MAXWELL'S EQUATIONS WITH IMPEDANCE BOUNDARY CONDITIONS: DISCONTINUOUS GALERKIN AND REDUCED BASIS METHODS

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Abstract. We consider Maxwell's equations with impedance boundary conditions on a polyhedron with polyhedral holes. Well-posedness of the variational formulation is proven and a discontinuous Galerkin (dG) approximation is introduced. We prove well-posedness of the dG problem as well as a priori error estimates.

Next, we use the frequency ω as a parameter in a multi-query context. For this purpose, we derive a Reduced Basis Method (RBM) based upon the dG formulation as well as the corresponding a posteriori error bound. Numerical results indicate the efficiency and the robustness of the scheme.

1. Introduction

This paper is concerned with the analysis and the efficient numerical solution of the time-harmonic Maxwell's equations on a simply-connected polyhedron which may have polyhedral holes. These holes can be seen as rigid bodies of perfectly conducting material, whereas on the exterior we impose an impedance boundary condition.

We start by proving well-posedness of the variational formulation in Section 2. Although this a is more or less standard application of the Lax-Milgram theorem, the verification of coercivity and boundedness at least under our (mild) assumptions is not trivial and we were not able to find a proof in the literature. Hence, we detail all arguments.

Next, in Section 3, we construct a discontinuous Galerkin (dG) numerical method to obtain a discretization of the electric field density $\boldsymbol{E}$. We detail well-posedness (which is again not trivially seen) and the a priori convergence analysis. These results are a generalization of [21]. Specifically, in §3.1 we use an interior penalty dG flux and derive a consistent discrete variational formulation in the non-conforming space of piecewise affine functions. The corresponding sesquilinear form is shown to be continuous and coercive w.r.t. an appropriate energy norm in §3.2. This provides the foundation for our error analysis in §3.3, where we show convergence at an optimal rate w.r.t. the energy norm.

Our main objective is to solve the time-harmonic Maxwell problem for several different values of the frequency ω. Hence, we obtain *parameterized* Maxwell's

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equations, where we seek a numerical approximation for many values of the frequency. For this kind of multi-query problems (i.e., solving the same problem for many different values of the parameter), the Reduced Basis Method (RBM) has become a well-accepted efficient numerical scheme, in particular for parameterized partial differential equations (pde's). Roughly speaking, the RBM is based upon a separation into offline and online computations as well as the availability of a detailed, but possibly costly numerical model, e.g., with a fine mesh size h and a huge number $\mathcal{N} = \mathcal{N}_h$ of unknowns. Using this detailed model and an efficiently computable error bound allows one to determine "bad" parameter values, say $\omega_1, \dots, \omega_N$, $N \ll \mathcal{N}$, in the offline phase by maximizing the error estimator w.r.t. the parameter ω[a]. For these ω_i, the detailed solution $\boldsymbol{\xi}_i := \boldsymbol{E}_h(\omega_i)$ is computed in the offline phase and stored. Then, the set $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N\}$ is called *reduced basis*, which is used in the online phase to compute an approximation $\boldsymbol{E}_N(\omega)$ for a new parameter value $\omega \neq \omega_i$. The already mentioned a posteriori error bound gives rise to a certified reduced numerical approximation.

There are several papers dealing with RBMs for different versions of Maxwell's equations, see [9, 10, 14, 15, 18, 19, 28], just to mention a few. However, to the best of our knowledge, the case treated in this paper has not been considered so far. For the following reasons we think that the presented framework is particularly interesting:

- The domain Ω, on which we consider the pde, is non-convex, so that the solution cannot be expected to have maximal regularity and hence H^1-conforming finite elements may not be appropriate, whereas a dG approach seems adequate.
- Changing the frequency ω, i.e., interpreting it as a parameter may also change the mathematical properties of the pde. If "critical" parameter values are not known a priori, RBM variants such as local RBM [23] or hp-RBM [12, 13] are at least not straightforward to apply.
- In the literature, usually perfectly conducting material and corresponding boundary conditions on all of $\partial\Omega$ have been considered. Instead, we use impedance boundary conditions on the outward part Σ of $\partial\Omega$, see below.

Section 4 contains construction and analysis of a RBM for the above Maxwell setting. Finally, Section 5 is devoted to the presentation of our numerical results that show efficiency and robustness of our approach.

2. Model problem

We consider an electromagnetic cavity problem on a bounded, simply-connected Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$ with M disjoint connected boundary parts $\Gamma_1, \dots,$ Γ_{M-1}, Σ. Note that Σ is the boundary of the only unbounded connected component of the complement $\mathbb{R}^3 \setminus \bar{\Omega}$ – the "interface" to the exterior. At the interior boundaries $\Gamma_1, \dots, \Gamma_{M-1}$ the domain Ω is assumed to be surrounded by perfectly conducting material. At the exterior boundary Σ the electromagnetic field satisfies an impedance boundary condition. Following the approach presented, e.g., in [25] we obtain the following boundary value problem for the case of a *time-harmonic*

[a]In the RBM literature, usually the letter μ is used for denoting the parameter. Since we consider the frequency ω as the relevant parameter and here μ denotes the magnetic permeability, we use ω to denote the parameter.

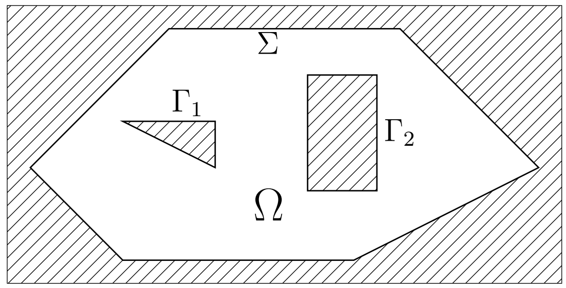


FIGURE 1. Cross section of a possible 3d-domain Ω

electromagnetic wave propagation ($\boldsymbol{n}$ being the outward normal)

$$\nabla \times (\mu^{-1} \nabla \times \boldsymbol{E}) - (\omega^2 \varepsilon + i\omega\sigma)\boldsymbol{E} = i\omega\sqrt{\varepsilon_0}\boldsymbol{J_a} \qquad \text{in } \Omega, \tag{2.1a}$$

$$\boldsymbol{n} \times \boldsymbol{E} = 0 \qquad \text{on } \Gamma_1, \ldots, \Gamma_{M-1}, \tag{2.1b}$$

$$(\mu^{-1} \nabla \times \boldsymbol{E}) \times \boldsymbol{n} - i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\boldsymbol{E}_T = \mu_0^{-1}\boldsymbol{g} \qquad \text{on } \Sigma. \tag{2.1c}$$

As usual, $\mu_0 = 4\pi \cdot 10^{-7}$ Hm^{-1} and $\varepsilon_0 \approx 8.854 \cdot 10^{-12}$ Fm^{-1} denote the magnetic permeability and electric permittivity in vacuum, respectively. For a vector-valued function $\boldsymbol{u} \in H(\mathrm{curl}, \Omega) := \left\{\boldsymbol{v} \in L^2\left(\Omega; \mathbb{C}^3\right) \,\middle|\, \nabla \times \boldsymbol{v} \in L^2\left(\Omega; \mathbb{C}^3\right)\right\}$ the "tangential components trace" $\boldsymbol{u}_T$ on $\partial\Omega$ is defined as $\boldsymbol{u}_T := (\boldsymbol{n} \times \boldsymbol{u}|_{\partial\Omega}) \times \boldsymbol{n}$, cf. [7], [25, Theorem 3.31]. Finally, $\lambda > 0$ is a constant parameter depending on the intensity of the impedance. In addition, we make the following assumptions on the model.

Assumption 2.1. *a) The magnetic permeability μ satisfies $\mu^{-1} \in W^{1,\infty}(\Omega; \mathbb{R})$ and there exist constants μ_-, $\mu_+ > 0$ such that*

$$0 < \mu_- \leq \operatorname*{ess\,inf}_{\boldsymbol{x} \in \Omega} \mu(\boldsymbol{x}) \leq \operatorname*{ess\,sup}_{\boldsymbol{x} \in \Omega} \mu(\boldsymbol{x}) \leq \mu_+ < +\infty. \tag{2.2}$$

b) We assume $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R})$ and that Ω is a conductor, i.e, there exist constants ε_-, ε_+, σ_-, $\sigma_+ > 0$ with

$$0 < \varepsilon_- \leq \operatorname*{ess\,inf}_{\boldsymbol{x} \in \Omega} \varepsilon(\boldsymbol{x}) \leq \operatorname*{ess\,sup}_{\boldsymbol{x} \in \Omega} \varepsilon(\boldsymbol{x}) \leq \varepsilon_+ < +\infty, \tag{2.3}$$

$$0 < \sigma_- \leq \operatorname*{ess\,inf}_{\boldsymbol{x} \in \Omega} \sigma(\boldsymbol{x}) \leq \operatorname*{ess\,sup}_{\boldsymbol{x} \in \Omega} \sigma(\boldsymbol{x}) \leq \sigma_+ < +\infty. \tag{2.4}$$

c) $\boldsymbol{J_a} \in H(\mathrm{div}, \Omega) := \left\{\boldsymbol{u} \in L^2\left(\Omega; \mathbb{C}^3\right) \,\middle|\, \nabla \cdot \boldsymbol{u} \in L^2(\Omega; \mathbb{C})\right\}$.
d) $\boldsymbol{g} \in L^2_t(\Sigma; \mathbb{C}^3) := \left\{\boldsymbol{v} \in L^2\left(\Sigma; \mathbb{C}^3\right) \mid \boldsymbol{n} \cdot \boldsymbol{v} = 0 \text{ a.e. on } \Sigma\right\}$.

Proceeding as in [25, Ch. 4], one can establish the following variational formulation of (2.1a)–(2.1c): Given $\omega > 0$, find $\boldsymbol{E} \in X$ such that

$$a_e(\boldsymbol{E}, \boldsymbol{v}; \omega) = f(\boldsymbol{v}; \omega) \qquad \forall \boldsymbol{v} \in X, \tag{2.5}$$

with the trial and test space defined as

$$X := \{\boldsymbol{u} \in H(\mathrm{curl}, \Omega) \,|\, \boldsymbol{n} \times \boldsymbol{u} = 0 \text{ on } \Gamma_1, \ldots, \Gamma_{M-1};\ \boldsymbol{u}_T \in L^2(\Sigma; \mathbb{C}^3) \text{ on } \Sigma\}, \tag{2.6}$$

the sesquilinear form

$$\begin{aligned} a_e(\boldsymbol{E}, \boldsymbol{v}; \omega) := {}& (\mu^{-1} \nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{v})_\Omega - \omega^2(\varepsilon\boldsymbol{E}, \boldsymbol{v})_\Omega - i\omega(\sigma\boldsymbol{E}, \boldsymbol{v})_\Omega \\ & - i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\langle \boldsymbol{E}_T, \boldsymbol{v}_T \rangle_\Sigma, \end{aligned} \tag{2.7}$$

and the right-hand side

$$f(\boldsymbol{v};\,\omega) := i\omega\sqrt{\varepsilon_0}(\boldsymbol{J_a},\boldsymbol{v})_\Omega + \mu_0^{-1}\langle \boldsymbol{g},\boldsymbol{v}_T\rangle_\Sigma, \tag{2.8}$$

where $(\cdot,\cdot)_\Omega$ denotes the inner product on $L^2\left(\Omega;\mathbb{C}^3\right)$, and for $\boldsymbol{g}\in L^2_t(\Sigma;\mathbb{C}^3)$ and $\boldsymbol{v}\in H(\mathrm{curl},\Omega)$, $\langle\cdot,\cdot_T\rangle_\Sigma$ is the dual pairing w.r.t. the pivot space $L^2(\Sigma;\mathbb{C}^3)$, i.e., $\langle \boldsymbol{g},\boldsymbol{v}_T\rangle_\Sigma := \int_\Sigma \boldsymbol{g}\cdot\overline{\boldsymbol{v}}_T\,\mathrm{d}S$. As shown in [25, Thm. 4.1], X is a Hilbert space with inner product $(\boldsymbol{u},\boldsymbol{v})_X := (\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v})_\Omega + (\boldsymbol{u},\boldsymbol{v})_\Omega + \langle \boldsymbol{u}_T,\boldsymbol{v}_T\rangle_\Sigma$, $\boldsymbol{u},\boldsymbol{v}\in X$, and induced norm $\|\boldsymbol{u}\|_X^2 := \|\nabla\times\boldsymbol{u}\|^2_{L^2(\Omega)^3} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^3} + \|\boldsymbol{u}_T\|^2_{L^2(\Sigma)^3}$, $\boldsymbol{u}\in X$. The next theorem states well-posedness of (2.5).

