}} = (\tilde{q}, \tilde{w})_{\tilde{W}}$ for all $\tilde{w} \in \tilde{W}'$, and with $R : W \to W'$ the space-time Riesz operator defined by $\langle Rw, q\rangle_{W'\times W} = (w, q)_W$ for all $q \in W$. In particular, we have $\|\tilde{R}\tilde{w}\|_{\tilde{W}'} = \|\tilde{w}\|_{\tilde{W}}$ and $\|Rw\|_{W'} = \|w\|_W$. For later reference, we note that $(w, q)_W = (Rw, Rq)_{W'}$ and similarly $(\tilde{q}, \tilde{w})_{\tilde{W}} = (\tilde{R}\tilde{q}, \tilde{R}\tilde{w})_{\tilde{W}'}$. Then, we get the following result.

Proposition 4.8. *Assume that* $\tilde{W}' \hookrightarrow \tilde{H} \hookrightarrow \tilde{W}$ *is a Gelfand triple such that* $\tilde{M} \subset \tilde{W}'$*. Moreover, assume the existence of a linear mapping* $\tilde{D} : \tilde{W} \to V$ *with (i)* $\tilde{b}(\tilde{D}\tilde{p}, \tilde{q}) = (\tilde{p}, \tilde{q})_{\tilde{W}}$ *for all* $\tilde{p}, \tilde{q} \in \tilde{M}$ *and (ii)* $\|\tilde{D}\tilde{p}\|_V \leq C_{\tilde{D}} \|\tilde{p}\|_{\tilde{W}}$ *for all* $\tilde{p} \in \tilde{M}$*. Then,* K *is* X*-compatible with* $(Dp)(t) := \tilde{D}(\tilde{R}^{-1}[(Rp)(t)])$ *for* $t \in \bar{I}$ *and* $p \in M$*.*

Remark 4.9. *Note that the condition* $\tilde{M} \subset \tilde{W}'$ *is stronger than* $\tilde{M} \subset \tilde{H}$*.*

Proof. For $p \in M$, we have $Rp \in W'$, thus $(Rp)(t) \in \tilde{W}'$ and $\tilde{R}^{-1}[(Rp)(t)] \in \tilde{W}$ for all $t \in \bar{I}$, so that D is well-defined on M. In order to verify (1) in Definition 3.10, we get for $p, q \in M$ by (4.10), (i) and the definitions of R and $\tilde{R}$

$$\begin{aligned} b(Dp,q) &= \int_0^T \tilde{b}((Dp)(t), q(t))\, dt = \int_0^T \tilde{b}\big(\tilde{D}(\tilde{R}^{-1}[(Rp)(t)]), q(t)\big)\, dt \\ &= \int_0^T \big(\tilde{R}^{-1}[(Rp)(t)], q(t)\big)_{\tilde{W}}\, dt = \int_0^T \langle (Rp)(t), q(t)\rangle_{\tilde{W}'\times\tilde{W}}\, dt \\ &= \langle Rp, q\rangle_{W'\times W} = (p,q)_W, \end{aligned}$$

which proves (1). In order to show (2), we first note that for $v \in X$, we have

$$\begin{aligned} \|v(T)\|_H^2 &= \int_0^T \frac{d}{dt}\|v(t)\|_H^2\, dt = 2\int_0^T \langle \dot{v}(t), v(t)\rangle_{V'\times V}\, dt \\ &\le 2\int_0^T \|\dot{v}(t)\|_{V'}\, \|v(t)\|_V\, dt \le 2\|\dot{v}\|_{Y'}\, \|v\|_Y \le \|\dot{v}\|_{Y'}^2 + \|v\|_Y^2, \end{aligned}$$

so that we get for $p \in M$

$$|\!|\!| Dp |\!|\!|_X^2 \le 2\|D\dot{p}\|_{Y'}^2 + 2\|Dp\|_Y^2 = 2\int_0^T \|(D\dot{p})(t)\|_{V'}^2\, dt + 2\int_0^T \|(Dp)(t)\|_V^2\, dt.$$

