REDUCED BASIS METHODS FOR QUADRATICALLY NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH STOCHASTIC INFLUENCES

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ABSTRACT. We present a new approach to solve nonlinear parametric partial differential equations (PPDEs) with stochastic coefficients, which is based on the Reduced Basis (RB) method. It is imposed that the problem formulation allows for an affine decomposition in the (deterministic) parameter. The uncertainties in the coefficients are modeled using Karhunen-Loève expansions. The nonlinearity is resolved using Newton's method with Fréchet derivatives. For the computation of a linear output functional, its first and second moment, and its variance, we additionally solve some linear dual problems. Statistical analysis is then done using the Monte Carlo (MC) method.

We investigate offline/online decompositions that enable efficient computations independent of the number of degrees of freedom of the full system. We provide efficient and rigorous a-posteriori error bounds for the state variable, the output functional, and the statistical quantities based upon the Brezzi-Rappaz-Raviart theory, dealing however with additional errors as a result of the uncertainty modeling.

We present numerical experiments for a stationary quadratic convection-diffusion problem to illustrate different aspects of the proposed method.

1. INTRODUCTION

Stochastic partial differential equations (PDEs) are widely used to model e.g. physical, medical, or economical problems containing stochastic uncertainties. One distinguishes between stochastic PDEs involving the Itô calculus and PDEs with stochastic influences such as uncertain, random or unknown coefficients. In the present paper, we deal with problems of the latter kind, e.g. porous media flows. To describe additional deterministic dependencies, including e.g. geometric variations, model parameters or forces, we include deterministic parameters to the PDE.

Widely studied solution techniques for PDEs with stochastic influences include different Galerkin methods [14]. So called stochastic finite elements use weak formulations in space and probability. Alternatively, weak solutions in space are combined with Monte Carlo evaluations.

In this context, we integrate the Reduced Basis Method (RBM) which has been intensively studied for PDEs that have to be evaluated repeatedly for many instances of deterministic parameters, see e.g. [7, 15, 16]. It is based upon an offline-online decomposition, where the offline construction of the reduced basis, involving solutions of the full problem, is separated from the efficient online simulations that are independent of the dimension of the full problem.

The online procedure includes the evaluation of efficient and rigorous a-posteriori error bounds for the state and outputs.

The key condition for the efficiency of the RBM is the availability of an affine decomposition of the system in parametric and/or stochastic dependencies from spatial and time variables. For deterministic problems, such decompositions can be obtained using the Empirical Interpolation Method (EIM) [1, 17]. For the stochastic case, we apply the Karhunen-Loève (KL) expansion [12, 13], where the involved random variables are modeled using polynomial chaos (PC) expansions [22]. For combinations of parametric and stochastic problems, extensions of the EIM have been developed in [19].

Deterministic parametrized quadratically nonlinear problems and RBM have been studied for affine problems e.g. in [20] and non-affine problems e.g. in [5]. The analysis is based on the Brezzi-Rappaz-Raviart (BRR) theory [3, 4]. RBM for stochastic parametrized linear problems have been studied in [2, 8]. The evaluation of statistical outputs such as second moment or variance requires good approximation procedures for quadratic output functionals which has been developed in [8] for this special case. In general, quadratic output functionals in the RB context are introduced in [9, 10].

In the present paper, we combine the methods for quadratic deterministic and linear stochastic problems for the case of a given affine decomposition w.r.t. the deterministic parameter. The affine decomposition w.r.t. the stochastic dependency is obtained using the KL expansion. Consequently, especially the error analysis of state and linear output functional is very similar to [20] and [5], whereas the analysis of quadratic and statistical outputs is strongly based upon [8].

We begin in Section 2 with the introduction of the general variational formulation and its Frèchet derivative for the class of problems we are dealing with. Furthermore, we briefly describe the Karhunen-Loève expansion and introduce the desired random and statistical outputs of interest. In Section 3, we present the nonlinear primal RB formulation of the problem and appropriate linear dual RB problems used for the RB approximation of the different outputs of interest. The a-posteriori analysis of the error of state and outputs is developed in Section 4 and the offline-online decomposition in Section 5, where also evaluation procedures for the inf-sup and continuity constants are presented, which are needed for the evaluation of the error bounds. Finally, in Section 6, we provide numerical experiments for a stationary quadratic convection-diffusion problem.

2. PRELIMINARIES

2.1. **Variational formulation.** Let $D \subset \mathbb{R}^d$ denote an open, bounded, spatial domain, $\mathcal{P} \subset \mathbb{R}^p$ a set of deterministic parameters, and $(\Omega, \mathfrak{A}, \mathbb{P})$ a probability space. For some finite element (FE) subspace $X \subset H^1(D)$ of dimension $\dim(X) = \mathcal{N}$, let $a_0 : X \times X \times \mathcal{M} \to \mathbb{R}$, $\mathcal{M} := \mathcal{P} \times \Omega$, be a bilinear form w.r.t. the first two arguments, $a_1 : X \times X \times X \times \mathcal{M} \to \mathbb{R}$ a trilinear form w.r.t. the first three arguments, and let $f : X \times \mathcal{M} \to \mathbb{R}$ be linear and bounded. We assume uniformly boundedness of a_0 and a_1, i.e. for $(\mu, \omega) \in \mathcal{M}$, there are continuity constants $0 < \rho_0(\mu, \omega) < \bar\rho_0 < \infty$ and $0 < \rho_1(\mu, \omega) < \bar\rho_1 < \infty$ such that

$$(2.1) \qquad |a_0(u, v; \mu, \omega)| \ \leq \ \rho_0(\mu, \omega) \, \|u\|_X \, \|v\|_X, \quad u, v \in X,$$

$$(2.2) \qquad |a_1(u, w, v; \mu, \omega)| \ \leq \ \rho_1(\mu, \omega) \, \|u\|_X \, \|w\|_X \, \|v\|_X, \quad u, w, v \in X.$$

For $(\mu, \omega) \in \mathcal{M}$ and $w, v \in X$, we define

$$(2.3) \qquad g(w, v; \mu, \omega) \ := \ a_0(w, v; \mu, \omega) + a_1(w, w, v; \mu, \omega) - f(v; \mu, \omega)$$

and solve the nonlinear, parametrized and random variational problem

$$(2.4) \qquad g(u(\mu,\omega),v;\mu,\omega) \; = \; 0, \quad \forall v \in X.$$

For the moment, we assume the existence of a solution of (2.4) for each pair (μ,ω). A detailed proof is given in Section 4, following the well known Brezzi-Rappaz-Raviart (BRR) theory [3].

2.2. **Affine decomposition via Karhunen-Loève expansion.** In order to achieve computational efficiency of a RBM, we assume g to allow for an affine decomposition in the deterministic parameter μ, namely

$$(2.5) \qquad g(w,v;\mu,\omega) \; = \; \sum_{q=1}^{Q} \theta_q(\mu)\big[\bar{g}_q(w,v) + g_q(w,v;\omega)\big],$$

where $\bar{g}_q : X \times X \to \mathbb{R}$ are bounded and denote the expectations of the terms in brackets and $g_q : X \times X \times \Omega \to \mathbb{R}$ have zero mean and represent the fluctuating parts. To separate also stochastic and spatial dependencies, we express $g_q(w,v;\omega)$ using Karhunen-Loève expansions [12, 13], and obtain

$$(2.6) \qquad g_q(w,v;\omega) \; = \; \sum_{k=0}^{\infty} \xi_{q,k}(\omega)\, g_{q,k}(w,v), \quad q = 1, ..., Q.$$

The random variables $\xi_{q,k} : \Omega \to \mathbb{R}$ are uncorrelated and have zero mean and unit variance. The bilinear forms $g_{q,k} : X \times X \to \mathbb{R}$ are bounded and the magnitude typically decreases exponentially fast in k. For numerical purposes, one usually restricts the infinite sums by some sufficiently large $K < \infty$, leading to truncated forms g_q^K and thereby g^K. The corresponding solution of the truncated form of (2.4) is denoted by $u_K(\mu,\omega)$.

In practice, one may have different numbers Q of affine terms for a_0, a_1 and f, and one may truncate each of the respective decomposed forms at different values of K. However, for notational convenience, we do not explicitly specify all dependencies but indicate them just by Q and K, respectively. Furthermore, an index K indicates that the expression denotes or is based upon truncated systems.

2.3. **Newton iteration.** We iteratively solve (2.4) or the respective truncated problem using Newton's method. The Frèchet derivative of g at some point $z \in X$ is given by

$$(2.7) \qquad dg(u,v;\mu,\omega)[z] \; = \; a_0(u,v;\mu,\omega) + a_1(u,z,v;\mu,\omega) + a_1(z,u,v;\mu,\omega)$$

and the respective truncated form is denoted by dg^K. For some initial guess $u_K^{[0]}(\mu,\omega)$, we solve

$$(2.8) \qquad dg^K(\delta u_K^{[i]}(\mu,\omega),v;\mu,\omega)[u_K^{[i]}(\mu,\omega)] \; = \; -g^K(u_K^{[i]}(\mu,\omega),v;\mu,\omega), \quad \forall v \in X$$

and evaluate the Newton update $u_K^{[i+1]}(\mu,\omega) = u_K^{[i]}(\mu,\omega) + \delta u_K^{[i]}(\mu,\omega)$.

2.4. **Output of interest.** Often, one is not only interested in the state $u(\mu,\omega)$ but also in some output functional $s(\mu,\omega) := \ell(u(\mu,\omega);\mu)$, where $\ell : X \times \mathcal{P} \to \mathbb{R}$ denotes a parametric linear form. Furthermore, we may be interested in the squared functional $s^2(\mu,\omega) := (\ell(u(\mu,\omega),\mu))^2$.

Besides these random outputs, we want to evaluate some statistical quantities such as first and second moment of $s(\mu,\omega)$, denoted by $\mathbb{M}_1(\mu) := \mathbb{E}[s(\mu,\cdot)]$ and $\mathbb{M}_2(\mu) := \mathbb{E}[s^2(\mu,\cdot)]$, respectively. Additionally, we need the squared first moment $\mathbb{M}_1^2(\mu) = (\mathbb{E}[s(\mu,\cdot)])^2$ to evaluate the variance, given by $\mathbb{V}(\mu) = \mathbb{M}_2(\mu) - \mathbb{M}_1^2(\mu)$.