Theorem 2.2 (Existence and uniqueness of $\boldsymbol{E}$)**.** *Suppose Assumption 2.1 is satisfied. Then, there exists a unique solution $\boldsymbol{E}\in X$ to (2.5) and a constant $C>0$ depending on ω, μ, ε, σ, and λ such that $\|\boldsymbol{E}\|_X \le C\left(\|\boldsymbol{J_a}\|_{L^2(\Omega)^3} + \|\boldsymbol{g}\|_{L^2(\Sigma)^3}\right)$.*

Proof. The claim follows from a complex-valued version of the Lax-Milgram lemma, (e.g. [25, Lem. 2.21]), since $a_e : X\times X\to\mathbb{C}$ in (2.7) is continuous and coercive and $f : X\to\mathbb{C}$ in (2.8) is bounded. In fact, fix $\omega>0$ and let $\boldsymbol{u}$, $\boldsymbol{v}\in X$, then

$$\begin{aligned}
|a_e(\boldsymbol{u},\boldsymbol{v};\,\omega)| &= |(\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v})_\Omega - \omega^2(\varepsilon\boldsymbol{u},\boldsymbol{v})_\Omega - i\omega(\sigma\boldsymbol{u},\boldsymbol{v})_\Omega \\
&\qquad - i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\langle \boldsymbol{u}_T,\boldsymbol{v}_T\rangle_\Sigma| \\
&\le \mu_-^{-1}\|\nabla\times\boldsymbol{u}\|_{L^2(\Omega)^3}\|\nabla\times\boldsymbol{v}\|_{L^2(\Omega)^3} + \left(\omega^2\varepsilon_+ + \omega\sigma_+\right)\|\boldsymbol{u}\|_{L^2(\Omega)^3}\|\boldsymbol{v}\|_{L^2(\Omega)^3} \\
&\quad + \omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\,\|\boldsymbol{u}_T\|_{L^2(\Sigma)^3}\|\boldsymbol{v}_T\|_{L^2(\Sigma)^3} \\
&\le \max\left\{\mu_-^{-1},\left(\omega^2\varepsilon_+ + \omega\sigma_+\right),\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\right\}\|\boldsymbol{u}\|_X\|\boldsymbol{v}\|_X.
\end{aligned}$$

This shows continuity. To prove coercivity we have:

$$\begin{aligned}
|a_e(\boldsymbol{u},\boldsymbol{u};\,\omega)| &= \Big|(\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{u})_\Omega - \omega^2(\varepsilon\boldsymbol{u},\boldsymbol{u})_\Omega - i\omega(\sigma\boldsymbol{u},\boldsymbol{u})_\Omega \\
&\qquad - i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\langle \boldsymbol{u}_T,\boldsymbol{u}_T\rangle_\Sigma\Big| \\
&= \Bigg[\left((\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{u})_\Omega - \omega^2(\varepsilon\boldsymbol{u},\boldsymbol{u})_\Omega\right)^2 \\
&\qquad + \left(\omega(\sigma\boldsymbol{u},\boldsymbol{u})_\Omega + \omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\|\boldsymbol{u}_T\|^2_{L^2(\Sigma)^3}\right)^2\Bigg]^{1/2} \\
&\ge \Big[(\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{u})^2_\Omega - 2\omega^2(\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{u})_\Omega(\varepsilon\boldsymbol{u},\boldsymbol{u})_\Omega \\
&\qquad + \omega^4(\varepsilon\boldsymbol{u},\boldsymbol{u})^2_\Omega + \omega^2(\sigma\boldsymbol{u},\boldsymbol{u})^2_\Omega + \omega^2\lambda^2\varepsilon_0\mu_0^{-1}\|\boldsymbol{u}_T\|^4_{L^2(\Sigma)^3}\Big]^{1/2}.
\end{aligned}$$

Young's inequality gives for $\delta\in(0,1)$

$$\begin{aligned}
|a_e(\boldsymbol{u},\boldsymbol{u};\,\omega)| &\ge \Big[(1-\delta)(\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{u})^2_\Omega + \left(1-\delta^{-1}\right)\omega^4(\varepsilon\boldsymbol{u},\boldsymbol{u})^2_\Omega \\
&\qquad + \omega^2(\sigma\boldsymbol{u},\boldsymbol{u})^2_\Omega + \omega^2\lambda^2\varepsilon_0\mu_0^{-1}\|\boldsymbol{u}_T\|^4_{L^2(\Sigma)^3}\Big]^{1/2}.
\end{aligned}$$

The fact that $\sqrt{a+b} \geq \frac{1}{\sqrt{2}}(\sqrt{a}+\sqrt{b})$ for $a, b \geq 0$, together with (2.2), (2.3) and (2.4) in Assumption 2.1, leads to

$$\begin{aligned}
|a_e(\boldsymbol{u},\boldsymbol{u};\,\omega)| &\geq \frac{1}{\sqrt{2}}\Big[(1-\delta)(\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{u})_\Omega^2 + \left(1-\delta^{-1}\right)\omega^4(\varepsilon\boldsymbol{u},\boldsymbol{u})_\Omega^2 \\
&\qquad + \omega^2(\sigma\boldsymbol{u},\boldsymbol{u})_\Omega^2\Big]^{1/2} + \frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\|\boldsymbol{u}_T\|^2_{L^2(\Sigma)^3} \\
&\geq \frac{1}{\sqrt{2}}\left[\frac{1-\delta}{\mu_+^2}\|\nabla\times\boldsymbol{u}\|^4_{L^2(\Omega)^3} + \left(\omega^2\sigma_-^2 - \left(\delta^{-1}-1\right)\omega^4\varepsilon_+^2\right)\|\boldsymbol{u}\|^4_{L^2(\Omega)^3}\right]^{1/2} \\
&\qquad + \frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\|\boldsymbol{u}_T\|^2_{L^2(\Sigma)^3}.
\end{aligned}$$

Choose $1 > \delta > \frac{\omega^2\varepsilon_+^2}{\omega^2\varepsilon_+^2+\sigma_-^2} > 0$, e.g., $\delta = \frac{\omega^2\varepsilon_+^2+\sigma_-^2/2}{\omega^2\varepsilon_+^2+\sigma_-^2}$,

$$\begin{aligned}
|a_e(\boldsymbol{u},\boldsymbol{u};\,\omega)| &\geq \frac{1}{\sqrt{2}}\Big[\frac{\sigma_-^2}{2(\omega^2\varepsilon_+^2+\sigma_-^2)\mu_+^2}\|\nabla\times\boldsymbol{u}\|^4_{L^2(\Omega)^3} + \frac{\omega^4\sigma_-^2\varepsilon_+^2+\omega^2\sigma_-^4}{2\omega^2\varepsilon_+^2+\sigma_-^2}\|\boldsymbol{u}\|^4_{L^2(\Omega)^3}\Big]^{\frac{1}{2}} \\
&\qquad + \frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\|\boldsymbol{u}_T\|^2_{L^2(\Sigma)^3} \\
&\geq \frac{1}{2}\left(\frac{\sigma_-^2}{2(\omega^2\varepsilon_+^2+\sigma_-^2)\mu_+^2}\right)^{\frac{1}{2}}\|\nabla\times\boldsymbol{u}\|^2_{L^2(\Omega)^3} \\
&\qquad + \frac{1}{2}\left(\frac{\omega^4\sigma_-^2\varepsilon_+^2+\omega^2\sigma_-^4}{2\omega^2\varepsilon_+^2+\sigma_-^2}\right)^{\frac{1}{2}}\|\boldsymbol{u}\|^2_{L^2(\Omega)^3} + \frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\|\boldsymbol{u}_T\|^2_{L^2(\Sigma)^3} \\
&\geq \min\left\{\frac{\sigma_-}{2\mu_+\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}}, \frac{1}{2}\left(\frac{\omega^4\sigma_-^2\varepsilon_+^2+\omega^2\sigma_-^4}{2\omega^2\varepsilon_+^2+\sigma_-^2}\right)^{\frac{1}{2}}, \frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\right\}\|\boldsymbol{u}\|^2_X.
\end{aligned}$$

For the right-hand side, standard arguments yield for all $\boldsymbol{v} \in X$ the estimate $|f(\boldsymbol{v};\,\omega)| \leq \left(\omega\sqrt{\varepsilon_0}\|\boldsymbol{J_a}\|_{L^2(\Omega)^3} + \mu_0^{-1}\|\boldsymbol{g}\|_{L^2(\Sigma)^3}\right)\|\boldsymbol{v}\|_X$, which proves the claim. □

3. Discontinous Galerkin approximation

In this section we introduce a discontinuous Galerkin (dG) formulation of the impedance boundary value problem (2.1a)–(2.1c) in order to approximate the solution to (2.5). For this purpose, we adapt an interior penalty numerical flux in [21] (there for the special case of a perfectly conducting boundary, $\boldsymbol{n}\times\boldsymbol{E} = 0$ on all of $\partial\Omega$, and constant material parameters $\mu \equiv \mu_0$, $\varepsilon \equiv \varepsilon_0$ and $\sigma \equiv 0$).

3.1. **Interior penalty dG formulation.**

The derivation of the dG formulation follows the ideas of [4], where a general dG approach for elliptic problems using different numerical fluxes is described. Instead of the Laplace operator we have to investigate the curl-curl operator.