Since $\|\phi\|_{V'} \le \varrho^2\|\phi\|_V$, $\phi \in V$, with the embedding constant ϱ from (4.1), we obtain $|\!|\!| Dp |\!|\!|_X^2 \le 2\varrho^4 \int_0^T \|(D\dot{p})(t)\|_V^2\, dt + 2\int_0^T \|(Dp)(t)\|_V^2\, dt$. For any $q \in L_2(I;\tilde{W})$ it holds by (ii) that

$$\begin{aligned} \int_0^T \|(Dq)(t)\|_V^2\, dt &= \int_0^T \|\tilde{D}(\tilde{R}^{-1}[(Rq)(t)])\|_V^2 dt \le C_{\tilde{D}}^2 \int_0^T \|\tilde{R}^{-1}[(Rq)(t)]\|_{\tilde{W}}^2 dt \\ &= C_{\tilde{D}}^2 \int_0^T \|(Rq)(t)\|_{\tilde{W}'}^2 dt = C_{\tilde{D}}^2 \|Rq\|_{L_2(I;\tilde{W}')}. \end{aligned}$$

Using this, we get

$$\begin{aligned} |\!|\!| Dp |\!|\!|_X^2 &\le 2\varrho^4 C_{\tilde{D}}^2 \|R\dot{p}\|_{L_2(I;\tilde{W}')} + 2C_{\tilde{D}}^2 \|Rp\|_{L_2(I;\tilde{W}')} \\ &\le 2C_{\tilde{D}}^2 \max\{1,\varrho^4\}\|Rp\|_{H^1(I;\tilde{W}')}^2 = 2C_{\tilde{D}}^2 \max\{1,\varrho^4\}\|Rp\|_{W'}^2 \\ &= 2C_{\tilde{D}}^2 \max\{1,\varrho^4\}\|p\|_W^2, \end{aligned}$$

i.e., (2) in Definition 3.10 with $C_D = \sqrt{2}C_{\tilde{D}} \max\{1,\varrho^2\}$. □

4.3.4. *Projection onto the cone.* Now, we comment on the definition of the projector $\pi : W \to M$. With the above notation at hand, for any fixed $t \in \bar{I}$, we assume the existence of some $\tilde{\pi}(t) : \tilde{W} \to \tilde{M}$ (i.e., a projector in space only) satisfying

$$(4.11)\quad \langle \tilde{q} - \tilde{\pi}(t)(\tilde{q}), \tilde{\eta}\rangle_{\tilde{W}} \le 0, \tilde{q} \in \tilde{W}, \tilde{\eta} \in \tilde{M},\ \ \tilde{\pi}(t)(\tilde{\sigma}(t)) = 0,\ \langle \tilde{q}, \tilde{\sigma}(t)\rangle_{\tilde{\pi}} \le 0, \tilde{q} \in \tilde{M},$$

where $\tilde{\sigma}(t) \in \tilde{W}$ is the Riesz representation of the inequality residual, which means that $\langle \tilde{\sigma}(t), \tilde{q}\rangle_{\tilde{W}} = \tilde{b}(u(t), \tilde{q}) - \langle g(t), \tilde{q}\rangle_{\tilde{W}'\times\tilde{W}}$ for $\tilde{q} \in \tilde{W}$.[j] We will now show that there exists a $\pi : W \to M$, such that it defines a projection and that the necessary conditions for the error estimation (3.7) are fulfilled.

Lemma 4.10. *Let $\tilde{\pi}(t) : \tilde{W} \to \tilde{M}$ satisfy* (4.11) *for all $t \in \bar{I}$ such that $\tilde{\pi}(t)$ is uniformly bounded, Fréchet differentiable with uniformly bounded $\tilde{\pi}'(t) := \frac{d}{dt}\tilde{\pi}(t)$. Then $[\pi(q)](t) := \tilde{\pi}(t)[\tilde{R}^{-1}\hat{q}(t)]$, $q \in W$, $\hat{q} := Rq$, is a projector $\pi : W \to M$.*

[j]Note, that here we have a slight abuse of notation using the tilde for space-only quantities. Even though $\tilde{\pi}(t)$ is a projection with respect to space only, the dependence on t describes its temporal evolution, i.e., $\tilde{\pi}$ defined by $(\tilde{\pi})(t) := \tilde{\pi}(t)$ is a space-time object.