3. REDUCED BASIS SYSTEM

In this section, we introduce reduced primal and dual systems that are used to derive good approximations of the desired random and statistical outputs of interest. For the construction of the dual problems, we combine the ideas of [5, 20], where dual problems for quadratically nonlinear problems with linear outputs are derived, and [8], where dual formulations for linear problems in combination with quadratic and statistical outputs are introduced.

3.1. The primal-dual formulation for linear outputs.
We create a reduced basis from solutions $\zeta_n := u_K(\mu_n, \omega_n)$ for some appropriate parameter set $\{\mu_n, \omega_n\}_{n=1}^N \in \mathcal{M}^N$, $N \ll \mathcal{N}$. The reduced space is given by $X_N = \text{span}(\{\zeta_n\}_{n=1}^N) \subset X$. Due to the affine decomposition of g and dg, it is possible to assemble and solve the reduced system

$$(3.9) \qquad g^K(u_{N,K}(\mu, \omega), v; \mu, \omega) = 0, \qquad \forall v \in X_N.$$

for each $(\mu, \omega) \in \mathcal{M}$ with $\mathcal{N}$-independent computational complexity $\mathcal{O}(QKN^3I)$, where I denotes the number of Newton iterations. We also introduce a linear dual problem in full and reduced form,

$$(3.10) \qquad dg(v, p^{(1)}(\mu, \omega); \mu, \omega)[\tfrac{1}{2}(u(\mu, \omega) + u_{N,K}(\mu, \omega))] = -\ell(v; \mu), \quad \forall v \in X,$$

$$(3.11) \qquad dg^K(v, p_{N,K}^{(1)}(\mu, \omega); \mu, \omega)[u_{N,K}(\mu, \omega)] = -\ell(v; \mu), \quad \forall v \in \tilde{X}_N^{(1)},$$

with solutions $p^{(1)}(\mu, \omega) \in X$ and $p_{N,K}^{(1)}(\mu, \omega) \in \tilde{X}_N^{(1)}$, respectively. The superscript $^{(1)}$ is motivated by the fact that we will introduce further dual problems later on. The reduced dual space $\tilde{X}_N^{(1)}$ of dimension $\tilde{N}^{(1)} \ll \mathcal{N}$ is constructed analogously to X_N as the span of solutions of (3.10) for appropriate parameter pairs (μ, ω). The complexity to solve the dual problem corresponds to just one Newton iteration of the primal problem. Here and in the following, the index N indicates that the expression denotes or is based on reduced systems. We do not explicitly indicate the dependencies on the different dimensions of the primal and dual reduced systems. For notational simplicity, we also omit the parameter pair (μ, ω) in many cases, where it does not affect the understanding.

Let $r_{\text{RB}}(v; \mu, \omega) := g^K(u_{N,K}(\mu, \omega), v; \mu, \omega)$ be the residual of the reduced primal problem for some $v \in X$. We define the RB approximation of the linear output $s(\mu, \omega)$ and its corresponding linear statistical output, the first moment $\mathbb{M}_1(\mu)$, by

$$(3.12) \qquad s_{N,K}(\mu, \omega) := \ell(u_{N,K}; \mu) + r_{\text{RB}}(p_{N,K}^{(1)}; \mu, \omega),$$

$$(3.13) \qquad \mathbb{M}_{1,NK}(\mu) := \mathbb{E}[s_{N,K}(\mu, \cdot)],$$

respectively, where $r_{\text{RB}}(p_{N,K}^{(1)}(\mu, \omega); \mu, \omega)$ has been added as a correction term to improve the approximation. In Section 4, we will provide error bounds to show that this choice leads to good results.

3.2. The dual formulations for quadratic outputs.
As mentioned in Section 2.4, we are also interested in the squared output $s^2(\mu, \omega)$. Since the straightforward approximation $(s_{N,K}(\mu, \omega))^2$ does not lead to accurate results, we define $\ell^{(2)}(\mu, \omega) := 2s_{N,K}(\mu, \omega)\ell(v; \mu)$ and introduce the additional linear dual problems, full and reduced,

$$(3.14) \qquad dg(v, p^{(2)}(\mu, \omega); \mu, \omega)[\tfrac{1}{2}(u(\mu, \omega) + u_{N,K}(\mu, \omega))] = -\ell^{(2)}(v; \mu, \omega), \forall v \in X,$$

$$(3.15) \qquad dg^K(v, p_{N,K}^{(2)}(\mu, \omega); \mu, \omega)[u_{N,K}(\mu, \omega)] = -\ell^{(2)}(v; \mu, \omega), \forall v \in \tilde{X}_N^{(2)},$$

with solutions $p^{(2)}(\mu, \omega) \in X$ and $p_{N,K}^{(2)}(\mu, \omega) \in \tilde{X}_N^{(2)}$, respectively, using some appropriate reduced dual space $\tilde{X}_N^{(2)}$ of dimension $\tilde{N}^{(2)} \ll \mathcal{N}$. The RB approximation of the quadratic

output $s^2(\mu, \omega)$ and its corresponding statistical output, the second moment $\mathbb{M}_2(\mu)$, are then defined by

$$(3.16) \qquad s_{N,K}^{[2]}(\mu, \omega) := (s_{N,K})^2 + 2s_{N,K}\, r_{\mathrm{RB}}(p_{N,K}^{(1)}; \mu, \omega) - r_{\mathrm{RB}}(p_{N,K}^{(2)}; \mu, \omega),$$

$$(3.17) \qquad \mathbb{M}_{2,NK}(\mu) := \mathbb{E}\left[s_{N,K}^{[2]}(\mu, \cdot)\right],$$

i.e., we add two additional correction terms compared to the straightforward approximation.

3.3. The dual formulation for the variance approximation.

To develop good approximations of the variance $\mathbb{V}(\mu) = \mathbb{M}_2(\mu) - \mathbb{M}_1^2(\mu)$, it remains to find RB estimates of $\mathbb{M}_1^2(\mu)$. We define $\ell^{(3)}(\mu, \omega) := 2\mathbb{M}_{1,NK}(\mu, \omega)\ell(v; \mu)$ and introduce the additional linear dual problems, full and reduced,

$$(3.18) \qquad dg(v, p^{(3)}(\mu, \omega); \mu, \omega)[\tfrac{1}{2}(u(\mu, \omega) + u_{N,K}(\mu, \omega))] = -\ell^{(3)}(v; \mu, \omega),\ \forall v \in X,$$

$$(3.19) \qquad dg^K(v, p_{N,K}^{(3)}(\mu, \omega); \mu, \omega)[u_{N,K}(\mu, \omega)] = -\ell^{(3)}(v; \mu, \omega),\ \forall v \in \tilde{X}_N^{(3)},$$

with solutions $p^{(3)}(\mu, \omega) \in X$ and $p_{N,K}^{(3)}(\mu, \omega) \in \tilde{X}_N^{(3)}$, respectively, using some appropriate reduced dual space $\tilde{X}_N^{(3)}$ of dimension $\tilde{N}^{(3)} \ll \mathcal{N}$. The RB approximations of the squared first moment $\mathbb{M}_1^2(\mu)$ and the variance $\mathbb{V}(\mu)$ are then given by

$$(3.20) \qquad \mathbb{M}_{1,NK}^{[2]}(\mu) := (\mathbb{M}_{1,NK})^2 + 2\mathbb{M}_{1,NK}\mathbb{E}\left[r_{\mathrm{RB}}(p_{N,K}^{(1)})\right] - \mathbb{E}\left[r_{\mathrm{RB}}(p_{N,K}^{(3)})\right],$$

$$(3.21) \qquad \mathbb{V}_{NK}(\mu) := \mathbb{E}\left[s_{N,K}^{[2]}(\mu, \cdot)\right] - \mathbb{M}_{1,NK}^{[2]}(\mu)(\mu),$$

respectively. Analogously to (3.16), we added two correction terms.

In our numerical experiments, we have observed that it is sufficient to use the same reduced space for the second and third dual problem, i.e. $\tilde{X}_N^{(2)} = \tilde{X}_N^{(3)}$. Hence, we just solve

$$(3.22) \qquad dg^K(v, p_{N,K}^{(4)}(\mu, \omega); \mu, \omega)[u_{N,K}(\mu, \omega)] = -2\ell(v; \mu, \omega), \quad \forall v \in \tilde{X}_N^{(2)}$$

for $p_{N,K}^{(4)}(\mu, \omega) \in \tilde{X}_N^{(2)}$ such that $p_{N,K}^{(2)} = s_{N,K} \cdot p_{N,K}^{(4)}$ and $p_{N,K}^{(3)} = \mathbb{M}_{1,NK} \cdot p_{N,K}^{(4)}$.

4. A-POSTERIORI ANALYSIS

Parts of the following analysis are based on the Brezzi-Rappaz-Raviart (BRR) theory [3, 4] which has already been used in the RB context for affine deterministic problems, e.g. [20], and non-affine deterministic problems, e.g. [5]. Consequently, especially the analysis in Sections 4.2 to 4.4 is very similar to parts of the mentioned publications. The analysis of quadratic and statistical outputs is based on [8], where the linear stochastic case has been discussed.

Under the assumption that solutions $u(\mu, \omega)$ of (2.4) and $u_{N,K}(\mu, \omega)$ of (3.9) exist, we define the inf-sup constant $\beta(\mu, \omega)$ as

$$(4.23) \qquad \beta(\mu, \omega) := \inf_{w \in X} \sup_{v \in X} \frac{dg(w, v; \mu, \omega)[u_{N,K}(\mu, \omega)]}{\|w\|_X \|v\|_X}.$$

We furthermore assume the existence of some $\beta_0 > 0$ such that $\beta(\mu, \omega) > \beta_0$ for all $(\mu, \omega) \in \mathcal{M}$. Existence and uniqueness of solutions of the dual problems (3.11), (3.15) and (3.19) follows immediately. We furthermore assume the availability of a positive lower bound $\beta_{\mathrm{LB}}(\mu, \omega)$ of the inf-sup constant $\beta(\mu, \omega)$ and an efficient evaluation procedure, compare Section 5.2.