First, we introduce the auxiliary function $\boldsymbol{q} \in L^2(\Omega;\mathbb{C}^3)$ satisfying $\mu\boldsymbol{q} = \nabla\times\boldsymbol{E}$ a.e. in Ω, so that instead of (2.1a)–(2.1c) we can consider the first-order system

$$\nabla\times\boldsymbol{q} - (\omega^2\varepsilon + i\omega\sigma)\boldsymbol{E} = i\omega\sqrt{\varepsilon_0}\boldsymbol{J_a} \quad \text{in } \Omega, \tag{3.1a}$$

$$\mu\boldsymbol{q} = \nabla\times\boldsymbol{E} \quad \text{in } \Omega, \tag{3.1b}$$

$$\boldsymbol{n}\times\boldsymbol{E} = 0 \quad \text{on } \Gamma_1,\ldots,\Gamma_{M-1}, \tag{3.1c}$$

$$\boldsymbol{q} \times \boldsymbol{n} - i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\boldsymbol{E}_T = \mu_0^{-1}\boldsymbol{g} \qquad \text{on } \Sigma. \tag{3.1d}$$

We will follow the standard discontinuous Galerkin approach now:

1. Partition of the polyhedron Ω into a finite set of elements.
2. Multiply (3.1a) and (3.1b) with test functions, integrate over Ω and use integration by parts on each element.
3. In the integrals over the elemental boundaries replace $\boldsymbol{q}$ and $\boldsymbol{E}$ by their numerical fluxes $\boldsymbol{q}_h^*$ and $\boldsymbol{E}_h^*$.
4. Again, integrate (3.1b) by parts on each element.

For this purpose, let $\mathcal{T}_h$ be a shape-regular mesh of tetrahedra covering the polyhedral domain Ω. For each element $T \in \mathcal{T}_h$ we define h_T as the diameter of the smallest sphere containing T, and for $\mathcal{T}_h$ we define the mesh size as $h := \max_{T\in\mathcal{T}_h} h_T$. Furthermore, let $\mathcal{F}_h$ denote the set of all faces in $\mathcal{T}_h$, $\mathcal{F}_h^I$ the set of all *interior* faces,

$$\mathcal{F}_h^I := \mathcal{F}_h \cap \Omega, \quad \text{and} \quad \mathcal{F}_h^B := \mathcal{F}_h \cap \partial\Omega$$

the set of all faces in the mesh on the *boundary* $\partial\Omega$. We partition $\mathcal{F}_h^B$ in accordance to the two different boundary conditions (3.1c) and (3.1d),

$$\mathcal{F}_h^\Gamma := \mathcal{F}_h \cap (\Gamma_1 \cup \ldots \cup \Gamma_{M-1}), \quad \mathcal{F}_h^\Sigma := \mathcal{F}_h \cap \Sigma.$$

The size of each face $F \in \mathcal{F}_h$ is measured by the diameter h_F of the smallest circle containing F. In this context, we additionally define the function $\mathtt{h}$ by

$$\mathtt{h} : \bigcup_{F\in\mathcal{F}_h} F \to \mathbb{R}_{>0}, \qquad \mathtt{h}(\boldsymbol{x}) := \sum_{F\in\mathcal{F}_h} h_F\, \chi_F(\boldsymbol{x}). \tag{3.2}$$

For the dG formulation we will need the following definitions of the tangential jump and average across an interface $F \in \mathcal{F}_h$ between two tetrahedra $T^L \neq T^R$, which are well-defined for functions $\boldsymbol{u} \in \{\boldsymbol{f} \in L^2(\Omega;\mathbb{C}^3) \,|\, \boldsymbol{f}|_T \in C^0(T;\mathbb{C}^3)\ \forall T \in \mathcal{T}_h\}$,

$$[\![\boldsymbol{u}]\!] := \begin{cases} \boldsymbol{n}_{T^L} \times \boldsymbol{u}|_{T^L} + \boldsymbol{n}_{T^R} \times \boldsymbol{u}|_{T^R} & \text{on } F \in \mathcal{F}_h^I,\ F \subset \partial T^L \cap \partial T^R, \\ \boldsymbol{n} \times \boldsymbol{u} & \text{on } F \in \mathcal{F}_h^B, \end{cases}$$

$$\{\!\{\boldsymbol{u}\}\!\} := \begin{cases} \frac{1}{2}\left(\boldsymbol{u}|_{T^L} + \boldsymbol{u}|_{T^R}\right) & \text{on } F \in \mathcal{F}_h^I,\ F \subset \partial T^L \cap \partial T^R, \\ \boldsymbol{u} & \text{on } F \in \mathcal{F}_h^B. \end{cases}$$

Here, $\boldsymbol{n}_{T^L}$ denotes the outward normal of the tetrahedron T^L, $\boldsymbol{n}_{T^R}$ the one of T^R. In this context, we recall the so-called "dG magic formula", cf. [20, eq. (3.1)],

$$\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{n}_T \times \boldsymbol{u}, \boldsymbol{v}\rangle_{\partial T} = \sum_{F\in\mathcal{F}_h^I} \langle [\![\boldsymbol{u}]\!], \{\!\{\boldsymbol{v}\}\!\}\rangle_F - \sum_{F\in\mathcal{F}_h^I} \langle \{\!\{\boldsymbol{u}\}\!\}, [\![\boldsymbol{v}]\!]\rangle_F + \sum_{F\in\mathcal{F}_h^B} \langle \boldsymbol{n} \times \boldsymbol{u}, \boldsymbol{v}\rangle_F \tag{3.3}$$

for $\boldsymbol{u}$, $\boldsymbol{v} \in \{\boldsymbol{f} \in L^2(\Omega;\mathbb{C}^3) \,|\, \boldsymbol{f}|_T \in C^0(T,\mathbb{C}^3)\ \forall T \in \mathcal{T}_h\}$ and abbreviate:

$$\langle\cdot,\cdot\rangle_{\mathcal{F}_h} := \sum_{F\in\mathcal{F}_h} \langle\cdot,\cdot\rangle_F, \quad \langle\cdot,\cdot\rangle_{\mathcal{F}_h^B} := \sum_{F\in\mathcal{F}_h^B} \langle\cdot,\cdot\rangle_F, \quad \langle\cdot,\cdot\rangle_{\mathcal{F}_h^I} := \sum_{F\in\mathcal{F}_h^I} \langle\cdot,\cdot\rangle_F,$$

$$\langle\cdot,\cdot\rangle_{\mathcal{F}_h^\Sigma} := \sum_{F\in\mathcal{F}_h^\Sigma} \langle\cdot,\cdot\rangle_F, \quad \langle\cdot,\cdot\rangle_{\mathcal{F}_h^\Gamma} := \sum_{F\in\mathcal{F}_h^\Gamma} \langle\cdot,\cdot\rangle_F, \quad \langle\cdot,\cdot\rangle_{\mathcal{F}_h^{I\cup\Gamma}} := \sum_{F\in\mathcal{F}_h^I\cup\mathcal{F}_h^\Gamma} \langle\cdot,\cdot\rangle_F.$$

Following the steps mentioned above, we multiply (3.1a) with the complex conjugate of a test function $\boldsymbol{v}_h$ in the finite dimensional space

$$V_h := \left\{ \boldsymbol{\psi} \in L^2(\Omega;\mathbb{C}^3) \,\middle|\, \boldsymbol{\psi}|_T \in \mathcal{P}^1(T;\mathbb{C})^3 \ \forall T \in \mathcal{T}_h \right\}, \tag{3.4}$$

where $\mathcal{P}^1(T;\mathbb{C})$ denotes the space of affine-linear complex-valued functions on T. Assuming that $\boldsymbol{q}_h$ and $\boldsymbol{E}_h$ are also in V_h, integrating over Ω leads to

$$(\nabla_h \times \boldsymbol{q}_h, \boldsymbol{v}_h)_\Omega - \omega^2(\varepsilon \boldsymbol{E}_h, \boldsymbol{v}_h)_\Omega - i\omega(\sigma \boldsymbol{E}_h, \boldsymbol{v}_h)_\Omega = i\omega\sqrt{\varepsilon_0}(\boldsymbol{J_a}, \boldsymbol{v}_h)_\Omega. \tag{3.5}$$

Here, $\nabla_h \times$ denotes the elementwise curl operator, i.e., $(\nabla_h \times \boldsymbol{q}_h, \boldsymbol{v}_h)_\Omega := \sum_{T\in\mathcal{T}_h} (\nabla \times \boldsymbol{q}_h, \boldsymbol{v}_h)_T$, which is well-defined for functions in V_h. By using integration by parts on each element $T \in \mathcal{T}_h$, substituting $\boldsymbol{q}_h$ with its numerical flux $\boldsymbol{q}_h^*$ in the integrals over the elemental boundaries and applying the dG formula (3.3), we obtain

$$\begin{aligned}
(\nabla_h \times \boldsymbol{q}_h, \boldsymbol{v}_h)_\Omega &= \sum_{T\in\mathcal{T}_h} (\nabla \times \boldsymbol{q}_h, \boldsymbol{v}_h)_T = \sum_{T\in\mathcal{T}_h} \left[(\boldsymbol{q}_h, \nabla \times \boldsymbol{v}_h)_T + \langle \boldsymbol{n}_T \times \boldsymbol{q}_h, \boldsymbol{v}_h \rangle_{\partial T} \right] \\
&= (\boldsymbol{q}_h, \nabla_h \times \boldsymbol{v}_h)_\Omega + \sum_{T\in\mathcal{T}_h} \langle \boldsymbol{n}_T \times \boldsymbol{q}_h^*, \boldsymbol{v}_h \rangle_{\partial T} \\
&= (\boldsymbol{q}_h, \nabla_h \times \boldsymbol{v}_h)_\Omega + \sum_{F\in\mathcal{F}_h^I} \left[\langle [\![\boldsymbol{q}_h^*]\!], \{\!\{\boldsymbol{v}_h\}\!\} \rangle_F - \langle \{\!\{\boldsymbol{q}_h^*\}\!\}, [\![\boldsymbol{v}_h]\!] \rangle_F \right] + \sum_{F\in\mathcal{F}_h^B} \langle \boldsymbol{n} \times \boldsymbol{q}_h^*, \boldsymbol{v}_h \rangle_F.
\end{aligned} \tag{3.6}$$

Now we adopt a similar procedure to the second equation (3.1b). Therefore, let $\boldsymbol{\phi}_h \in \left\{ \boldsymbol{f} \in L^2(\Omega;\mathbb{C}^3) \,\middle|\, \boldsymbol{f}|_T \in C^0(T;\mathbb{C}^3) \cap H^1(T,\mathbb{C}^3) \ \forall T \in \mathcal{T}_h \right\}$. Then,

$$\begin{aligned}
(\mu \boldsymbol{q}_h, \boldsymbol{\phi}_h)_\Omega &= (\nabla_h \times \boldsymbol{E}_h, \boldsymbol{\phi}_h)_\Omega = \sum_{T\in\mathcal{T}_h} (\nabla \times \boldsymbol{E}_h, \boldsymbol{\phi}_h)_T \\
&= \sum_{T\in\mathcal{T}_h} \left[(\boldsymbol{E}_h, \nabla \times \boldsymbol{\phi}_h)_T + \langle \boldsymbol{n}_T \times \boldsymbol{E}_h, \boldsymbol{\phi}_h \rangle_{\partial T} \right] \\
&= \sum_{T\in\mathcal{T}_h} \left[(\boldsymbol{E}_h, \nabla \times \boldsymbol{\phi}_h)_T + \langle \boldsymbol{n}_T \times \boldsymbol{E}_h^*, \boldsymbol{\phi}_h \rangle_{\partial T} \right] \\
&= \sum_{T\in\mathcal{T}_h} \left[(\nabla \times \boldsymbol{E}_h, \boldsymbol{\phi}_h)_T + \langle \boldsymbol{n}_T \times (\boldsymbol{E}_h^* - \boldsymbol{E}_h), \boldsymbol{\phi}_h \rangle_{\partial T} \right] \\
&= (\nabla_h \times \boldsymbol{E}_h, \boldsymbol{\phi}_h)_\Omega + \sum_{F\in\mathcal{F}_h^I} \left[\langle [\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!], \{\!\{\boldsymbol{\phi}_h\}\!\} \rangle_F - \langle \{\!\{\boldsymbol{E}_h^* - \boldsymbol{E}_h\}\!\}, [\![\boldsymbol{\phi}_h]\!] \rangle_F \right] \\
&\qquad + \sum_{F\in\mathcal{F}_h^B} \langle \boldsymbol{n} \times (\boldsymbol{E}_h^* - \boldsymbol{E}_h), \boldsymbol{\phi}_h \rangle_F.
\end{aligned} \tag{3.7}$$