Proof. We need to show that $\pi(q) \in M$ for all $q \in W$. To this end,

$$
\begin{aligned}
\|\pi(q)\|^2_{H^1(I;\tilde{W})} &= \int_0^T \{\|\frac{d}{dt}\pi(q)(t)\|^2_{\tilde{W}} + \|\pi(q)(t)\|^2_{\tilde{W}}\}dt \\
&= \int_0^T \{\|\frac{d}{dt}[\tilde{\pi}(t)(\tilde{R}^{-1}\hat{q}(t))]\|^2_{\tilde{W}} + \|\tilde{\pi}(t)(\tilde{R}^{-1}\hat{q}(t))\|^2_{\tilde{W}}\}dt \\
&\leq \int_0^T \Big\|\tilde{\pi}'(t)(\tilde{R}^{-1}\hat{q}(t)) + \tilde{\pi}(t)\Big(\tilde{R}^{-1}\big(\frac{d}{dt}\hat{q}(t)\big)\Big)\Big\|^2_{\tilde{W}} \, dt + \|\tilde{\pi}\|^2_\infty \int_0^T \|\hat{q}(t)\|^2_{\tilde{W}'} dt \\
&\leq 2\|\tilde{\pi}'\|^2_\infty \int_0^T \|\tilde{R}^{-1}\hat{q}(t)\|^2_{\tilde{W}} + 2\|\tilde{\pi}\|^2_\infty \int_0^T \big\|\tilde{R}^{-1}\big(\frac{d}{dt}\hat{q}(t)\big)\big\|^2_{\tilde{W}} dt + \|\tilde{\pi}\|^2_\infty \|\hat{q}\|^2_{L_2(I;\tilde{W}')} \\
&\leq 2\|\tilde{\pi}'\|^2_\infty \|\hat{q}\|^2_{L_2(I;\tilde{W}')} + \|\tilde{\pi}\|^2_\infty \Big(2\|\frac{d}{dt}\hat{q}\|^2_{L_2(I;\tilde{W}')} + \|\hat{q}\|^2_{L_2(I;\tilde{W}')}\Big) \\
&\leq (3\|\tilde{\pi}\|^2_\infty + 2\|\tilde{\pi}'\|^2_\infty)\|\hat{q}\|^2_{H^1(I;\tilde{W}')} < \infty,
\end{aligned}
$$

for $q \in W$. Thus, $\pi(q) \in H^1(I;\tilde{W}) \cap C(\bar{I};\tilde{M})$ and therefore $\pi(q) \in M$. □

Of course, the assumptions of Lemma 4.10 are valid in the case that the cone is stationary. This also holds for the assumptions of the next statement. For this, we recall that the Riesz representation $R : W \to W'$ can also be described as $(Rw, \xi)_{W'} = \langle w, \xi \rangle_{W \times W'}$. Then, we get:

Lemma 4.11. *Under the assumptions of Lemma 4.10, the above defined projector* $\pi : W \to M$ *also fulfills* (3.7).

Proof. We start by proving (3.7a), i.e., $(q - \pi(q), \eta)_W \leq 0$ for all $q \in W$, $\eta \in M$. We have, recalling that $\hat{q} := Rq$,

$$
\begin{aligned}
(q - \pi(q), \eta)_W &= \int_0^T \langle q(t) - \pi(q)(t), R\eta(t)\rangle_{\tilde{W}\times\tilde{W}'} \, dt \\
&= \int_0^T \langle q(t) - \tilde{\pi}(t)(\tilde{R}^{-1}\hat{q}(t)), R\eta(t)\rangle_{\tilde{W}\times\tilde{W}'} \, dt \\
&= \int_0^T (q(t) - \tilde{\pi}(t)(\tilde{R}^{-1}\hat{q}(t)), \eta(t))_{\tilde{W}} \, dt \\
&= \int_0^T (\tilde{R}^{-1}\hat{q}(t) - \tilde{\pi}(t)(\tilde{R}^{-1}\hat{q}(t)), \eta(t))_{\tilde{W}} \, dt \leq 0,
\end{aligned}
$$

by the first condition in (4.11) for the projector in space $\tilde{\pi}(\mathrm{t})$. To prove the second condition (3.7b), we start by using the inequality residual

$$
\begin{aligned}
(\sigma, q)_W = b(u, q) - \langle g, q\rangle_{W'\times W} &= \int_0^T \{\tilde{b}(u(t), q(t)) - \langle g(t), q(t)\rangle_{\tilde{W}'\times\tilde{W}}\}dt \\
&= \int_0^T (\tilde{\sigma}(t), q(t))_{\tilde{W}} \, dt
\end{aligned}
$$

By definition of the space-time projector, we have $[\pi(\sigma)](t) = \tilde{\pi}(t)(\tilde{R}^{-1}R(\sigma(t)))$ and therefore $\hat{\sigma} = R\sigma \in W = H^1_{\{0\}}(I;\tilde{W}')$. By applying the Riesz operator, we get

$$
(\sigma, q)_W = \langle \hat{\sigma}, q\rangle_{W'\times W} = \int_0^T \langle \hat{\sigma}(t), q(t)\rangle_{\tilde{W}'\times\tilde{W}} \, dt = \int_0^T (\tilde{R}^{-1}\hat{\sigma}(t), q(t))_{\tilde{W}} \, dt
$$