4.1. **Notation.** We first introduce some notation for the subsequent analysis. Let

$$e_{\mathrm{RB}}(\mu,\omega) := u_K(\mu,\omega) - u_{N,K}(\mu,\omega),$$
$$\tilde{e}_{\mathrm{RB}}^{(i)}(\mu,\omega) := p_K^{(i)}(\mu,\omega) - p_{N,K}^{(i)}(\mu,\omega), \quad i=1,2,3,$$

denote the error between the reduced primal and dual solutions and the corresponding solutions of the full but truncated systems, respectively. Furthermore, let

$$e(\mu,\omega) := u(\mu,\omega) - u_{N,K}(\mu,\omega),$$
$$\tilde{e}^{(i)}(\mu,\omega) := p^{(i)}(\mu,\omega) - p_{N,K}^{(i)}(\mu,\omega), \quad i=1,2,3,$$

denote the total error of the reduced primal and dual solutions, respectively. We define the RB residuals

$$r_{\mathrm{RB}}(v;\mu,\omega) := g^K(u_{N,K},v;\mu,\omega) = g^K(e_{\mathrm{RB}},v;\mu,\omega),$$
$$\tilde{r}_{\mathrm{RB}}^{(i)}(v;\mu,\omega) := dg^K(v,p_{N,K}^{(i)})[u_{N,K}] + \ell^{(i)}(v) = dg^K(v,\tilde{e}_{\mathrm{RB}}^{(i)})[u_{N,K}], \quad i=1,2,3,$$

as a "measure" of the error that results from the basis reduction. Additionally, we define some KL "residuals" indicating the truncation errors $g - g^K$ and $dg - dg^K$. To obtain truncation bounds independent of the actual random realization, we replace the random variables $\xi_{q,k}$, $k > K$ by some ϱ-quantile ξ_ϱ, i.e., we define some $0 \le \varrho \ll 1$ such that $|\xi_{q,k}| \le \xi_\varrho$ holds with probability $1 - \varrho$. We define

$$\delta_{\mathrm{KL}}(v;\mu,\omega) := \sum_{q=1}^{Q} |\theta_q(\mu)| \sum_{k=K+1}^{\infty} \xi_\varrho \, |g_{q,k}(u_{N,K},v)|,$$
$$\tilde{\delta}_{\mathrm{KL}}^{(i)}(v;\mu,\omega) := \sum_{q=1}^{Q} |\theta_q(\mu)| \sum_{k=K+1}^{\infty} \xi_\varrho \, |dg_{q,k}(v,p_{N,K}^{(i)})[u_{N,K}]|, \quad i=1,2,3.$$

For numerical purposes, the possibly infinte sums in the above definitions will be truncated as well at some large $K_{\max} > K$ such that the additional truncation error is negligible.

Since we replaced the random variables $\xi_{q,k}(\omega)$ by its ϱ-quantile ξ_ϱ, the KL residuals δ_{KL} and $\tilde{\delta}_{\mathrm{KL}}^{(i)}$ are not residuals in the classical sense but represent corresponding quantiles, i.e. $\delta_{\mathrm{KL}}(v) \ge |(g - g^K)(u_{N,K},v)|$ and $\tilde{\delta}_{\mathrm{KL}}^{(i)}(v) \ge |(dg - dg^K)(v,p_{N,K}^{(i)})[u_{N,K}]|$ holds with a certain probability. In many cases, the random variables $\xi_{q,k}(\omega)$ are bounded since the underlying problem restricts their variations. Then, we can choose $\varrho = 0$ and obtain rigorous bounds. Otherwise, ϱ should be sufficiently small to be negligible in the following analysis.

Based on the introduced residuals, we define RB and KL bounds for $i \in \{1,2,3\}$,

$$(4.24) \qquad \Delta_{\mathrm{RB}}(\mu,\omega) := \frac{1}{\beta_{\mathrm{LB}}} \sup_{v\in X} \left(\frac{r_{\mathrm{RB}}(v)}{\|v\|_X} \right), \quad \tilde{\Delta}_{\mathrm{RB}}^{(i)}(\mu,\omega) := \frac{1}{\beta_{\mathrm{LB}}} \sup_{v\in X} \left(\frac{\tilde{r}_{\mathrm{RB}}^{(i)}(v)}{\|v\|_X} \right),$$

$$(4.25) \qquad \Delta_{\mathrm{KL}}(\mu,\omega) := \frac{1}{\beta_{\mathrm{LB}}} \sup_{v\in X} \left(\frac{\delta_{\mathrm{KL}}(v)}{\|v\|_X} \right), \quad \tilde{\Delta}_{\mathrm{KL}}^{(i)}(\mu,\omega) := \frac{1}{\beta_{\mathrm{LB}}} \sup_{v\in X} \left(\frac{\tilde{\delta}_{\mathrm{KL}}^{(i)}(v)}{\|v\|_X} \right).$$

Before we provide the actual error bounds for the state and the outputs, we introduce a so called proximity indicator $\tau(\mu,\omega)$ which can be seen as a dimensionless measure of the residuals. Similarly to [5, 20], we define

$$(4.26) \qquad \tau(\mu,\omega) := 4 \frac{\rho_1(\mu,\omega)}{\beta_{\mathrm{LB}}(\mu,\omega)} \left(\Delta_{\mathrm{RB}}(\mu,\omega) + \Delta_{\mathrm{KL}}(\mu,\omega) \right),$$

where $\rho_1(\mu, \omega)$ is given by (2.2). For $\tau(\mu, \omega) < 1$, we furthermore define

$$(4.27) \qquad d(\mu, \omega) := \left(1 + \sqrt{1 - \tau(\mu, \omega)}\right)^{-1}$$

which will appear as a factor in the upcoming error bounds. It is easy to see that $d(\mu, \omega)$ is decreasing in $\tau(\mu, \omega)$ and takes values in the interval $[1/2, 1)$.

4.2. **Primal solution error.** For $\tau(\mu, \omega) < 1$, we define the bound

$$(4.28) \qquad \Delta(\mu, \omega) := 2d(\mu, \omega)\left(\Delta_{\mathrm{RB}}(\mu, \omega) + \Delta_{\mathrm{KL}}(\mu, \omega)\right).$$

Since $d(\mu, \omega)$ approaches $1/2$ for small τ, the bound $\Delta(\mu, \omega)$ approaches $\Delta_{\mathrm{RB}}(\mu, \omega) + \Delta_{\mathrm{KL}}(\mu, \omega)$ which corresponds to the bound in the linear case, see [8]. To show that $\Delta(\mu, \omega)$ is indeed an upper bound for the error of the reduced primal solution $u_{N,K}$, we need the following statement, which is introduced and proved almost analogously for deterministic problems in [5, 20].

Lemma 4.1. *For $(\mu, \omega) \in \mathcal{M}$ and $\tau(\mu, \omega) < 1$, the operator $\Phi : X \times \mathcal{M} \to X$ defined by*

$$dg(\Phi(w; \mu, \omega), v; \mu, \omega) = dg(w, v; \mu, \omega)[u_{N,K}(\mu, \omega)] - g(w, v; \mu, \omega) \quad \forall v \in X,$$

for a given $w \in X$, has a unique fixed point $w^*(\mu, \omega)$ in the ball $B(u_{N,K}(\mu, \omega), r(\mu, \omega)) \subset X$ with the radius $r(\mu, \omega) \in \left[\Delta(\mu, \omega), \beta_{LB}(\mu, \omega)(2\rho_1(\mu, \omega))^{-1}\right)$.

Proof. We omit all parameter dependencies for notational convenience. First, it is straightforward to show the identity

$$(4.29) \qquad g(w_2, v) - g(w_1, v) = dg(w_2 - w_1, v)[\tfrac{1}{2}(w_2 + w_1)]$$

and the inequality

$$dg(w, v)[z_2] - dg(w, v)[z_1] = a_1(w, z_2 - z_1, v) + a_1(z_2 - z_1, w, v)$$
$$(4.30) \qquad\qquad\qquad\qquad \leq 2\rho_1 \|w\|_X \|v\|_X \|z_2 - z_1\|_X,$$

using just the definition of g in (2.3) and dg in (2.7) and the continuity assumption (2.2). To prove the Lemma, we apply these results and use the Banach fixed point theorem. We first show that Φ is a contraction on $\bar{B}(u_{N,K}, r)$ for some $r > 0$. For $w_1, w_2 \in \bar{B}(u_{N,K}, r)$, we know that $\tfrac{1}{2}(w_2 + w_1) \in \bar{B}(u_{N,K}, r)$. Using (4.29), we obtain

$$dg(\Phi[w_2] - \Phi[w_1], v)[u_{N,K}] = dg(w_2 - w_1, v)[u_{N,K}] - (g(w_2, v) - g(w_1, v))$$
$$= dg(w_2 - w_1, v)[u_{N,K}] - dg(w_2 - w_1, v)[\tfrac{1}{2}(w_2 + w_1)].$$

Hence, applying (4.30) and the fact that $\tfrac{1}{2}(w_2 + w_1) \in \bar{B}(u_{N,K}, r)$,

$$|dg(\Phi[w_2] - \Phi[w_1], v)[u_{N,K}]| \leq 2\rho_1 \|w_2 - w_1\|_X \|v\|_X \|u_{N,K} - \tfrac{1}{2}(w_2 + w_1)\|_X$$
$$\leq 2r\rho_1 \|w_2 - w_1\|_X \|v\|_X.$$

We use this result and the inf-sup constant (4.23),

$$\|\Phi[w_2] - \Phi[w_1]\|_X \leq \frac{1}{\beta_{\mathrm{LB}}} \sup_{v \in X} \frac{dg(\Phi[w_2] - \Phi[w_1], v)[u_{N,K}]}{\|v\|_X} \leq \frac{2r\rho_1}{\beta_{\mathrm{LB}}} \|w_2 - w_1\|_X.$$