The Sobolev embedding $\mu^{-1} \in W^{1,\infty}(\Omega;\mathbb{R}) \hookrightarrow C^{0,1}(\bar{\Omega};\mathbb{R}) \subset C^0(\bar{\Omega};\mathbb{R})$, cf. [1, Lem. 4.28], shows that $\mu^{-1}\boldsymbol{\psi}_h \in C^0(T;\mathbb{C}^3) \cap H^1(T;\mathbb{C}^3)$ for each element $T \in \mathcal{T}_h$ and every function $\boldsymbol{\psi}_h \in V_h$. Hence, the following expressions are all well-defined,

$$\begin{aligned}
(\boldsymbol{q}_h, \boldsymbol{\psi}_h)_\Omega &= (\mu \boldsymbol{q}_h, \mu^{-1}\boldsymbol{\psi}_h)_\Omega \\
&= (\nabla_h \times \boldsymbol{E}_h, \mu^{-1}\boldsymbol{\psi}_h)_\Omega + \sum_{F\in\mathcal{F}_h^I} \langle [\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!], \{\!\{\mu^{-1}\boldsymbol{\psi}_h\}\!\} \rangle_F \\
&\qquad - \sum_{F\in\mathcal{F}_h^I} \langle \{\!\{\boldsymbol{E}_h^* - \boldsymbol{E}_h\}\!\}, [\![\mu^{-1}\boldsymbol{\psi}_h]\!] \rangle_F + \sum_{F\in\mathcal{F}_h^B} \langle \boldsymbol{n} \times (\boldsymbol{E}_h^* - \boldsymbol{E}_h), \mu^{-1}\boldsymbol{\psi}_h \rangle_F,
\end{aligned}$$

where we used (3.7). Inserting this into (3.6) and the result into (3.5) yields

$$
\begin{aligned}
i\omega\sqrt{\varepsilon_0}(\boldsymbol{J_a}, \boldsymbol{v}_h)_\Omega &= (\mu^{-1}\nabla_h \times \boldsymbol{E}_h, \nabla_h \times \boldsymbol{v}_h)_\Omega - \omega^2(\varepsilon\boldsymbol{E}_h, \boldsymbol{v}_h)_\Omega - i\omega(\sigma\boldsymbol{E}_h, \boldsymbol{v}_h)_\Omega \\
&\quad + \langle [\![\boldsymbol{E}_h^* - \boldsymbol{E}_h]\!], \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{v}_h\}\!\}\rangle_{\mathcal{F}_h^I} - \langle \{\!\{\boldsymbol{E}_h^* - \boldsymbol{E}_h\}\!\}, [\![\mu^{-1}\nabla_h \times \boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^I} \\
&\quad + \langle \boldsymbol{n} \times (\boldsymbol{E}_h^* - \boldsymbol{E}_h), \mu^{-1}\nabla_h \times \boldsymbol{v}_h\rangle_{\mathcal{F}_h^B} \\
&\quad + \langle [\![\boldsymbol{q}_h^*]\!], \{\!\{\boldsymbol{v}_h\}\!\}\rangle_{\mathcal{F}_h^I} - \langle \{\!\{\boldsymbol{q}_h^*\}\!\}, [\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^I} - \langle \boldsymbol{q}_h^*, \boldsymbol{n} \times \boldsymbol{v}_h\rangle_{\mathcal{F}_h^B}.
\end{aligned}
\tag{3.8}
$$

As fluxes $\boldsymbol{E}_h^*$ and $\boldsymbol{q}_h^*$ we choose interior penalty fluxes similar to the ones in [21]:

$$
\boldsymbol{E}_h^* := \begin{cases} \{\!\{\boldsymbol{E}_h\}\!\} & \text{on } F \in \mathcal{F}_h^I, \\ 0 & \text{on } F \in \mathcal{F}_h^\Gamma, \\ \boldsymbol{E}_h & \text{on } F \in \mathcal{F}_h^\Sigma, \end{cases} \quad \boldsymbol{q}_h^* := \begin{cases} \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{E}_h\}\!\} - \tau\, h_F^{-1}[\![\boldsymbol{E}_h]\!] & \text{on } F \in \mathcal{F}_h^I, \\ \mu^{-1}\nabla_h \times \boldsymbol{E}_h - \tau\, h_F^{-1}(\boldsymbol{n} \times \boldsymbol{E}_h) & \text{on } F \in \mathcal{F}_h^\Gamma, \end{cases}
$$

where $\tau > 0$ is a constant penalty parameter. From the tangential component of the flux $\boldsymbol{q}_h^*$ on Σ we require $\boldsymbol{n} \times \boldsymbol{q}_h^* := -\mu_0^{-1}\boldsymbol{g} - i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}(\boldsymbol{E}_h)_T$. Inserting these fluxes into (3.8), using the definition (2.8) of f and observing that $[\![\{\!\{\boldsymbol{u}\}\!\}]\!] = 0$, $[\![[\![\boldsymbol{u}]\!]]\!] = 0$, $\{\!\{\{\!\{\boldsymbol{u}\}\!\}\}\!\} = \{\!\{\boldsymbol{u}\}\!\}$, $\{\!\{[\![\boldsymbol{u}]\!]\}\!\} = [\![\boldsymbol{u}]\!]$, leads to the following equation

$$
\begin{aligned}
f(\boldsymbol{v}_h; \omega) &= (\mu^{-1}\nabla_h \times \boldsymbol{E}_h, \nabla_h \times \boldsymbol{v}_h)_\Omega - \omega^2(\varepsilon\boldsymbol{E}_h, \boldsymbol{v}_h)_\Omega - i\omega(\sigma\boldsymbol{E}_h, \boldsymbol{v}_h)_\Omega \\
&\quad - \langle [\![\boldsymbol{E}_h]\!], \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{v}_h\}\!\}\rangle_{\mathcal{F}_h^{I\cup\Gamma}} - \langle \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{E}_h\}\!\}, [\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}} \\
&\quad + \langle \tau \mathtt{h}^{-1}[\![\boldsymbol{E}_h]\!], [\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}} - i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\langle(\boldsymbol{E}_h)_T, (\boldsymbol{v}_h)_T\rangle_\Sigma \\
&=: a_h(\boldsymbol{E}_h, \boldsymbol{v}_h; \omega, \tau).
\end{aligned}
\tag{3.9}
$$

Then for given ω, $\tau > 0$ the discrete problem reads: Find $\boldsymbol{E}_h \in V_h$ such that

$$
a_h(\boldsymbol{E}_h, \boldsymbol{v}_h; \omega, \tau) = f(\boldsymbol{v}_h; \omega) \qquad \forall \boldsymbol{v}_h \in V_h. \tag{3.10}
$$

Remark 3.1. *Since $V_h \not\subset H(\mathrm{curl}, \Omega)$ and since we use a sesquilinear form a_h different from a_e in* (2.7)*, existence and uniqueness of a solution $\boldsymbol{E}_h$ to* (3.10) *are not obvious. We will return to this later on.*

Theorem 3.2 (Consistency)**.** *The formulation* (3.10) *is consistent, i.e., if $\boldsymbol{E}$ is the analytical solution to* (2.1a)–(2.1c)*, then $\boldsymbol{E}$ also satisfies $a_h(\boldsymbol{E}, \boldsymbol{v}_h; \omega, \tau) = f(\boldsymbol{v}_h; \omega)$ for all $\boldsymbol{v}_h \in V_h$.*

Proof. The proof follows the standard method to show consistency of discrete dG variational formulations that is mentioned, e.g., in [30] for other fluxes. Since, however, we were not able to find a proof in the literature for the fluxes chosen above, we state it here completely.

If $\boldsymbol{E}$ is a solution of (2.1a)–(2.1c), then the following tangential jumps vanish, $[\![\mu^{-1}\nabla \times \boldsymbol{E}]\!] = 0$ on $\mathcal{F}_h^I$, $[\![\boldsymbol{E}]\!] = 0$ on $\mathcal{F}_h^I \cup \mathcal{F}_h^\Gamma$, since $\boldsymbol{E}$ and $\mu^{-1}\nabla \times \boldsymbol{E}$ are functions in $H(\mathrm{curl}, \Omega)$ and $\boldsymbol{n} \times \boldsymbol{E} = 0$ on $\Gamma_1, \ldots, \Gamma_{M-1}$. This fact together with the identity

$$
\begin{aligned}
(\mu^{-1}\nabla \times \boldsymbol{E}, \nabla_h \times \boldsymbol{v}_h)_\Omega &= (\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}), \boldsymbol{v}_h)_\Omega - \langle [\![\mu^{-1}\nabla \times \boldsymbol{E}]\!], \{\!\{\boldsymbol{v}_h\}\!\}\rangle_{\mathcal{F}_h^I} \\
&\quad + \langle \{\!\{\mu^{-1}\nabla \times \boldsymbol{E}\}\!\}, [\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^I} - \langle \boldsymbol{n} \times (\mu^{-1}\nabla \times \boldsymbol{E}), \boldsymbol{v}_h\rangle_{\mathcal{F}_h^B}
\end{aligned}
$$

leads to the desired consistency:

$$
\begin{aligned}
&a_h(\boldsymbol{E}, \boldsymbol{v}_h; \omega, \tau) = (\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}), \boldsymbol{v}_h)_\Omega - \overbrace{\langle [\![\mu^{-1}\nabla \times \boldsymbol{E}]\!], \{\!\{\boldsymbol{v}_h\}\!\}\rangle_{\mathcal{F}_h^I}}^{=0} \\
&+ \langle \{\!\{\mu^{-1}\nabla \times \boldsymbol{E}\}\!\}, [\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^I} - \langle \boldsymbol{n} \times (\mu^{-1}\nabla \times \boldsymbol{E}), \boldsymbol{v}_h\rangle_{\mathcal{F}_h^\Gamma} - \langle \boldsymbol{n} \times (\mu^{-1}\nabla \times \boldsymbol{E}), \boldsymbol{v}_h\rangle_{\mathcal{F}_h^\Sigma} \\
&- ((\omega^2\varepsilon + i\omega\sigma)\boldsymbol{E}, \boldsymbol{v}_h)_\Omega - i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\langle \boldsymbol{E}_T, (\boldsymbol{v}_h)_T\rangle_\Sigma - \langle \{\!\{\mu^{-1}\nabla \times \boldsymbol{E}\}\!\}, [\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}}
\end{aligned}
$$

$$
\begin{aligned}
&- \underbrace{\langle [\![\boldsymbol{E}]\!], \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{v}_h\}\!\}\rangle_{\mathcal{F}_h^{I\cup\Gamma}}}_{=0} + \underbrace{\langle \tau \mathtt{h}^{-1}[\![\boldsymbol{E}]\!], [\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}}}_{=0} \\
&= (\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}) - (\omega^2\varepsilon + i\omega\sigma)\boldsymbol{E}, \boldsymbol{v}_h)_\Omega - \langle \boldsymbol{n} \times (\mu^{-1}\nabla \times \boldsymbol{E}), (\boldsymbol{v}_h)_T\rangle_\Sigma \\
&\qquad\qquad\qquad - i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\langle \boldsymbol{E}_T, (\boldsymbol{v}_h)_T\rangle_\Sigma \\
&= i\omega\sqrt{\varepsilon_0}(\boldsymbol{J_a}, \boldsymbol{v}_h)_\Omega + \mu_0^{-1}\langle \boldsymbol{g}, (\boldsymbol{v}_h)_T\rangle_\Sigma = f(\boldsymbol{v}_h; \omega)
\end{aligned}
$$

for all $\boldsymbol{v}_h \in V_h$, since $\langle \boldsymbol{n} \times (\mu^{-1}\nabla \times \boldsymbol{E}), \boldsymbol{v}_h\rangle_{\mathcal{F}_h^\Sigma} = \langle \boldsymbol{n} \times (\mu^{-1}\nabla \times \boldsymbol{E}), (\boldsymbol{v}_h)_T\rangle_\Sigma$. □