Thus we have $\tilde{R}^{-1}\hat{\sigma}(t) = \tilde{\sigma}(t)$ for $t \in I$ a.e., hence $\tilde{\sigma} \in W$ which means that $\sigma(t)$ and $\tilde{\sigma}(t)$ can be identified with each other for almost every $t \in I$. This proves the claim as $[\pi(\sigma)](t) = \tilde{\pi}(t)(\tilde{R}^{-1}\hat{\sigma}(t)) = \tilde{\pi}(t)(\tilde{\sigma}(t)) = 0$. We finish by proving (3.7c) for the space-time projector, i.e., $\langle q, \sigma\rangle_\pi \leq 0$ for all $q \in M$. Setting

$$\langle q, p\rangle_\pi := \int_0^T \langle q(t), p(t)\rangle_{\tilde{\pi}}\, dt, \quad p, q \in W,$$

we get for $q \in M$ that $\langle q, \sigma\rangle_\pi := \int_0^T \langle q(t), \sigma(t)\rangle_{\tilde{\pi}}\, dt = \int_0^T \langle q(t), \tilde{\sigma}(t)\rangle_{\tilde{\pi}}\, dt \leq 0$, as $q(t) \in \tilde{M}$ and the third condition of (4.11) holds. □

4.4. **Obstacle Problems.** Finally, we show a concrete example for obstacle problems for which we can apply the previous results. For obstacle problems, we have $K(t) = \{v \in Y : v(t) \geq g(t)\}$, where $g(t) \in H_0^1(\Omega) = \tilde{W}' = V$ and $\tilde{W} = V' = H^{-1}(\Omega)$. In this case, we have $\tilde{b}(\phi, \eta) = \langle \phi, \eta\rangle_{V \times V'}$, $\tilde{M} = V^+ := \{\phi \in V : \phi \leq 0 \text{ on } \Omega\}$, $\tilde{H} = H = L_2(\Omega)$, so that $\tilde{M} \subset \tilde{W}' \subset \tilde{H}$. Then, $M = H^1(I; V^+), W = (H^1_{\{0\}}(I; H^1_0(\Omega)))' \cong H^{-1}(I; H^{-1}(\Omega)) := \{g' : g \in L_2(I; H^{-1}(\Omega))\}$, recall (4.8). Since $W' \subset X$ we we can define D and π as above which yields a corresponding error estimator.

5. Conclusions

We have considered variational inequalities under milder conditions than coercivity on the involved bilinear form $a(\cdot, \cdot)$. This framework in particular includes space-time variational formulations of parabolic variational inequalities. We derive a new well-posedness results and prove an estimate for the error in terms of the (computable) residual. This error estimate is particularly useful for Reduced Basis Methods (RBMs). We have summarized our results in Table 5.1. It is worth mentioning that some results are with respect to the seminorm $[\![\cdot]\!]_X$ which is *not* a norm on X. Considering parabolic problems, this is natural as we cannot hope to control the norm of the derivative, i.e., $\|\dot{u}\|_{L_2(I;V')}$. On the other hand, $[\![\cdot]\!]_X$ is significantly stronger than $\|\cdot\|_Y$, in the parabolic case, e.g., it allows one to control the state at the final time. Finally, we have detailed our general construction for space-time parabolic variational inequalities.

References

[1] W. Arendt and K. Urban. Partielle Differenzialgleichungen. Spektrum Akademischer Verlag Heidelberg 2010.

[2] C. Bernardi, C. Canuto, and Y. Maday. Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem. SIAM J. Numer. Anal. **25**, no. 6, 1237-1271 (1988).

[3] R. Dautray and J.L. Lions. Mathematical analysis and numerical methods for science and technology. Vol. 5. Springer-Verlag, Berlin, 1992. Evolution problems I.

[4] L.C. Evans. Partial differential equations. Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI (2010).

[5] H.-G. Roos and C. Grossmann. Numerische Behandlung partieller Differentialgleichungen. Teubner-Verlag, Stuttgart, 3rd edition 2005.

[6] B. Haasdonk, J. Salomon, and B. Wohlmuth. A Reduced Basis Method for Parametrized Variational Inequalities. SIAM J. Numer. Anal. **50**, no. 5, 2656-2676 (2012).