Hence, Φ is a contraction for $0 < r < \beta_{\mathrm{LB}}/2\rho_1$. Next, we show that there is such a radius r such that Φ maps $\bar{B}(u_{N,K}, r)$ into itself. For $w \in \bar{B}(u_{N,K}, r)$. It holds with (4.29) that

$$dg(\Phi[w] - u_{N,K}, v) = dg(w - u_{N,K}, v)[u_{N,K}] - g(w, v)$$
$$= dg(w - u_{N,K}, v)[u_{N,K}] - (g(w, v) - g(u_{N,K}, v)) - g(u_{N,K}, v)$$
$$= dg(w - u_{N,K}, v)[u_{N,K}] - dg(w - u_{N,K}, v)[\tfrac{1}{2}(w + u_{N,K})] - g(u_{N,K}, v).$$

Using again (4.30), we obtain

$$|dg(w-u_{N,K},v)[u_{N,K}] - dg(w-u_{N,K},v)[\tfrac{1}{2}(w+u_{N,K})]| \;\leq\; \rho_1\|w-u_{N,K}\|_X^2\|v\|_X$$
$$\leq\; \rho_1 r^2\|v\|_X.$$

Furthermore, it is clear that

$$|g(u_{N,K},v)| \;\leq\; |(g-g^K)(u_{N,K},v)| + |g^K(u_{N,K},v)| \leq \delta_{\mathrm{KL}}(v) + |r_{\mathrm{RB}}(v)|.$$

Hence, using again the inf-sup constant (4.23), we get

$$\|\Phi[w]-u_{N,K}\|_X \;\leq\; \frac{1}{\beta_{\mathrm{LB}}} \sup_{v\in X} \frac{dg(\Phi[w]-u_{N,K},v)}{\|v\|_X} \;\leq\; \frac{\rho_1 r^2}{\beta_{\mathrm{LB}}} + (\Delta_{\mathrm{KL}} + \Delta_{\mathrm{RB}}).$$

Therefore, Φ maps $\bar{B}(u_{N,K},r)$ into itself for all r with $\rho_1 r^2\beta_{\mathrm{LB}}^{-1}+\Delta_{\mathrm{KL}}+\Delta_{\mathrm{RB}} < r$, which holds for $r \in [\Delta, \beta_{\mathrm{LB}}/(2\rho_1 d)]$. Since $d < 1$ by (4.27), Φ has a unique fixed point on $B(u_{N,K},r)$ for $r \in [\Delta, \beta_{\mathrm{LB}}/(2\rho_1))$. $\qquad\square$

Proposition 4.2. *For $\tau(\mu,\omega) < 1$, $(\mu,\omega) \in \mathcal{M}$, there exists a unique solution $u(\mu,\omega) \in B\big(u_{N,K}(\mu,\omega), \frac{\beta_{LB}(\mu,\omega)}{2\rho_1(\mu,\omega)}\big)$ of (2.4) such that $\|u(\mu,\omega) - u_{N,K}(\mu,\omega)\|_X \leq \Delta(\mu,\omega)$.*

Proof. The proof follows directly from Lemma 4.1. Since the fixed point of Φ solves (2.4), we have existence and uniqueness in $B(u_{N,K}, \frac{\beta_{\mathrm{LB}}}{2\rho_1})$. Furthermore, the fixed point is in the ball $B(u_{N,K},\Delta)$ which leads to the error bound. $\qquad\square$

At the beginning of Section 4, we assumed the existence of solutions $u(\mu,\omega)$ of (2.4) and $u_{N,K}(\mu,\omega)$ of (3.9). With Proposition 4.2, we can prove existence and local uniqueness of $u(\mu,\omega)$ a-posteriori, solving just the reduced problem and evaluating $\tau(\mu,\omega)$. However, the reduced basis has to be sufficiently large to fulfill the requirement $\tau(\mu,\omega) < 1$. This reflects the fact that we can not expect well-posedness of the nonlinear problem for all parameters μ and ω.

4.3. **Dual solution error.** For the dual solutions $p^{(i)}_{N,K}(\mu,\omega)$ of (3.11), (3.15) and (3.19), we define the bounds $\tilde{\Delta}^{(i)}(\mu,\omega)$, $i \in \{1,2,3\}$, by

$$(4.31)\quad \tilde{\Delta}^{(i)}(\mu,\omega) := 2d(\mu,\omega)\left(\tilde{\Delta}^{(i)}_{\mathrm{RB}}(\mu,\omega)+\tilde{\Delta}^{(i)}_{\mathrm{KL}}(\mu,\omega) + \frac{\rho_1(\mu,\omega)}{\beta_{\mathrm{LB}}(\mu,\omega)}\Delta(\mu,\omega)\|p^{(i)}_{N,K}(\mu,\omega)\|_X\right).$$

The last term of (4.31) can also be expressed in terms of τ and d and we obtain the alternative notation $\tilde{\Delta}^{(i)} = 2d(\tilde{\Delta}^{(i)}_{\mathrm{RB}} + \tilde{\Delta}^{(i)}_{\mathrm{KL}}) + d^2\tau\|p^{(i)}_{N,K}\|_X$.

Proposition 4.3. *For $\tau(\mu,\omega) < 1$, it holds that $\|p^{(i)}(\mu,\omega) - p^{(i)}_{N,K}(\mu,\omega)\|_X \leq \tilde{\Delta}^{(i)}(\mu,\omega)$ for $i \in \{1,2,3\}$, $(\mu,\omega) \in \mathcal{M}$.*

Proof. It is straightforward that

$$dg(v,\tilde{e}^{(i)})[u_{N,K}] \;=\; dg(v,p^{(i)})[\tfrac{1}{2}(u+u_{N,K})] - dg(v,p^{(i)})[\tfrac{1}{2}(u-u_{N,K})] - dg(v,p^{(i)}_{N,K})[u_{N,K}].$$

Let us consider the first and last term.

$$\big|dg(v,p^{(i)})[\tfrac{1}{2}(u+u_{N,K})] - dg(v,p^{(i)}_{N,K})[u_{N,K}]\big| \;=$$
$$\big|\ell^{(i)}(v) - dg^K(v,p^{(i)}_{N,K})[u_{N,K}] - (dg-dg^K)(v,p^{(i)}_{N,K})[u_{N,K}]\big| \;\leq\; \big|\tilde{r}^{(i)}_{\mathrm{RB}}(v)\big| + \big|\tilde{\delta}^{(i)}_{\mathrm{KL}}(v)\big|.$$

For the middle term, we use $p^{(i)} = \tilde{e}^{(i)} + p^{(i)}_{N,K}$ and inequality (4.30) to obtain

$$\big|dg(v,p^{(i)})[\tfrac{1}{2}(u-u_{N,K})]\big| \;\leq\; \rho_1\|e\|_X \left(\|\tilde{e}^{(i)}\|_X + \|p^{(i)}_{N,K}\|_X\right)\|v\|_X.$$

We combine these results to estimate the error $\tilde{e}^{(i)}$. Using the inf-sup condition (4.23), we obtain

$$\|\tilde{e}^{(i)}\|_X \leq \frac{1}{\beta_{\mathrm{LB}}} \sup_{v \in X} \frac{dg(v, \tilde{e}^{(i)})[u_{N,K}]}{\|v\|_X} \ \leq \ (\tilde{\Delta}_{\mathrm{RB}}^{(i)} + \tilde{\Delta}_{\mathrm{KL}}^{(i)}) + \frac{\rho_1}{\beta_{\mathrm{LB}}} \Delta(\|\tilde{e}^{(i)}\|_X + \|p_{N,K}^{(i)}\|_X),$$

$$\text{i.e.,} \quad \|\tilde{e}^{(i)}\|_X \left(1 - \frac{\rho_1}{\beta_{\mathrm{LB}}}\Delta\right) \ \leq \ (\tilde{\Delta}_{\mathrm{RB}}^{(i)} + \tilde{\Delta}_{\mathrm{KL}}^{(i)}) + \frac{\rho_1}{\beta_{\mathrm{LB}}}\Delta\|p_{N,K}^{(i)}\|_X.$$

Since $(1 - \frac{\rho_1}{\beta_{\mathrm{LB}}}\Delta) = \frac{1}{2d} > 0$, the claim is proven [1].

$\square$

4.4. **Linear output error.** In the subsequent sections, we provide bounds for the errors between the outputs defined in Section 2.4 and its approximations. In all proofs, we will omit the parameters (μ, ω) for notational compactness. In this section, we will provide error bounds for the approximations of the linear output $s(\mu, \omega)$ and the first moment $\mathbb{M}_1(\mu)$. However, we start with a statement that will be used in the proofs of all output error bounds.

Lemma 4.4. *Let $u(\mu, \omega)$ be the solution of (2.4), $u_{N,K}(\mu, \omega)$ the solution of (3.9) and $p^{(i)}(\mu, \omega)$, $i = 1, 2, 3$, the solutions of (3.10), (3.14) and (3.18), respectively. For $i \in \{1, 2, 3\}$, it holds that $\ell^{(i)}(u) - \ell^{(i)}(u_{N,K}) = g(u_{N,K}, p^{(i)})$.*

Proof. Since $\ell^{(i)}(u) - \ell^{(i)}(u_{N,K}) = \ell^{(i)}(e)$ and using the respective dual formulation (3.10), (3.14) or (3.18), we have

$$
\begin{aligned}
\ell^{(i)}(u) - \ell^{(i)}(u_{N,K}) &= -dg(e, p^{(i)})[\tfrac{1}{2}(u + u_{N,K})] \\
&= -a_0(e, p^{(i)}) - \tfrac{1}{2}a_1(e, u + u_{N,K}, p^{(i)}) - \tfrac{1}{2}a_1(u + u_{N,K}, e, p^{(i)}) \\
&= -a_0(u, p^{(i)}) - a_1(u, u, p^{(i)}) + a_0(u_{N,K}, p^{(i)}) + a_1(u_{N,K}, u_{N,K}, p^{(i)}) \\
&= -f(p^{(i)}) + a_0(u_{N,K}, p^{(i)}) + a_1(u_{N,K}, u_{N,K}, p^{(i)}) \\
&= g(u_{N,K}, p^{(i)}),
\end{aligned}
$$

which proves the postulated equality. $\square$

Let us now introduce the bound for the error between the linear output $s(\mu, \omega)$ and its approximation $s_{N,K}(\mu, \omega)$ defined in (3.12). We define the bound $\Delta^s(\mu, \omega)$ by

$$(4.32) \qquad \Delta^s(\mu, \omega) \ := \ \frac{\beta_{\mathrm{LB}}(\mu, \omega)}{2d(\mu, \omega)}\Delta(\mu, \omega)\tilde{\Delta}^{(1)}(\mu, \omega) \ + \ \delta_{\mathrm{KL}}(p_{N,K}^{(1)}(\mu, \omega); \mu, \omega).$$