3.2. **Continuity and coercivity.** The next step is to show coercivity of a_h on V_h and boundedness of an extension $\widetilde{a}_h$ to a vector space containing both, V_h and X, with respect to an energy norm on this space. Later on, these results will be the basis for an a priori error analysis. First, let us define the space $\widetilde{V}_h$ that relates the spaces V_h in (3.4) and X in (2.6):

$$\widetilde{V}_h := V_h + X = \left\{\boldsymbol{v} \in L^2(\Omega;\mathbb{C}^3) \,\middle|\, \exists \boldsymbol{w}_h \in V_h,\, \exists \boldsymbol{u} \in X : \boldsymbol{v} = \boldsymbol{w}_h + \boldsymbol{u}\right\}.$$

Note that the sum of the two vector spaces V_h and X is not direct since $\{0\} \neq V_h \cap X$. On $\widetilde{V}_h$ we introduce the following dG-norm:

$$\|\boldsymbol{v}\|_{\mathrm{DG}}^2 := \|\boldsymbol{v}\|_{L^2(\Omega)^3}^2 + \|\nabla_h \times \boldsymbol{v}\|_{L^2(\Omega)^3}^2 + \|\boldsymbol{v}_T\|_{L^2(\Sigma)^3}^2 + \|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}]\!]\|_{\mathcal{F}_h^{I\cup\Gamma}}^2 \tag{3.11}$$

for all $\boldsymbol{v} \in \widetilde{V}_h$, where $\mathtt{h}$ is the function which has been defined in (3.2) and

$$\|\boldsymbol{w}\|_{\mathcal{F}_h^{I\cup\Gamma}}^2 := \sum_{F \in \mathcal{F}_h^I \cup \mathcal{F}_h^\Gamma} \|\boldsymbol{w}\|_{L^2(F)^3}^2, \quad \boldsymbol{w} \in L^2\left(\mathcal{F}_h^I \cup \mathcal{F}_h^\Gamma; \mathbb{C}^3\right). \tag{3.12}$$

The following *inverse inequality* will be essential for the analysis of the dG scheme.

Lemma 3.3 (Inverse inequality)**.** *There exists a constant $C_{\mathrm{inv}} > 0$, independent of the mesh size h, such that for every $\boldsymbol{v}_h \in V_h$*

$$h_T \|\boldsymbol{v}_h\|_{L^2(\partial T)^3}^2 \leq C_{\mathrm{inv}} \|\boldsymbol{v}_h\|_{L^2(T)^3}^2 \quad \forall T \in \mathcal{T}_h. \tag{3.13}$$

Proof. According to [31, Theorem 4] with polynomial degree $p = 1$, for $\boldsymbol{v}_h \in V_h$ and $T \in \mathcal{T}_h$ it holds $\|\boldsymbol{v}_h\|_{L^2(\partial T)^3}^2 \leq \frac{8}{3}\frac{\text{surface area}(T)}{\text{volume}(T)}\|\boldsymbol{v}_h\|_{L^2(T)^3}^2$. This implies (3.13) since on shape-regular meshes, there exists a constant $C > 0$, independent of the element T, such that $\frac{\text{surface area}(T)}{\text{volume}(T)} \leq C\, h_T^{-1}$ for all $T \in \mathcal{T}_h$. □

Lemma 3.4. *For every $\boldsymbol{v}_h \in V_h$ it holds*

$$\|\mathtt{h}^{\frac{1}{2}}\{\!\{\boldsymbol{v}_h\}\!\}\|_{\mathcal{F}_h^{I\cup\Gamma}}^2 \leq C_{\mathrm{inv}} \|\boldsymbol{v}_h\|_{L^2(\Omega)^3}^2, \tag{3.14}$$

with $\|\cdot\|_{\mathcal{F}_h^{I\cup\Gamma}}$ as in (3.12) *and the constant C_{inv} in* (3.13) *of Lemma 3.3.*

Proof. Let $\boldsymbol{v}_h \in V_h$. Then we can estimate as follows

$$
\begin{aligned}
\|\mathtt{h}^{\frac{1}{2}}\{\!\{\boldsymbol{v}_h\}\!\}\|_{\mathcal{F}_h^{I\cup\Gamma}}^2 &= \sum_{F \in \mathcal{F}_h^I} \frac{h_F}{4}\left\|\boldsymbol{v}_h|_{T^L} + \boldsymbol{v}_h|_{T^R}\right\|_{L^2(F)^3}^2 + \sum_{F \in \mathcal{F}_h^\Gamma} h_F\|\boldsymbol{v}_h\|_{L^2(F)^3}^2 \\
&\leq \frac{1}{4}\sum_{F \in \mathcal{F}_h^I} h_F\left(\|\boldsymbol{v}_h|_{T^L}\|_{L^2(F)^3} + \|\boldsymbol{v}_h|_{T^R}\|_{L^2(F)^3}\right)^2 + \sum_{F \in \mathcal{F}_h^\Gamma} h_F\|\boldsymbol{v}_h\|_{L^2(F)^3}^2 \\
&\leq \frac{1}{2}\sum_{F \in \mathcal{F}_h^I} h_F\left(\|\boldsymbol{v}_h|_{T^L}\|_{L^2(F)^3}^2 + \|\boldsymbol{v}_h|_{T^R}\|_{L^2(F)^3}^2\right) + \sum_{F \in \mathcal{F}_h^\Gamma} h_F\|\boldsymbol{v}_h\|_{L^2(F)^3}^2
\end{aligned}
$$

$$\leq \frac{1}{2} \sum_{F \in \mathcal{F}_h^I} \Big(h_{T^L} \|\boldsymbol{v}_h|_{T^L}\|^2_{L^2(F)^3} + h_{T^R} \|\boldsymbol{v}_h|_{T^R}\|^2_{L^2(F)^3} \Big) + \sum_{F \in \mathcal{F}_h^B} h_F \|\boldsymbol{v}_h\|^2_{L^2(F)^3}$$
$$\leq \sum_{T \in \mathcal{T}_h} \sum_{\substack{F \in \mathcal{F}_h \\ F \in \partial T}} h_T \|\boldsymbol{v}_h|_T\|^2_{L^2(F)^3} = \sum_{T \in \mathcal{T}_h} h_T \|\boldsymbol{v}_h|_T\|^2_{L^2(\partial T)^3} \leq C_{\mathrm{inv}} \sum_{T \in \mathcal{T}_h} \|\boldsymbol{v}_h|_T\|^2_{L^2(T)^3}$$

where we used the inverse inequality (3.13) in the last step. This proves the lemma since the last term equals $C_{\mathrm{inv}} \|\boldsymbol{v}_h\|^2_{L^2(\Omega)^3}$. □

In order to extend a_h to $\widetilde{V}_h \times \widetilde{V}_h$, we need the following definition, [27, Sect. 3.5].

Definition 3.5. *Let Assumption 2.1 a) on μ be satisfied. For $\boldsymbol{u} \in \widetilde{V}_h$ we define the* lifting operator $\mathcal{L}_\mu(\boldsymbol{u}) \in V_h$ *via*

$$(\mathcal{L}_\mu(\boldsymbol{u}), \mu^{-1}\boldsymbol{v}_h)_\Omega = \langle [\![\boldsymbol{u}]\!], \{\!\{\mu^{-1}\boldsymbol{v}_h\}\!\}\rangle_{\mathcal{F}_h^I \cup \Gamma} \quad \forall \boldsymbol{v}_h \in V_h. \tag{3.15}$$

Remark 3.6. *For $\boldsymbol{u} \in \widetilde{V}_h$ existence and uniqueness of $\mathcal{L}_\mu(\boldsymbol{u}) \in V_h$ satisfying* (3.15) *follow from the complex-valued Riesz representation theorem, cf.* [24, Thm. 2.30]*: For μ as in Assumption 2.1 a) the form $(\cdot, \mu^{-1}\cdot)_\Omega = (\mu^{-1}\cdot, \cdot)_\Omega$ is an inner product on $L^2(\Omega; \mathbb{C}^3)$. $V_h \subset L^2(\Omega; \mathbb{C}^3)$ is a closed subspace and for every $\boldsymbol{u} \in \widetilde{V}_h$, the mapping $\ell^\mu_{\boldsymbol{u}}(\boldsymbol{v}_h) := \langle \{\!\{\mu^{-1}\boldsymbol{v}_h\}\!\}, [\![\boldsymbol{u}]\!]\rangle_{\mathcal{F}_h^I \cup \Gamma}$, $\boldsymbol{v}_h \in V_h$, is a bounded linear functional on V_h: Indeed, inequality* (3.14) *from above yields*

$$|\ell^\mu_{\boldsymbol{u}}(\boldsymbol{v}_h)| \leq \|\mathtt{h}^{\frac{1}{2}} \{\!\{\mu^{-1}\boldsymbol{v}_h\}\!\}\|_{\mathcal{F}_h^I \cup \Gamma} \|\mathtt{h}^{-\frac{1}{2}} [\![\boldsymbol{u}]\!]\|_{\mathcal{F}_h^I \cup \Gamma} \leq \mu_-^{-1} \|\mathtt{h}^{\frac{1}{2}} \{\!\{\boldsymbol{v}_h\}\!\}\|_{\mathcal{F}_h^I \cup \Gamma} \|\boldsymbol{u}\|_{\mathrm{DG}}$$
$$\leq \Big(\mu_-^{-1} \sqrt{C_{\mathrm{inv}}}\, \|\boldsymbol{u}\|_{\mathrm{DG}} \Big) \|\boldsymbol{v}_h\|_{L^2(\Omega)^3}.$$