[7] B. Haasdonk, J. Salomon, and B. Wohlmuth. Preprint version of [6], SimTech Preprint 2011-17, University of Stuttgart (2011).

[8] B. Haasdonk, J. Salomon, and B. Wohlmuth. A Reduced Basis Method for the Simulation of American Options. ENUMATH Proceedings, 2012, arXiv:1201.3289.

<table>
<tr><th rowspan="2"></th><th rowspan="2">coercive (known)</th><th colspan="2">non-coercive, Nečas (new)</th></tr>
<tr><th>inequality</th><th>saddle-point</th></tr>
<tr><td>existence</td><td>Thm. 2.1</td><td>Thm. 2.15
• symm. bounded
• weakly coercive</td><td rowspan="3">Equivalence by Lemma 3.2:
• dual cone form
• b inf-sup stable with respect to $\|\cdot\|_Y$</td></tr>
<tr><td>uniqueness</td><td>Thm. 2.1 ($\|\cdot\|_X$)</td><td>§2.4.2 ($\llbracket\cdot\rrbracket_X$)
• weakly coercive</td></tr>
<tr><td>stability</td><td>Thm. 2.1 ($\|\cdot\|_X$)</td><td>Thm. 2.16 ($\|\cdot\|_X$)</td></tr>
<tr><td>error/residual estimate</td><td>Prop. 3.3 (ineq.)
Prop. 3.5 (sad.-pt.)
($\|\cdot\|_X$)</td><td>Prop. 3.7 ($\llbracket\cdot\rrbracket_X$)
• symm. bounded
• weakly coercive</td><td>• symm. bounded
• weakly coercive
• dual cone form
Thm. 3.12 ($\llbracket\cdot\rrbracket_X$):
• X-compatible
Rem. 3.13 ($\llbracket\cdot\rrbracket_X$):
• strong inf-sup</td></tr>
</table>

TABLE 5.1. Summary of results and requirements.

[9] K. Ito and K. Kunisch. Parabolic variational inequalities: The Lagrange multiplier approach. J. Math. Pures Appl. **9**, 85, no. 3, 415-449 (2006).

[10] D. Kinderlehrer and G. Stampacchia. An introduction to variational inequalities and their applications. Reprint of the 1980 original. SIAM, Philadelphia, PA, 2000.

[11] J.-L. Lions and G. Stampacchia. Variational inequalities. Comm. Pure Appl. Math. **20**, 493-519 (1967).

[12] J. Nečas. Sur une méthode pour resoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle. Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. 16, 305-326 (1962)

[13] R.H. Nochetto, K.G. Siebert, and A. Veeser. Theory of Adaptive Finite Element Methods: an Introduction. In: Multiscale, Nonlinear and Adaptive Approximation, R. DeVore and A. Kunoth (eds.), Springer, 409-542 (2009).

[14] G. Rozza and A.T. Patera. Reduced Basis Approximation and A Posteriori Error Estimation for Parametrized Partial Differential Equations. Massachusetts Institute of Technology (MIT), 2006. `http://augustine.mit.edu`.

[15] M. Rudd and K. Schmitt. Variational inequalities of elliptic and parabolic type. Taiwanese J. Math. **6**, no. 3, 287-322 (2002).

[16] C. Schwab and R. Stevenson. Space-time adaptive wavelet methods for parabolic evolution problems. Math. Comp. **78**, 1293-1318 (2009).

[17] K. Urban and A.T. Patera. A New Error Bound for Reduced Basis Approximation of Parabolic Partial Differential Equations. C.R. Acad. Sci. Paris Series I, 350(3-4), 203-207 (2012).

[18] K. Urban and A.T. Patera. An Improved Error Bound for Reduced Basis Approximation of Linear Parabolic Problems . Math. Comput., online first (2013), doi: `http://dx.doi.org/10.1090/S0025-5718-2013-02782-2`.

SILKE GLAS, UNIVERSITY OF DUISBURG-ESSEN, CHAIR FOR ENERGY TRADING AND FINANCE, D-45117 ESSEN, GERMANY

E-mail address: `silke.glas@uni-due.de`

KARSTEN URBAN, UNIVERSITY OF ULM, INSTITUTE FOR NUMERICAL MATHEMATICS, HELMHOLTZSTRASSE 20, D-89069 ULM, GERMANY

E-mail address: `karsten.urban@uni-ulm.de`