Proposition 4.5. *For $\tau(\mu, \omega) < 1$, it holds that $|s(\mu, \omega) - s_{N,K}(\mu, \omega)| \leq \Delta^s(\mu, \omega)$.*

Proof. From Lemma 4.4, we know that $\ell(u) - \ell(u_{N,K}) = g(u_{N,K}, p^{(1)})$. Hence, with $s_{N,K}$ from (3.12), we obtain

$$
\begin{aligned}
s - s_{N,K} &= g(u_{N,K}, p^{(1)}) - g^K(u_{N,K}, p_{N,K}^{(1)}) \\
&= g^K(u_{N,K}, p^{(1)}) - g^K(u_{N,K}, p_{N,K}^{(1)}) + (g - g^K)(u_{N,K}, p^{(1)}) \\
&= g^K(u_{N,K}, \tilde{e}^{(1)}) + (g - g^K)(u_{N,K}, \tilde{e}^{(1)}) + (g - g^K)(u_{N,K}, p_{N,K}^{(1)}).
\end{aligned}
$$

---

[1] Since $(1 - \frac{\rho_1}{\beta_{\mathrm{LB}}}\Delta) = (1 - \tfrac{1}{2}d\tau) = (1 - \tfrac{1}{2}\frac{\tau}{1+\sqrt{1-\tau}}) = (\tfrac{2}{2} - \frac{1-\sqrt{1-\tau}}{2}) = \frac{1+\sqrt{1-\tau}}{2} = \frac{1}{2d}$.

We use the definition of the bounds introduced in Section 4.1 and estimate

$$
\begin{aligned}
|s - s_{N,K}| \;&\leq\; |r_{\mathrm{RB}}(\tilde{e}^{(1)})| + \delta_{\mathrm{KL}}(\tilde{e}^{(1)}) + \delta_{\mathrm{KL}}(p_{N,K}^{(1)}) \\
&\leq\; \beta_{\mathrm{LB}}\Delta_{\mathrm{RB}}\|\tilde{e}^{(1)}\|_X + \beta_{\mathrm{LB}}\Delta_{\mathrm{KL}}\|\tilde{e}^{(1)}\|_X + \delta_{\mathrm{KL}}(p_{N,K}^{(1)}) \\
&\leq\; \beta_{\mathrm{LB}}(\Delta_{\mathrm{RB}} + \Delta_{\mathrm{KL}})\tilde{\Delta}^{(1)} + \delta_{\mathrm{KL}}(p_{N,K}^{(1)}),
\end{aligned}
$$

which proves the claim. $\qquad\square$

With Proposition 4.5 at hand, it is clear that we can easily define a good bound for the error between the first moment $\mathbb{M}_1(\mu)$ and its approximation $\mathbb{M}_{1,NK}(\mu)$ as defined in (3.13). We define the bound $\Delta^{\mathbb{M}_1}(\mu)$ by

$$
(4.33) \qquad\qquad \Delta^{\mathbb{M}_1}(\mu) \;:=\; \mathbb{E}\left[\Delta^s(\mu,\cdot)\right].
$$

Corollary 4.6. *For $\mu \in \mathcal{P}$ and $\tau(\mu,\cdot) < 1$, it holds that $|\mathbb{M}_1(\mu) - \mathbb{M}_{1,NK}(\mu)| \leq \Delta^{\mathbb{M}_1}(\mu)$.*

Proof. The bound follows from Proposition 4.5 and Definition (3.13). $\qquad\square$

4.5. **Quadratic output error.** We continue with the quadratic outputs $s^2(\mu,\omega)$ and $\mathbb{M}_2(\mu)$ and start with the bound for the error between the squared output $s^2(\mu,\omega)$ and its approximation $s_{N,K}^{[2]}(\mu,\omega)$ from (3.16). We define the bound $\Delta^{s^2}(\mu,\omega)$ by

$$
(4.34) \quad \Delta^{s^2}(\mu,\omega) \;:=\; \left(\Delta^s(\mu,\omega)\right)^2 \;+\; \frac{\beta_{\mathrm{LB}}(\mu,\omega)}{2d(\mu,\omega)}\Delta(\mu,\omega)\tilde{\Delta}^{(2)}(\mu,\omega) + \delta_{\mathrm{KL}}(p_{N,K}^{(2)}(\mu,\omega);\mu,\omega).
$$

Proposition 4.7. *For $\tau(\mu,\omega) < 1$, it holds that $|s^2(\mu,\omega) - s_{N,K}^{[2]}(\mu,\omega)| \leq \Delta^{s^2}(\mu,\omega)$.*

Proof. With the definition of $s_{N,K}^{[2]}$ in (3.16), the output error is given by

$$
\begin{aligned}
s^2 - s_{N,K}^{[2]} \;&=\; s^2 - (s_{N,K})^2 - 2s_{N,K}\,r_{\mathrm{RB}}(p_{N,K}^{(1)}) + r_{\mathrm{RB}}(p_{N,K}^{(2)}) \\
&=\; (s - s_{N,K})^2 + 2s_{N,K}(s - s_{N,K}) - 2s_{N,K}\,r_{\mathrm{RB}}(p_{N,K}^{(1)}) + r_{\mathrm{RB}}(p_{N,K}^{(2)}).
\end{aligned}
$$

Using $s_{N,K} = \ell(u_{N,K}) - r_{\mathrm{RB}}(p_{N,K}^{(1)})$ from (3.12) yields

$$
2s_{N,K}(s - s_{N,K}) \;=\; 2s_{N,K}\left(\ell(u) - \ell(u_{N,K}) + r_{\mathrm{RB}}(p_{N,K}^{(1)})\right).
$$

Together, replacing $2s_{N,K}\ell$ by $\ell^{(2)}$, we have

$$
(4.35) \qquad s^2 - s_{N,K}^{[2]} \;=\; (s - s_{N,K})^2 \;+\; \ell^{(2)}(u) - \ell^{(2)}(u_{N,K}) + r_{\mathrm{RB}}(p_{N,K}^{(2)}).
$$

From Proposition 4.5, we know that $(s - s_{N,K})^2 \leq (\Delta^s)^2$. The second part of (4.35) can be estimated analogously to Proposition 4.5 by replacing ℓ by $\ell^{(2)}$ as well as $p^{(1)}$ by $p^{(2)}$ and with Lemma 4.4. We obtain

$$
\left|\ell^{(2)}(u) - \ell^{(2)}(u_{N,K}) + r_{\mathrm{RB}}(p_{N,K}^{(2)})\right| \;\leq\; \frac{\beta_{\mathrm{LB}}}{2d}\Delta\tilde{\Delta}^{(2)} + \delta_{\mathrm{KL}}(p_{N,K}^{(2)})
$$

which proves the claim. $\qquad\square$

Since the second moment $\mathbb{M}_2(\mu)$ and its approximation $\mathbb{M}_{2,NK}(\mu)$ defined in (3.17) are just the expectations of $s^2(\mu,\cdot)$ and $s_{N,K}^{[2]}(\mu,\cdot)$, respectively, it is clear that we can define the bound $\Delta^{\mathbb{M}_2}(\mu)$ by the expectation of $\Delta^{s^2}(\mu,\cdot)$, i.e.,

$$
(4.36) \qquad\qquad \Delta^{\mathbb{M}_2}(\mu) \;:=\; \mathbb{E}\left[\Delta^{s^2}(\mu,\cdot)\right].
$$

Corollary 4.8. *For $\mu \in \mathcal{P}$ and $\tau(\mu,\cdot) < 1$, it holds that $|\mathbb{M}_2(\mu) - \mathbb{M}_{2,NK}(\mu)| \leq \Delta^{\mathbb{M}_2}(\mu)$.*

Proof. Follows from Proposition 4.7 and Definition (3.17). $\qquad\square$

4.6. **Variance output error.** We start with the bound for the error between squared first moment $\mathbb{M}_1^2(\mu)$ and its approximation $\mathbb{M}_{1,NK}^{[2]}(\mu)$. We define the bound $\Delta^{\mathbb{M}_1^2}(\mu)$ by

$$(4.37) \quad \Delta^{\mathbb{M}_1^2}(\mu) \ := \ (\Delta^{\mathbb{M}_1}(\mu))^2 + \mathbb{E}\Big[\frac{\beta_{\mathrm{LB}}(\mu,\cdot)}{2d(\mu,\cdot)}\Delta(\mu,\cdot)\tilde{\Delta}^{(3)}(\mu,\cdot)\Big] + \mathbb{E}\Big[\delta_{\mathrm{KL}}(p_{N,K}^{(3)}(\mu,\cdot);\mu,\cdot)\Big].$$

Proposition 4.9. *For $\mu \in \mathcal{P}$ and $\tau(\mu,\cdot) < 1$, it holds $\left|\mathbb{M}_1^2(\mu) - \mathbb{M}_{1,NK}^{[2]}(\mu)\right| \le \Delta^{\mathbb{M}_1^2}(\mu)$.*

Proof. Analogously to Proposition 4.7, the output error is given by

$$\mathbb{M}_1^2 - \mathbb{M}_{1,NK}^{[2]} \ = \ (\mathbb{M}_1 - \mathbb{M}_{1,NK})^2 \ + \ \mathbb{E}\left[\ell^{(3)}(u) - \ell^{(3)}(u_{N,K}) + r_{\mathrm{RB}}(p_{N,K}^{(3)})\right]$$