Now we introduce the extended form $\widetilde{a}_h(\cdot, \cdot; \omega, \tau) : \widetilde{V}_h \times \widetilde{V}_h \to \mathbb{C}$ for ω, $\tau > 0$ as

$$\begin{aligned} \widetilde{a}_h(\boldsymbol{u}, \boldsymbol{v}; \omega, \tau) := {} & (\mu^{-1}\nabla_h \times \boldsymbol{u}, \nabla_h \times \boldsymbol{v})_\Omega - \omega^2(\varepsilon \boldsymbol{u}, \boldsymbol{v})_\Omega - i\omega(\sigma \boldsymbol{u}, \boldsymbol{v})_\Omega \\ & - (\mathcal{L}_\mu(\boldsymbol{u}), \mu^{-1}\nabla_h \times \boldsymbol{v})_\Omega - (\mu^{-1}\nabla_h \times \boldsymbol{u}, \mathcal{L}_\mu(\boldsymbol{v}))_\Omega \\ & + \langle \tau \mathtt{h}^{-1} [\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]\rangle_{\mathcal{F}_h^I \cup \Gamma} - i\omega\lambda \sqrt{\varepsilon_0 \mu_0^{-1}} \langle \boldsymbol{u}_T, \boldsymbol{v}_T \rangle_\Sigma. \end{aligned} \tag{3.16}$$

Remark 3.7. *Note that on $V_h \times V_h$ the sesquilinear form $\widetilde{a}_h(\cdot, \cdot; \omega, \tau)$ in* (3.16) *equals $a_h(\cdot, \cdot; \omega, \tau)$ in* (3.9)*.*

Theorem 3.8 (Continuity)**.** *Let Assumption 2.1 be satisfied. Then for all ω, $\tau > 0$ there is a constant $\gamma = \gamma(\mu, \varepsilon, \sigma, \lambda, \omega, \tau) > 0$ with $|\widetilde{a}_h(\boldsymbol{u}, \boldsymbol{v}; \omega, \tau)| \leq \gamma \|\boldsymbol{u}\|_{\mathrm{DG}} \|\boldsymbol{v}\|_{\mathrm{DG}}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \widetilde{V}_h$, i.e., a_h is bounded on $\widetilde{V}_h \times \widetilde{V}_h$ w.r.t. the dG-norm in* (3.11)*.*

To prove Theorem 3.8 we will need the μ^{-1}-orthogonal L^2-projection onto V_h.

Definition 3.9. *Let Assumption 2.1 a) on μ be satisfied. For $\boldsymbol{u} \in L^2(\Omega; \mathbb{C}^3)$ the projection $\Pi_\mu \boldsymbol{u} \in V_h$ is defined as the following Riesz representative:*

$$(\mu^{-1}\Pi_\mu \boldsymbol{u}, \boldsymbol{v}_h)_\Omega = (\mu^{-1}\boldsymbol{u}, \boldsymbol{v}_h)_\Omega \quad \forall \boldsymbol{v}_h \in V_h. \tag{3.17}$$

Note that Π_μ satisfies the stability estimate

$$\|\Pi_\mu \boldsymbol{u}\|_{L^2(\Omega)^3} \leq \frac{\mu_+}{\mu_-} \|\boldsymbol{u}\|_{L^2(\Omega)^3} \quad \forall \boldsymbol{u} \in L^2(\Omega; \mathbb{C}^3), \tag{3.18}$$

since it is readily seen that $\mu_+^{-1} \|\Pi_\mu \boldsymbol{u}\|^2_{L^2(\Omega)^3} \leq (\mu^{-1}\Pi_\mu \boldsymbol{u}, \Pi_\mu \boldsymbol{u})_\Omega = (\mu^{-1}\boldsymbol{u}, \Pi_\mu \boldsymbol{u})_\Omega \leq \mu_-^{-1} \|\boldsymbol{u}\|_{L^2(\Omega)^3} \|\Pi_\mu \boldsymbol{u}\|_{L^2(\Omega)^3}$.

Proof of Theorem 3.8. Fix ω, $\tau > 0$ and let $\boldsymbol{u}$, $\boldsymbol{v} \in \widetilde{V}_h$. The properties (2.2), (2.3), (2.4) of μ, ε and σ lead to

$$
\begin{aligned}
\big|a_h(\boldsymbol{u},\boldsymbol{v};\,\omega,\tau)\big| &= \Big|(\mu^{-1}\nabla_h\times\boldsymbol{u},\nabla_h\times\boldsymbol{v})_\Omega-\omega^2(\varepsilon\boldsymbol{u},\boldsymbol{v})_\Omega-i\omega(\sigma\boldsymbol{u},\boldsymbol{v})_\Omega\\
&\quad-(\mathcal{L}_\mu(\boldsymbol{u}),\mu^{-1}\nabla_h\times\boldsymbol{v})_\Omega-(\mu^{-1}\nabla_h\times\boldsymbol{u},\mathcal{L}_\mu(\boldsymbol{v}))_\Omega\\
&\quad+\langle\tau\mathtt{h}^{-1}[\![\boldsymbol{u}]\!],[\![\boldsymbol{v}]\!]\rangle_{\mathcal{F}_h^I\cup\Gamma}-i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\langle\boldsymbol{u}_T,\boldsymbol{v}_T\rangle_\Sigma\Big|\\
&\le\mu_-^{-1}\|\nabla_h\times\boldsymbol{u}\|_{L^2(\Omega)^3}\|\nabla_h\times\boldsymbol{v}\|_{L^2(\Omega)^3}+\big(\omega^2\varepsilon_++\omega\sigma_+\big)\|\boldsymbol{u}\|_{L^2(\Omega)^3}\|\boldsymbol{v}\|_{L^2(\Omega)^3}\\
&\quad+\|\mu^{-1}\mathcal{L}_\mu(\boldsymbol{u})\|_{L^2(\Omega)^3}\|\nabla_h\times\boldsymbol{v}\|_{L^2(\Omega)^3}+\|\nabla_h\times\boldsymbol{u}\|_{L^2(\Omega)^3}\|\mu^{-1}\mathcal{L}_\mu(\boldsymbol{v})\|_{L^2(\Omega)^3}\\
&\quad+\tau\,\|\mathtt{h}^{-\frac12}[\![\boldsymbol{u}]\!]\|_{\mathcal{F}_h^I\cup\Gamma}\|\mathtt{h}^{-\frac12}[\![\boldsymbol{v}]\!]\|_{\mathcal{F}_h^I\cup\Gamma}+\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\,\|\boldsymbol{u}_T\|_{L^2(\Sigma)^3}\|\boldsymbol{v}_T\|_{L^2(\Sigma)^3}\\
&\le\left(\mu_-^{-1}+\omega^2\varepsilon_++\omega\sigma_++2\mu_+\mu_-^{-2}\sqrt{C_{\mathrm{inv}}}+\tau+\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\right)\|\boldsymbol{u}\|_{\mathrm{DG}}\,\|\boldsymbol{v}\|_{\mathrm{DG}}
\end{aligned}
$$

because we can estimate as follows (since $\mathcal{L}_\mu(\boldsymbol{u})\in V_h$ and $\Pi_\mu\boldsymbol{w}\in V_h$)

$$
\begin{aligned}
\|\mu^{-1}\mathcal{L}_\mu(\boldsymbol{u})\|_{L^2(\Omega)^3}&=\sup_{\boldsymbol{w}\in L^2(\Omega)^3}\frac{(\mu^{-1}\mathcal{L}_\mu(\boldsymbol{u}),\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{L^2(\Omega)^3}}=\sup_{\boldsymbol{w}\in L^2(\Omega)^3}\frac{(\mathcal{L}_\mu(\boldsymbol{u}),\mu^{-1}\Pi_\mu\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{L^2(\Omega)^3}}\\
&=\sup_{\boldsymbol{w}\in L^2(\Omega)^3}\frac{\langle[\![\boldsymbol{u}]\!],\{\!\{\mu^{-1}\Pi_\mu\boldsymbol{w}\}\!\}\rangle_{\mathcal{F}_h^I\cup\Gamma}}{\|\boldsymbol{w}\|_{L^2(\Omega)^3}}\le\|\mathtt{h}^{-\frac12}[\![\boldsymbol{u}]\!]\|_{\mathcal{F}_h^I\cup\Gamma}\sup_{\boldsymbol{w}\in L^2(\Omega)^3}\frac{\|\mathtt{h}^{\frac12}\mu^{-1}\{\!\{\Pi_\mu\boldsymbol{w}\}\!\}\|_{\mathcal{F}_h^I\cup\Gamma}}{\|\boldsymbol{w}\|_{L^2(\Omega)^3}}\\
&\le\mu_-^{-1}\sqrt{C_{\mathrm{inv}}}\,\|\boldsymbol{u}\|_{\mathrm{DG}}\sup_{\boldsymbol{w}\in L^2(\Omega)^3}\frac{\|\Pi_\mu\boldsymbol{w}\|_{L^2(\Omega)^3}}{\|\boldsymbol{w}\|_{L^2(\Omega)^3}}\le\mu_+\mu_-^{-2}\sqrt{C_{\mathrm{inv}}}\|\boldsymbol{u}\|_{\mathrm{DG}}.
\end{aligned}
$$

In this calculation we used the definitions (3.15) and (3.17) of the operators $\mathcal{L}_\mu$ and Π_μ as well as the estimates (3.14) and (3.18). This proves the assertion. □

Theorem 3.10 (Coercivity)**.** *Let Assumption 2.1 be satisfied. Then there exists a penalty parameter $\tau^*>0$ such that for all $\tau>\tau^*$, and $\omega>0$ there is a constant $\alpha=\alpha(\mu,\varepsilon,\sigma,\lambda,\omega,\tau)>0$ with $|a_h(\boldsymbol{v}_h,\boldsymbol{v}_h;\,\omega,\tau)|\ge\alpha\,\|\boldsymbol{v}_h\|_{\mathrm{DG}}^2$ for all $\boldsymbol{v}_h\in V_h$.*

Proof. Fix $\omega>0$ and let $\boldsymbol{v}_h\in V_h$. Then,

$$
\begin{aligned}
\big|a_h(\boldsymbol{v}_h,\boldsymbol{v}_h;\,\omega,\tau)\big|&=\big|(\mu^{-1}\nabla_h\times\boldsymbol{v}_h,\nabla_h\times\boldsymbol{v}_h)_\Omega-\omega^2(\varepsilon\boldsymbol{v}_h,\boldsymbol{v}_h)_\Omega-i\omega(\sigma\boldsymbol{v}_h,\boldsymbol{v}_h)_\Omega\\
&\quad-\langle[\![\boldsymbol{v}_h]\!],\{\!\{\mu^{-1}\nabla_h\times\boldsymbol{v}_h\}\!\}\rangle_{\mathcal{F}_h^I\cup\Gamma}-\langle\{\!\{\mu^{-1}\nabla_h\times\boldsymbol{v}_h\}\!\},[\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^I\cup\Gamma}\\
&\quad+\langle\tau\mathtt{h}^{-1}[\![\boldsymbol{v}_h]\!],[\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^I\cup\Gamma}-i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\,\|(\boldsymbol{v}_h)_T\|_{L^2(\Sigma)^3}^2\big|.
\end{aligned}
$$