From Corollary 4.6, we know $(\mathbb{M}_1 - \mathbb{M}_{1,NK})^2 \le (\Delta^{\mathbb{M}_1})^2 = (\mathbb{E}\,[\Delta^s])^2$. We estimate the remaining term analogously to Proposition 4.5, replacing ℓ by $\ell^{(3)}$ as well as $p^{(1)}$ by $p^{(3)}$ and with Lemma 4.4. We obtain

$$\left|\ell^{(3)}(u) - \ell^{(3)}(u_{N,K}) + r_{\mathrm{RB}}(p_{N,K}^{(3)})\right| \ \le \ \frac{\beta_{\mathrm{LB}}}{2d}\Delta\tilde{\Delta}^{(3)} + \delta_{\mathrm{KL}}(p_{N,K}^{(3)})$$

and the claim follows directly. $\qquad\square$

From the above results, it is clear that the variance error could directly be bounded by

$$(4.38) \qquad\qquad |\mathbb{V}(\mu) - \mathbb{V}_{NK}(\mu)| \ \le \ \Delta^{\mathbb{M}_2}(\mu) + \Delta^{\mathbb{M}_1^2}(\mu).$$

However, we can derive more precise error bounds. Analogously to Section 4.1, we define dual RB and KL residuals $\tilde{r}_{\mathrm{RB}}^{(4)}(v;\mu,\omega)$ and $\tilde{\delta}_{\mathrm{KL}}^{(4)}(v;\mu,\omega)$, replacing $p_{N,K}^{(i)}$ by $(p_{N,K}^{(2)} - p_{N,K}^{(3)})$,

$$\tilde{r}_{\mathrm{RB}}^{(4)}(v;\mu,\omega) \ := \ dg^K(v, p_{N,K}^{(2)} - p_{N,K}^{(3)})[u_{N,K}] + \ell^{(i)}(v)$$

$$\tilde{\delta}_{\mathrm{KL}}^{(4)}(v;\mu,\omega) \ := \ \sum_{q=1}^Q |\theta_q(\mu)| \sum_{k=K+1}^\infty \xi_\varrho\,|dg_{q,k}(v, p_{N,K}^{(2)} - p_{N,K}^{(3)})[u_{N,K}]|.$$

The corresponding bounds read

$$\tilde{\Delta}_{\mathrm{RB}}^{(4)}(\mu,\omega) \ := \ \beta_{\mathrm{LB}}^{-1}(\mu,\omega)\sup_{v\in X}\big(\tilde{r}_{\mathrm{RB}}^{(4)}(v;\mu,\omega)/\|v\|_X\big),$$

$$\tilde{\Delta}_{\mathrm{KL}}^{(4)}(\mu,\omega) \ := \ \beta_{\mathrm{LB}}^{-1}(\mu,\omega)\sup_{v\in X}\big(\tilde{\delta}_{\mathrm{KL}}^{(4)}(v;\mu,\omega)/\|v\|_X\big).$$

As a consequence of Proposition 4.3, we obtain

$$\|\tilde{e}^{(2)} - \tilde{e}^{(3)}\|_X \le \tilde{\Delta}^{(4)} \ := \ 2d\Big(\tilde{\Delta}_{\mathrm{RB}}^{(4)} + \tilde{\Delta}_{\mathrm{KL}}^{(4)}\Big) + 2d\frac{\rho_1}{\beta_{\mathrm{LB}}}\Delta\|p_{N,K}^{(2)} - p_{N,K}^{(3)}\|_X$$

and define the variance error bound $\Delta^{\mathbb{V}}(\mu)$ by

$$(4.39) \quad \begin{aligned} \Delta^{\mathbb{V}}(\mu) \ := \ & \mathbb{E}\left[(\Delta^s(\mu,\cdot))^2\right] + (\Delta^{\mathbb{M}_1}(\mu))^2 \\ & + \ \mathbb{E}\Big[\frac{\beta_{\mathrm{LB}}(\mu,\cdot)}{2d(\mu,\cdot)}\Delta(\mu,\cdot)\tilde{\Delta}^{(4)}(\mu,\cdot)\Big] + \mathbb{E}\Big[\delta_{\mathrm{KL}}(p_{N,K}^{(2)}(\mu,\cdot) - p_{N,K}^{(3)}(\mu,\cdot);\mu,\cdot)\Big]. \end{aligned}$$

Proposition 4.10. *For $\mu \in \mathcal{P}$ and $\tau(\mu,\cdot) < 1$, it holds that $|\mathbb{V}(\mu) - \mathbb{V}_{NK}(\mu)| \le \Delta^{\mathbb{V}}(\mu)$.*

Proof. From Propositions 4.7 and 4.9, we know

$$\mathbb{V} - \mathbb{V}_{NK} \; = \; \mathbb{E}\left[(s - s_{N,K})^2\right] \; - \; (\mathbb{M}_1 - \mathbb{M}_{1,NK})^2$$
$$+ \; \mathbb{E}\left[\ell^{(2)}(u) - \ell^{(2)}(u_{N,K}) + r_{\mathrm{RB}}(p_{N,K}^{(2)})\right]$$
$$- \; \mathbb{E}\left[\ell^{(3)}(u) - \ell^{(3)}(u_{N,K}) + r_{\mathrm{RB}}(p_{N,K}^{(3)})\right]$$

and the first two terms can be bounded by $\mathbb{E}\left[(\Delta^s)^2\right]$ and $(\Delta^{\mathbb{M}_1})^2$, respectively. From Lemma 4.4 and the definition of the residual r_{RB}, we know that

$$\ell^{(i)}(u) - \ell^{(i)}(u_{N,K}) + r_{\mathrm{RB}}(p_{N,K}^{(i)}) \; = \; g(u_{N,K}, p^{(i)}) - g^K(u_{N,K}, p_{N,K}^{(i)}), \quad i = 2, 3.$$

We subtract the two expressions and follow again the proof of Proposition 4.5. The claim follows directly using the above definitions. $\square$

5. OFFLINE-ONLINE DECOMPOSITION

In the RB context, one distinguishes between expensive offline computations that have to be done once to create the reduced system and repeatedly performed online computations, see e.g. [8, 15]. The aim of the RBM are online evaluation procedures of state, outputs and corresponding error bounds independent of the dimension $\mathcal{N}$ of X. In this section, we describe the offline-online decomposition and provide the respective complexities.

For the $\mathcal{N}$-independence, it is of crucial importance to efficiently evaluate the continuity constant $\rho_1(\mu, \omega)$ from (2.2) and the inf-sup constant $\beta_{\mathrm{LB}}(\mu, \omega)$ from (4.23). We start with an evaluation procedure for the continuity constant.

5.1. **Continuity constant.** The derivation of the continuity constant $\rho_1(\mu, \omega)$ from (2.2) is commonly done using Hölder's inequality and applying the Sobolev embedding theorem [6, 18], where the existence of a so called Sobolev embedding constant ρ_X with $\|v\|_4 \leq \rho_X \|v\|_X$ for all $v \in X$ is shown. However, the actual derivation of $\rho_1(\mu, \omega)$ depends on the specific form of the trilinear form a_1. Here, we exemplarily provide the derivation strategy for a specific trilinear form that also (but not only) covers the example problem discussed in Section 6. Let a_1 be given by

$$a_1(u, w, v; \mu, \omega) \; := \; \int_D \vec{\nu}(\mu, \omega) \cdot \nabla u w v \; = \; \int_D \nu_1(\mu, \omega) u_x w v + \int_D \nu_2(\mu, \omega) u_y w v,$$

where $\nu(\mu, \omega) : D \times \mathcal{M} \to \mathbb{R}^2$ denotes some parametric spatial stochastic process. For the first part, omitting ν for one moment, we apply Hölder's inequality twice,

$$\int_D u_x w v \; \leq \; \left[\int_D (u_x)^2\right]^{1/2} \left[\int_D (wv)^2\right]^{1/2} \; \leq \; \left[\int_D (u_x)^2\right]^{1/2} \left[\int_D (ww)^2\right]^{1/4} \left[\int_D (vv)^2\right]^{1/4}.$$

Analogously, we estimate the second part. For $\bar{\nu}(\mu, \omega) := \max_{i \in \{1,2\}} \|\nu_i(\mu, \omega)\|_\infty$, we directly obtain the bound $a_1(u, w, v; \mu, \omega) \leq \bar{\nu}(\mu, \omega)(\|u_x\|_2 + \|u_y\|_2)\|w\|_4\|v\|_4$. Using Young's inequality, we can easily show that $\|u_x\|_2 + \|u_y\|_2 \leq \sqrt{2}\|u\|_X$. Hence,

$$a_1(u, w, v; \mu, \omega) \; \leq \; \sqrt{2}\bar{\nu}(\mu, \omega)\|u\|_X\|w\|_4\|v\|_4.$$

Now, we apply the Sobolev embedding theorem and obtain the desired continuity constant $\rho_1(\mu, \omega) := \sqrt{2}\bar{\nu}(\mu, \omega)\rho_X^2$. Suppose $\vec{\nu}$ allows for an affine decomposition in the parameters (μ, ω), it is clear that $\bar{\nu}$ and therefore ρ_1 can be decomposed as well with the same number of affine terms. Hence, the online evaluation of $\rho_1(\mu, \omega)$ can be done efficiently.

It remains to compute the Sobolev embedding constant ρ_X which involves the solution of a nonlinear eigenproblem of the form

$$(5.40) \qquad \int_D \phi^3 v \; = \; \lambda \cdot (\phi, v)_X, \quad \forall v \in X, \quad \|\phi\|_X = 1.$$

The solution of (5.40) can be obtained using e.g. fixed point or homotopy procedures [20]. The Sobolev embedding constant ρ_X is then given by $(\lambda_{\max})^{1/4}$. The evaluation can be done offline.

5.2. **Inf-sup constant.** For the evaluation of the inf-sup constant, we refer to the successive constraint method (SCM) [11] that can almost directly be applied to the stochastic case. However, due to the KL truncation, we have to subtract a correction term. Let $\beta_{\mathrm{LB}}^K(\mu, \omega)$ be a lower bound of the inf-sup constant with respect to the truncated form $dg^K(w, v; \mu, \omega)[u_{N,K}(\mu, \omega)]$. We furthermore define

$$\Delta_{\mathrm{KL}}^{\beta}(\mu, \omega) \; := \; \sup_{w \in X} \sup_{v \in X} \left(\sum_{q=1}^{Q} |\theta_q(\mu)| \sum_{k=K+1}^{K_{\max}} \xi_\varrho \frac{|dg_{q,k}(v,w)[u_{N,K}]|}{\|w\|_X \|v\|_X} \right)$$

and obtain the lower bound (c.f. [5, 8])

$$\beta_{\mathrm{LB}}(\mu, \omega) \; := \; \beta_{\mathrm{LB}}^K(\mu, \omega) - \Delta_{\mathrm{KL}}^{\beta}(\mu, \omega) \; \leq \; \beta(\mu, \omega).$$

In [11], it is shown that the online evaluation of $\beta_{\mathrm{LB}}^K(\mu, \omega)$ is independent of $\mathcal{N}$. However, it involves the solution of a linear program with about $(QKN)^2/2$ degrees of freedom. One can show that the online evaluation of $\Delta_{\mathrm{KL}}^{\beta}(\mu, \omega)$ is of complexity $\mathcal{O}(Q(K_{\max} - K)N)$. The combined offline evaluations for $\beta_{\mathrm{LB}}^K(\mu, \omega)$ and $\Delta_{\mathrm{KL}}^{\beta}(\mu, \omega)$ include $QK_{\max}N$ eigenvalue problems of the full dimension $\mathcal{N}$.

5.3. **The offline complexity.** To generate the reduced basis, we use a Greedy algorithm as it is well known in the RB context [21]. Suppose we use a training set of n_{train} samples, we need $\mathcal{O}(N \cdot n_{\mathrm{train}})$ the online run-time for the basis selection procedure. Furthermore, the evaluation of the actual basis is of complexity $\mathcal{O}(IQK_{\mathrm{truth}}N\mathcal{N})$, where I is the number of used Newton iterations, assuming that the FE "truth" uses K_{truth} terms of the KL expansion. The complexity to compute the matrices and vectors of the reduced system is $\mathcal{O}(QK_{\max}N^3)$. For the evaluation of the Δ_{KL} and Δ_{RB} error bounds, we evaluate $\mathcal{O}(QK_{\max}N^2)$ Riesz representators, one for each affine term of the residuals, and its pairwise inner products. I.e., the complexity reads $\mathcal{O}(Q^2K_{\max}^2N^4\mathcal{N})$. We store the reduced system matrices and vectors and the Riesz representators inner products, i.e., the storage complexity is $\mathcal{O}(Q^2K_{\max}^2N^4)$.

5.4. **The online complexity.** Let us summarize the online run-time complexity to assemble and solve the reduced system for one parameter pair $(\mu, \omega) \in \mathcal{M}$ and to evaluate outputs and error bounds. Let I denote again the number of Newton iterations. In each iteration, we have to assemble and solve the reduced primal system which is of complexity $\mathcal{O}(QKN^3)$ and $\mathcal{O}(N^3)$, respectively. The evaluation of the residuals r_{RB} — needed as correction terms for the outputs — is done in $\mathcal{O}(QKN^3)$ as well. Furthermore, we need to assemble and solve the linear dual problems with complexity $\mathcal{O}(QKN^3 + N^3)$, i.e. the complexity of just one Newton iteration. For the error bounds, we first evaluate β_{LB}, solving a linear program with about $(QKN)^2/2$ degrees of freedom. The evaluation of ρ_1 can be done in $\mathcal{O}(QK)$. For the δ_{KL}-error bounds, we need $Q(K_{\max} - K)$ matrix-vector multiplications, i.e., the complexity is $\mathcal{O}(Q(K_{\max} - K)N^2)$. For the Δ_{KL} and Δ_{RB} error bounds, we have to assemble the inner products of the Riesz representators with the total complexity $\mathcal{O}((QK_{\max}N^2)^2)$. Hence, the overall complexity reads $\mathcal{O}(IQKN^3) + \mathcal{O}(Q^2K_{\max}^2N^4)$.
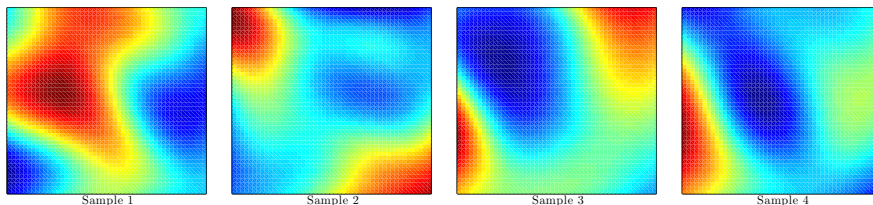
FIGURE 6.1. Four random realizations of $\tilde{\kappa}$
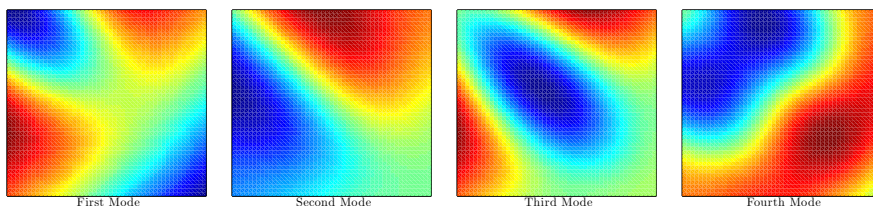
FIGURE 6.2. First four modes of $\tilde{\kappa}$

The storage complexity is $\mathcal{O}(QK_{\max}N^3)$ for all reduced matrices, vectors and Riesz inner products. Suppose we use M realizations to evaluate Monte Carlo statistical outputs. Then, we have an additional storage complexity of $\mathcal{O}(M)$ to store certain RB outputs or we need to reevaluate the respective quantities when needed. However, using the less precise variance error bound (4.38), it is possible to evaluate all quantities with an additional storage complexity of just $\mathcal{O}(1)$. For more details, we refer to [8].

6. NUMERICAL EXPERIMENT

In this section, we consider a two-dimensional stationary convection-diffusion process in a porous medium. We model the concentration or mass of a physical quantity transported through a wet sandstone. The diffusivity depends on the porosity, modeled by some spatial stochastic process, and the water saturation of the sandstone, given by some deterministic parameter. The nonlinear convective term includes the gradient of the concentration together with a given dominant direction and a scalar intensity factor given by another deterministic parameter.

Let $D = [0,1]^2$ denote the physical domain of the sandstone and $(\Omega, \mathcal{F}, P)$ some probability space. The porosity, i.e. the rate of pore space within some control volume, is denoted by the spatial stochastic process $\kappa : D \times \Omega \to [0,1]$ and is assumed to be smooth in space. Furthermore, the global water saturation in the pores is given by $\mu_1 \in [0.05, 1.00]$. Let $\eta_{\mathrm{s}} = 0.04$ be the diffusivity constant of pure (theoretically imporous) sandstone and $\eta_{\mathrm{w}} = 3.10$, $\eta_{\mathrm{a}} = 1.20$ the respective diffusivity constants of water and air. With these notations, the diffusivity of a wet sandstone is assumed to be

$$(6.41) \quad \begin{aligned} \eta(x; \mu, \omega) &= \eta_{\mathrm{s}} \cdot (1 - \kappa(x; \omega)) + (\mu_1 \eta_{\mathrm{w}} + (1 - \mu_1)\eta_{\mathrm{a}})\kappa(x; \omega) \\ &= \eta_{\mathrm{s}} + (-\eta_{\mathrm{s}} + \mu_1 \eta_{\mathrm{w}} + (1 - \mu_1)\eta_{\mathrm{a}})\kappa(x; \omega). \end{aligned}$$

We denote the scaled dominant convection direction by $\vec{\nu}(\mu_2) = \frac{\mu_2}{\sqrt{2}}\binom{1}{1}$, where $\mu_2 \in [0.2, 1.0]$. Finally, we introduce a random zero mean Neumann outlet condition $\gamma(\omega)$ at one part of the boundary. For $\mu := (\mu_1, \mu_2) \in \mathcal{D} := [0.05, 1.00] \times [0.20, 1.00]$ and $\mathcal{M} := \mathcal{D} \times \Omega$, the PDE reads
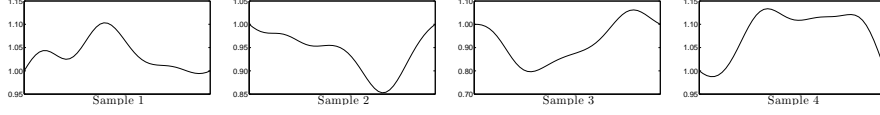
FIGURE 6.3. Four random realizations of γ
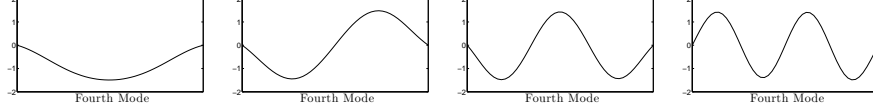
FIGURE 6.4. First four modes of γ

as follows: for given $(\mu, \omega) \in \mathcal{M}$, find $u(\mu, \omega)$ such that

$$(6.42) \quad \begin{cases} -\nabla \cdot \big(\eta(\mu_1, \omega)\, \nabla u(\mu, \omega)\big) + \vec{\nu}(\mu_2) \cdot \nabla u\, u & = & 0 \quad \text{in } D, \\ u(\mu, \omega) & = & 0 \quad \text{on } \Gamma_{\mathrm{D}}, \\ n \cdot \big(\eta(\mu_1, \omega)\, \nabla u(\mu, \omega)\big) & = & 0 \quad \text{on } \Gamma_{\mathrm{N}}, \\ n \cdot \big(\eta(\mu_1, \omega)\, \nabla u(\mu, \omega)\big) & = & \gamma(\omega) \quad \text{on } \Gamma_{\mathrm{out}}. \end{cases}$$