First, we use the inverse triangle inequality and Hölder's inequality to obtain

$$
\begin{aligned}
\big|a_h(\boldsymbol{v}_h,\boldsymbol{v}_h;\,\omega,\tau)\big|&\ge\big|(\mu^{-1}\nabla_h\times\boldsymbol{v}_h,\nabla_h\times\boldsymbol{v}_h)_\Omega-\omega^2(\varepsilon\boldsymbol{v}_h,\boldsymbol{v}_h)_\Omega\\
&\quad+\tau\|\mathtt{h}^{-\frac12}[\![\boldsymbol{v}_h]\!]\|_{\mathcal{F}_h^I\cup\Gamma}^2-i\omega(\sigma\boldsymbol{v}_h,\boldsymbol{v}_h)_\Omega-i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\,\|(\boldsymbol{v}_h)_T\|_{L^2(\Sigma)^3}^2\big|\\
&\quad-2\,\|\mathtt{h}^{-\frac12}[\![\boldsymbol{v}_h]\!]\|_{\mathcal{F}_h^I\cup\Gamma}\|\mathtt{h}^{\frac12}\mu^{-1}\{\!\{\nabla_h\times\boldsymbol{v}_h\}\!\}\|_{\mathcal{F}_h^I\cup\Gamma}.
\end{aligned}
$$

The definition of the absolute value of complex numbers yields

$$
\begin{aligned}
&=\Bigg(\Big[(\mu^{-1}\nabla_h\times\boldsymbol{v}_h,\nabla_h\times\boldsymbol{v}_h)_\Omega-\omega^2(\varepsilon\boldsymbol{v}_h,\boldsymbol{v}_h)_\Omega+\tau\|\mathtt{h}^{-\frac12}[\![\boldsymbol{v}_h]\!]\|_{\mathcal{F}_h^I\cup\Gamma}^2\Big]^2\\
&\quad+\Big[\omega^2(\sigma\boldsymbol{v}_h,\boldsymbol{v}_h)_\Omega^2+2\omega^2\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\,(\sigma\boldsymbol{v}_h,\boldsymbol{v}_h)_\Omega\,\|(\boldsymbol{v}_h)_T\|_{L^2(\Sigma)^3}^2
\end{aligned}
$$

$$+ \omega^2\lambda^2\varepsilon_0\mu_0^{-1}\|(\boldsymbol{v}_h)_T\|^4_{L^2(\Sigma)^3}\Big]\Big)^{\frac{1}{2}} - 2\,\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|_{\mathcal{F}_h^{I\cup\Gamma}}\|\mathtt{h}^{\frac{1}{2}}\mu^{-1}\{\!\{\nabla_h\times\boldsymbol{v}_h\}\!\}\|_{\mathcal{F}_h^{I\cup\Gamma}}.$$

Now we use property (2.2) of μ

$$\begin{aligned}
\geq \Bigg(&\Big[(\mu^{-1}\nabla_h\times\boldsymbol{v}_h,\nabla_h\times\boldsymbol{v}_h)_\Omega + \tau\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}}\Big]^2\\
&- 2\omega^2(\varepsilon\boldsymbol{v}_h,\boldsymbol{v}_h)_\Omega\Big[(\mu^{-1}\nabla_h\times\boldsymbol{v}_h,\nabla_h\times\boldsymbol{v}_h)_\Omega + \tau\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}}\Big]\\
&+ \omega^4(\varepsilon\boldsymbol{v}_h,\boldsymbol{v}_h)^2_\Omega + \omega^2(\sigma\boldsymbol{v}_h,\boldsymbol{v}_h)^2_\Omega + \omega^2\lambda^2\varepsilon_0\mu_0^{-1}\|(\boldsymbol{v}_h)_T\|^4_{L^2(\Sigma)^3}\Bigg)^{\frac{1}{2}}\\
&- 2\mu_-^{-1}\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|_{\mathcal{F}_h^{I\cup\Gamma}}\|\mathtt{h}^{\frac{1}{2}}\{\!\{\nabla_h\times\boldsymbol{v}_h\}\!\}\|_{\mathcal{F}_h^{I\cup\Gamma}}
\end{aligned}$$

as well as $\sqrt{x+y}\geq\frac{1}{\sqrt{2}}(\sqrt{x}+\sqrt{y})$ for x, $y\geq 0$ and we apply Young's inequality twice, for $a\in(0,1)$, $\delta>0$

$$\begin{aligned}
\geq \frac{1}{\sqrt{2}}\Bigg(&(1-a)\Big[(\mu^{-1}\nabla_h\times\boldsymbol{v}_h,\nabla_h\times\boldsymbol{v}_h)_\Omega + \tau\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}}\Big]^2\\
&- \big(a^{-1}-1\big)\,\omega^4(\varepsilon\boldsymbol{v}_h,\boldsymbol{v}_h)^2_\Omega + \omega^2(\sigma\boldsymbol{v}_h,\boldsymbol{v}_h)^2_\Omega\Bigg)^{\frac{1}{2}} + \frac{\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}}{\sqrt{2}}\,\|(\boldsymbol{v}_h)_T\|^2_{L^2(\Sigma)^3}\\
&- \delta^{-1}\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}} - \delta\mu_-^{-2}\|\mathtt{h}^{\frac{1}{2}}\{\!\{\nabla_h\times\boldsymbol{v}_h\}\!\}\|^2_{\mathcal{F}_h^{I\cup\Gamma}}.
\end{aligned}$$

We note that $\big(a^{-1}-1\big)>0$ for $a\in(0,1)$ and use the properties (2.3) and (2.4) of ε and σ in Assumption 2.1 as well as the inverse inequality (3.14):

$$\begin{aligned}
\geq \frac{1}{\sqrt{2}}\Bigg(&(1-a)\Big[(\mu^{-1}\nabla_h\times\boldsymbol{v}_h,\nabla_h\times\boldsymbol{v}_h)_\Omega + \tau\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}}\Big]^2\\
&+ \omega^2\left(\sigma_-^2 - \big(a^{-1}-1\big)\,\omega^2\varepsilon_+^2\right)\|\boldsymbol{v}_h\|^4_{L^2(\Omega)^3}\Bigg)^{\frac{1}{2}} + \frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\,\|(\boldsymbol{v}_h)_T\|^2_{L^2(\Sigma)^3}\\
&- \delta^{-1}\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}} - \delta C_{\mathrm{inv}}\mu_-^{-2}\|\nabla_h\times\boldsymbol{v}_h\|^2_{L^2(\Omega)^3}.
\end{aligned}$$

Once again $\sqrt{x+y}\geq\frac{1}{\sqrt{2}}(\sqrt{x}+\sqrt{y})$ together with the boundedness (2.2) of μ gives

$$\begin{aligned}
&\geq \frac{1}{2}\Bigg(\sqrt{1-a}\Big[\mu_+^{-1}\|\nabla_h\times\boldsymbol{v}_h\|^2_{L^2(\Omega)^3} + \tau\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}}\Big]\\
&\qquad + \omega\left(\sigma_-^2 - \big(a^{-1}-1\big)\,\omega^2\varepsilon_+^2\right)^{\frac{1}{2}}\|\boldsymbol{v}_h\|^2_{L^2(\Omega)^3}\Bigg) + \frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\,\|(\boldsymbol{v}_h)_T\|^2_{L^2(\Sigma)^3}\\
&\qquad - \delta^{-1}\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}} - \delta C_{\mathrm{inv}}\mu_-^{-2}\|\nabla_h\times\boldsymbol{v}_h\|^2_{L^2(\Omega)^3}\\
&= \|\nabla_h\times\boldsymbol{v}_h\|^2_{L^2(\Omega)^3}\left(\frac{\sqrt{1-a}}{2\mu_+} - \delta C_{\mathrm{inv}}\mu_-^{-2}\right)\\
&\qquad + \|\boldsymbol{v}_h\|^2_{L^2(\Omega)^3}\,\frac{\omega\left(\sigma_-^2 - \big(a^{-1}-1\big)\,\omega^2\varepsilon_+^2\right)^{\frac{1}{2}}}{2} + \|(\boldsymbol{v}_h)_T\|^2_{L^2(\Sigma)^3}\,\frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\\
&\qquad + \|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}}\left(\frac{\tau\sqrt{1-a}}{2} - \delta^{-1}\right).
\end{aligned}$$

We may choose $1 > a > \frac{\omega^2\varepsilon_+^2}{\omega^2\varepsilon_+^2+\sigma_-^2}$, e.g., $a = \frac{\omega^2\varepsilon_+^2+\frac{\sigma_-^2}{2}}{\omega^2\varepsilon_+^2+\sigma_-^2}$ and obtain

$$
\begin{aligned}
&= \|\nabla_h \times \boldsymbol{v}_h\|^2_{L^2(\Omega)^3}\left(\frac{1}{2\mu_+}\left(\frac{\sigma_-^2}{2(\omega^2\varepsilon_+^2+\sigma_-^2)}\right)^{\frac12} - \delta C_{\mathrm{inv}}\mu_-^{-2}\right)\\
&\quad + \|\boldsymbol{v}_h\|^2_{L^2(\Omega)^3}\frac{\omega}{2}\left(\sigma_-^2 - \frac{\sigma_-^2}{2\omega^2\varepsilon_+^2+\sigma_-^2}\omega^2\varepsilon_+^2\right)^{\frac12} + \|(\boldsymbol{v}_h)_T\|^2_{L^2(\Sigma)^3}\frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\\
&\quad + \|\mathtt{h}^{-1}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}}\left(\frac{\tau}{2}\left(\frac{\sigma_-^2}{2(\omega^2\varepsilon_+^2+\sigma_-^2)}\right)^{\frac12} - \delta^{-1}\right)\\
&= \|\nabla_h \times \boldsymbol{v}_h\|^2_{L^2(\Omega)^3}\left(\frac{\sigma_-}{2\mu_+\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}} - \delta C_{\mathrm{inv}}\mu_-^{-2}\right)\\
&\quad + \|\boldsymbol{v}_h\|^2_{L^2(\Omega)^3}\frac{\omega}{2}\left(\frac{\omega^2\varepsilon_+^2\sigma_-^2+\sigma_-^4}{2\omega^2\varepsilon_+^2+\sigma_-^2}\right)^{\frac12} + \|(\boldsymbol{v}_h)_T\|^2_{L^2(\Sigma)^3}\frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\\
&\quad + \|\mathtt{h}^{-1}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}}\left(\frac{\tau\sigma_-}{2\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}} - \delta^{-1}\right).
\end{aligned}
$$

Choose $0 < \delta < \frac{\mu_-^2\sigma_-}{2C_{\mathrm{inv}}\mu_+\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}}$. $\delta = \frac{\mu_-^2\sigma_-}{4C_{\mathrm{inv}}\mu_+\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}}$ yields

$$
\begin{aligned}
&= \|\nabla_h \times \boldsymbol{v}_h\|^2_{L^2(\Omega)^3}\frac{\sigma_-}{4\mu_+\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}}\\
&\quad + \|\boldsymbol{v}_h\|^2_{L^2(\Omega)^3}\frac{\omega}{2}\left(\frac{\omega^2\varepsilon_+^2\sigma_-^2+\sigma_-^4}{2\omega^2\varepsilon_+^2+\sigma_-^2}\right)^{\frac12} + \|(\boldsymbol{v}_h)_T\|^2_{L^2(\Sigma)^3}\frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\\
&\quad + \|\mathtt{h}^{-1}[\![\boldsymbol{v}_h]\!]\|^2_{\mathcal{F}_h^{I\cup\Gamma}}\left(\frac{\tau\sigma_-}{2\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}} - \frac{4C_{\mathrm{inv}}\mu_+\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}}{\mu_-^2\sigma_-}\right).
\end{aligned}
$$