In the weak form, this leads to the trilinear form $a_1(w, z, v; \mu) = \int_D \vec{\nu}(\mu_2) \cdot \nabla w\, z\, v$, the bilinear form $a_0(w, v; \mu, \omega) = \int_D \eta(\mu_1, \omega) \nabla w \nabla v$, and the linear form $f(v; \omega) = \int_{\Gamma_{\mathrm{out}}} \gamma(\omega) v$. We define $\theta_1(\mu) := \eta_s$ and $\theta_2(\mu) := -\eta_s + \mu_1 \eta_{\mathrm{w}} + (1 - \mu_1)\eta_{\mathrm{a}}$ using (6.41), as well as $\theta_3(\mu) := \mu_2$, and $\theta_4(\mu) := 1$. Hence, the affine decompositions w.r.t. μ of a_0, a_1 and f are given by

$$a_0(w, v; \mu, \omega) = \theta_1(\mu) \int_D \nabla w \nabla v + \theta_2(\mu) \int_D \kappa(\omega) \nabla w \nabla v,$$

$$a_1(w, z, v; \mu) = \theta_3(\mu) \int_D \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \nabla w\, z\, v,$$

$$f(v; \omega) = \theta_4(\mu) \int_{\Gamma_{\mathrm{out}}} \gamma(\omega)\, v.$$

Let $\bar{\kappa}(x)$ denote the mean of the porosity $\kappa(x; \cdot)$ and $\tilde{\kappa}(x; \omega) := \kappa(x; \omega) - \bar{\kappa}(x)$ its stochastic part with zero mean and the KL expansion $\tilde{\kappa}(x; \omega) = \sum_{k=1}^{K_{\kappa,\max}} \xi_k^\kappa(\omega) \kappa_k(x)$, where $\bar{\kappa}(x) \equiv 0.62$ is supposed to be constant in space. Four random realizations and the first four KL modes of $\tilde{\kappa}$ are provided in Figure 6 and 6.2, respectively. Analogously, we have the KL expansion for the zero mean outlet given by $\gamma(x; \omega) = \sum_{k=1}^{K_{\gamma,\max}} \xi_k^\gamma(\omega) \gamma_k(x)$. Four random realizations and the first four KL modes of $\tilde{\kappa}$ are provided in Figure 6.3 and 6.4, respectively. The KL eigenvalues of $\tilde{\kappa}$ and γ are plotted in Figure 6.5. For the truth, we used $K_{\kappa,\mathrm{truth}} = 78$ terms to specify $\tilde{\kappa}$ and $K_{\gamma,\mathrm{truth}} = 18$ terms to specify γ. In the reduced setting, $K_\kappa = 30$ and $K_\gamma = 11$ terms are used, respectively. The error is measured using $K_{\kappa,\max} = 47$ and $K_{\gamma,\max} = 15$, respectively, such that the additional truncation error is negligible compared to the given error tolerance. In total, the affine decomposition of g w.r.t. μ and ω consists of $3 + K_{\kappa,\mathrm{truth}} + K_{\gamma,\mathrm{truth}} = 99$ terms, the affine decomposition of dg of $2 + K_{\kappa,\mathrm{truth}} = 80$ terms, and the respective truncated forms of $3 + K_\kappa + K_\gamma = 44$ and $2 + K_\kappa = 32$.

The output of interest is assumed to be the average concentration at the "output" boundary Γ_{out}, i.e., for $\ell(v) = \int_{\Gamma_{\mathrm{out}}} v$, we define the output $s(\mu, \omega) := \ell(u(\mu, \omega))$. Furthermore, we are interested in its mean, second moment and variance.
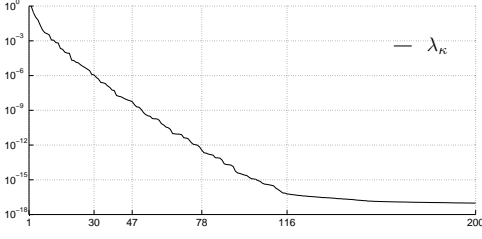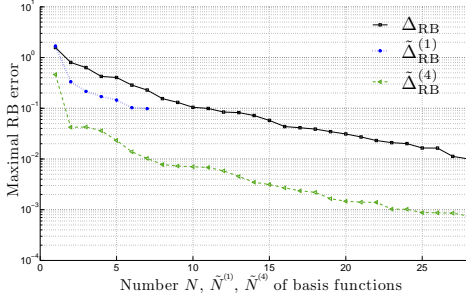
(a) Eigenvalues of the KL expansion of $\tilde{\kappa}$ and KL truncation values $K_\kappa = 30$, $K_{\kappa,\max} = 47$, $K_{\kappa,\mathrm{truth}} = 78$
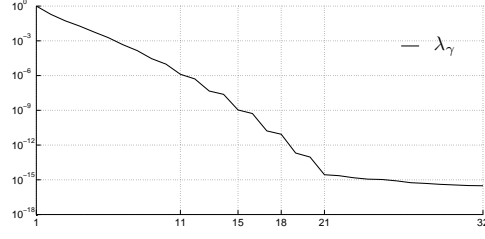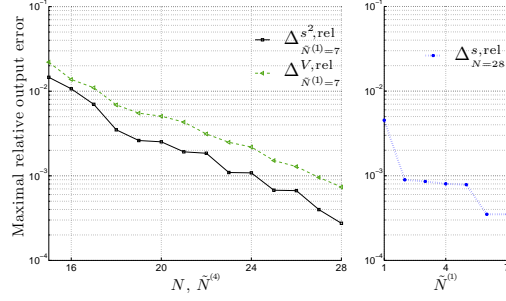
(b) Eigenvalues of the KL expansion of γ and KL truncation values $K_\gamma = 11$, $K_{\gamma,\max} = 15$, $K_{\gamma,\mathrm{truth}} = 18$

FIGURE 6.5. Eigenvalues and truncation values of the Karhunen-Loève expansions

(a) Error decay of primal solution u and dual solution $p^{(1)}$, and of $(p^{(2)} - p^{(3)})$

(b) Relative error decay of outputs s, s^2 and $\mathbb{V}$ without δ_{KL}-contributions, respectively

FIGURE 6.6. Greedy error decay

For the "truth", we use a finite element space $X \subset \{v \in H^1(D) \,|\, v = 0 \text{ on } \Gamma_D\}$ with linear Lagrange elements and $\mathcal{N} = 3191$ degrees of freedom. For the corresponding H^1-norm $\|\cdot\|_X$, we evaluate the Sobolev embedding constant $\rho_X = \sup_{v \in X} \|v\|_4 / \|v\|_X$ as described in Section 5.1 and obtain $\rho_x = 0.60077$.

For the basis construction, we use a greedy algorithm such that $\tilde{X}_N^{(2)} = \tilde{X}_N^{(3)}$. Figure 6.6(a) shows the decay of the maximal RB error bounds of the primal and dual solutions u and $p^{(1)}$ as well as the difference of the additional dual solutions $p^{(2)} - p^{(3)}$. For $(N, \tilde{N}^{(1)}, \tilde{N}^{(2)}) = (28, 7, 28)$, the error bounds of the desired outputs fall below the given tolerance $tol = 10^{-3}$ for all (μ, ω) in the training sample. The decay of the output error bounds is provided in Figure 6.6(b), omitting the δ_{KL}-parts that do not decrease in N and are therefore ignored in the greedy procedure. We parallely created X_N and $\tilde{X}_N^{(2)}$ used for reduced solution of s^2 and $\mathbb{V}$, assuming that $\tilde{N}^{(1)}$ is already large enough such that the terms $(\Delta^s)^2$ and $(\Delta^{\mathbb{M}_1})^2$ in the respective error bounds Δ^{s^2} and $\Delta^{\mathbb{V}}$ are sufficiently small. For $N \geq 15$ primal basis functions, we obtained $\tau < 1$ for all (μ, ω) in the training set. Then, we created $\tilde{X}_N^{(1)}$ such that Δ^s and $\Delta^{\mathbb{M}_1}$ indeed become sufficiently small. Since N was already large, only a small number of $\tilde{N}^{(1)} = 7$ basis functions were needed.

To compare truth and reduced solutions, we used a 3.06 GHz Intel Core 2 Duo processor, 4 GB RAM. We used Matlab 7.9.0 (R2012a) to run reduced simulations and Matlab 7.9.0 with

the link to Comsol 3.5a for the truth. For the (rather small) truth with $\mathcal{N} = 3.191$, we already achieved a speedup factor of about 33 from full to reduced simulations, where in the reduced case, the evaluations of all error bounds are included. Tests with finer meshes and hence larger $\mathcal{N}$ for the full solutions showed that the desired error tolerance can still be reached with the same numbers of basis functions. E.g. for $\mathcal{N} = 12.555$ and $(N, \tilde{N}^{(1)}, \tilde{N}^{(2)}) = (28, 7, 28)$, the speedup factor was about 96.

In Table 6.1, we compare the presented method to evaluate variances $\mathbb{V}_{NK}$ and the error bound $\Delta^{\mathbb{V}}$ with two alternative procedures. Neither of the two needs additional dual problems. The simplest method just uses the estimations $|s^2 - (s_{N,K})^2| = |(s - s_{N,K})(s + s_{N,K})| \leq \Delta^s(\Delta^s + 2|s_{N,K}|)$ and analogously $|\mathbb{M}_1^2 - (\mathbb{M}_{1,NK})^2| \leq \Delta^{\mathbb{M}_1}(\Delta^{\mathbb{M}_1} + 2|\mathbb{M}_{1,NK}|)$. For the more sophisticated method, we refer to [2]. We see that our variance evaluation and the error bounds produce much sharper results. Compared to the "simple" method, the bound is about 24 times smaller, compared to the "sophisticated" method, it still is over 4 times smaller. The costs, on the other hand, increase only moderately. The evaluation of the additional dual problem corresponds to just one Newton iteration of the primal problem.

	average error bound	factor
simple	$7.436 \cdot 10^{-3}$	$23.95 \cdot \Delta^{\mathbb{V}}$
sophisticated	$1.347 \cdot 10^{-3}$	$4.341 \cdot \Delta^{\mathbb{V}}$
$\Delta^{\mathbb{V}}$	$3.104 \cdot 10^{-4}$	$1.000 \cdot \Delta^{\mathbb{V}}$

TABLE 6.1. Comparison of different variance error bounds for a test set of 256 parameters, using 10.000 random samples for each parameter.

7. CONCLUSION

We have extended the theory of RBMs for linear PDEs with stochastic influences [8] to the nonlinear case, using the BRR theory and the results from [5, 20]. It is demonstrated that the RB methodology allows us to deal with large nonlinear parameterized systems involving significant stochastic deviations. We have shown that quadratic outputs such as second moment and variance can be also evaluated efficiently using an additional linear dual problem.

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