Now set $\tau^* := \frac{2\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}}{\sigma_-}\frac{4C_{\mathrm{inv}}\mu_+\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}}{\mu_-^2\sigma_-}$ and choose the penalty parameter $\tau > \tau_*$, e.g., $\tau = \frac{2\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}}{\sigma_-}\left(\frac{4C_{\mathrm{inv}}\mu_+\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}}{\mu_-^2\sigma_-} + \frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\right)$. Then,

$$
\geq \min\left\{\frac{\sigma_-}{4\mu_+\sqrt{2(\omega^2\varepsilon_+^2+\sigma_-^2)}}, \frac{\omega}{2}\left(\frac{\omega^2\varepsilon_+^2\sigma_-^2+\sigma_-^4}{2\omega^2\varepsilon_+^2+\sigma_-^2}\right)^{\frac12}, \frac{\omega\lambda\sqrt{\varepsilon_0}}{\sqrt{2\mu_0}}\right\}\|\boldsymbol{v}_h\|^2_{\mathrm{DG}},
$$

and everything is proven. □

Corollary 3.11 (Existence and uniqueness of $\boldsymbol{E}_h$)**.** *Let Assumption 2.1 be satisfied. Then for every frequency $\omega > 0$, and $\tau > \tau^*$ there exists a unique function $\boldsymbol{E}_h \in V_h$ solving* (3.10)*, i.e., $a_h(\boldsymbol{E}_h, \boldsymbol{v}_h; \omega, \tau) = f(\boldsymbol{v}_h; \omega)$ for all $\boldsymbol{v}_h \in V_h$.*

Proof. Since V_h is finite dimensional, it is a Hilbert space w.r.t. the inner product

$$
\begin{aligned}
(\boldsymbol{u}_h, \boldsymbol{v}_h)_{\mathrm{DG}} &:= (\boldsymbol{u}_h, \boldsymbol{v}_h)_\Omega + (\nabla_h \times \boldsymbol{u}_h, \nabla_h \times \boldsymbol{v}_h)_\Omega\\
&\quad + \langle(\boldsymbol{u}_h)_T, (\boldsymbol{v}_h)_T\rangle_\Sigma + \langle \mathtt{h}^{-1}[\![\boldsymbol{u}_h]\!], [\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}}, \quad \boldsymbol{u}_h, \boldsymbol{v}_h \in V_h.
\end{aligned}
$$

Due to Theorem 3.8 and Theorem 3.10, the form $a_h(\cdot,\cdot;\omega,\tau)$ is bounded and coercive w.r.t. the induced norm $\|\cdot\|_{\mathrm{DG}} = \sqrt{(\cdot,\cdot)_{\mathrm{DG}}}$ in (3.11) for any $\omega > 0$, and $\tau > \tau^*$ so that by Lax-Milgram there exists a unique function $\boldsymbol{E}_h \in V_h$ satisfying (3.10). $\square$

3.3. **Error in the energy norm.** For the error analysis we will need some properties of the projection Π_μ which has been introduced in Definition 3.9. In order to derive them we introduce the L^2-orthogonal projection Π_h onto the space V_h. For $\boldsymbol{w} \in L^2(\Omega;\mathbb{C}^3)$ the projection $\Pi_h \boldsymbol{w} \in V_h$ satisfies

$$(\Pi_h \boldsymbol{w}, \boldsymbol{v}_h)_\Omega = (\boldsymbol{w}, \boldsymbol{v}_h)_\Omega \qquad \forall \boldsymbol{v}_h \in V_h. \tag{3.19}$$

Lemma 3.12. *For a node $\boldsymbol{a}$ in the mesh $\mathcal{T}_h$ define $\Delta_{\boldsymbol{a}}$ as the set of all tetrahedra sharing this vertex. Then there exists a constant $C > 0$, independent of T and h_T, such that for any $\boldsymbol{v} \in L^2(\Omega;\mathbb{C}^3)$ with $\boldsymbol{v}|_{\Delta_{\boldsymbol{a}}} \in H^t(\Delta_{\boldsymbol{a}};\mathbb{C}^3)$, $t \in \left(\frac{1}{2}, 1\right]$, any $T \in \mathcal{T}_h$ and any vertex $\boldsymbol{a}$ of T the projection Π_h in (3.19) satisfies*

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|^2_{L^2(T)^3} + h_T \|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|^2_{L^2(\partial T)^3} \le C h_T^{2t} \|\boldsymbol{v}\|^2_{H^t(\Delta_{\boldsymbol{a}})^3}. \tag{3.20}$$

Proof. In two dimensions, this theorem follows from the properties of the interpolation operator in [11, Thm. 1] or [5, Thm. 2.1 and Rem. 4]. As mentioned in [17, Sect. A.3], the proof can be directly adopted to the case of higher dimensions. $\square$

The estimate (3.20) for the L^2-orthogonal projection Π_h on V_h provides us now also with an error bound for the μ^{-1}-orthogonal projection Π_μ in Definition 3.9.

Lemma 3.13. *There exists a constant $C = C(\mu) > 0$, independent of T and h_T, such that for any $\boldsymbol{v} \in L^2(\Omega;\mathbb{C}^3)$ with $\boldsymbol{v}|_{\Delta_{\boldsymbol{a}}} \in H^t(\Delta_{\boldsymbol{a}};\mathbb{C}^3)$, $t \in \left(\frac{1}{2}, 1\right]$, any $T \in \mathcal{T}_h$ and any vertex $\boldsymbol{a}$ of T the projection Π_μ in (3.17) satisfies*

$$\|\boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|^2_{L^2(T)^3} + h_T \|\boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|^2_{L^2(\partial T)^3} \le C h_T^{2t} \|\boldsymbol{v}\|^2_{H^t(\Delta_{\boldsymbol{a}})^3}. \tag{3.21}$$

Proof. Let $T \in \mathcal{T}_h$, $\boldsymbol{a}$ be a vertex of T and $\boldsymbol{v} \in H^t(\Delta_{\boldsymbol{a}};\mathbb{C}^3)$ for some $t \in \left(\frac{1}{2}\right]$.

$$\begin{aligned}
\mu_+^{-1} \|\Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|^2_{L^2(T)^3} &\le (\mu^{-1}(\Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v}), \Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v})_T \\
&= (\mu^{-1}\Pi_h \boldsymbol{v}, \Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v})_T - (\mu^{-1}\boldsymbol{v}, \Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v})_T \\
&= (\mu^{-1}(\Pi_h \boldsymbol{v} - \boldsymbol{v}), \Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v})_T \le \mu_-^{-1} \|\Pi_h \boldsymbol{v} - \boldsymbol{v}\|_{L^2(T)^3} \|\Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|_{L^2(T)^3}
\end{aligned}$$

and, therefore,

$$\|\Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|_{L^2(T)^3} \le \frac{\mu_+}{\mu_-} \|\Pi_h \boldsymbol{v} - \boldsymbol{v}\|_{L^2(T)^3}, \quad \text{as well as} \tag{3.22}$$

$$\begin{aligned}
\|\boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|_{L^2(T)^3} &\le \|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{L^2(T)^3} + \|\Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|_{L^2(T)^3} \\
&\le \left(1 + \frac{\mu_+}{\mu_-}\right) \|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{L^2(T)^3} \le \left(1 + \frac{\mu_+}{\mu_-}\right) C h_T^t \, \|\boldsymbol{v}\|_{H^t(\Delta_{\boldsymbol{a}})^3}
\end{aligned}$$

for any corner $\boldsymbol{a}$ of $T \in \mathcal{T}_h$ by Lemma 3.12. For the L^2-norm on ∂T we may use the inverse inequality (3.13), (3.20) for Π_h and (3.22). Thus,

$$\begin{aligned}
h_T \|\boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|^2_{L^2(\partial T)^3} &\le h_T \left(\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{L^2(\partial T)^3} + \|\Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|_{L^2(\partial T)^3}\right)^2 \\
&\le 2h_T \|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|^2_{L^2(\partial T)^3} + 2h_T \|\Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|^2_{L^2(\partial T)^3} \\
&\le 2h_T \|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|^2_{L^2(\partial T)^3} + 2C_{\mathrm{inv}} \|\Pi_h \boldsymbol{v} - \Pi_\mu \boldsymbol{v}\|^2_{L^2(T)^3} \\
&\le 2h_T \|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|^2_{L^2(\partial T)^3} + 2C_{\mathrm{inv}} \frac{\mu_+^2}{\mu_-^2} \|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|^2_{L^2(T)^3}
\end{aligned}$$

$$\leq C\left(h_T\|\boldsymbol{v}-\Pi_h\boldsymbol{v}\|^2_{L^2(\partial T)^3}+\|\boldsymbol{v}-\Pi_h\boldsymbol{v}\|^2_{L^2(T)^3}\right)\leq Ch_T^{2t}\|\boldsymbol{v}\|^2_{H^t(\Delta_{\boldsymbol{a}})^3},$$

so that the lemma is proven. ☐

For ω, $\tau>0$ we define the *residual* for $\boldsymbol{v}_h\in V_h$ by

$$r_h(\boldsymbol{v}_h;\,\omega,\tau):=\widetilde{a}_h(\boldsymbol{E},\boldsymbol{v}_h;\,\omega,\tau)-f(\boldsymbol{v}_h;\,\omega)=\widetilde{a}_h(\boldsymbol{E}-\boldsymbol{E}_h,\boldsymbol{v}_h;\,\omega,\tau),$$

where $\boldsymbol{E}$ denotes the exact solution of (2.1a)–(2.1c) and $\boldsymbol{E}_h$ the dG approximation as a solution of (3.10). We are able to estimate the absolute value of the residual r_h as follows.

Proposition 3.14. *Let Assumption 2.1 be satisfied, and $\boldsymbol{E}$ be the unique exact solution to* (2.1a)–(2.1c) *for a frequency $\omega>0$ with $\nabla\times\boldsymbol{E}\in H^t(\Omega;\mathbb{C}^3)$ for some $t\in\left(\frac{1}{2},1\right]$. Then, for $\tau>0$, the residual can be expressed as*

$$r_h(\boldsymbol{v}_h;\,\omega,\tau)=\langle\{\!\{\mu^{-1}(\nabla\times\boldsymbol{E}-\Pi_\mu(\nabla\times\boldsymbol{E}))\}\!\},[\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}}\quad\forall\boldsymbol{v}_h\in V_h. \tag{3.23}$$

In addition, the following estimate holds

$$|r_h(\boldsymbol{v}_h;\,\omega,\tau)|\leq Ch^t\|\boldsymbol{v}_h\|_{\mathrm{DG}}\|\nabla\times\boldsymbol{E}\|_{H^t(\Omega)^3}\quad\forall\boldsymbol{v}_h\in V_h.$$

Proof. In order to derive representation (3.23), let ω, $\tau>0$, and $\boldsymbol{v}_h\in V_h$.

$$\begin{aligned}
r_h(\boldsymbol{v}_h;\,\omega,\tau)&=\widetilde{a}_h(\boldsymbol{E},\boldsymbol{v}_h;\,\omega,\tau)-f(\boldsymbol{v}_h;\,\omega)\\
&=(\mu^{-1}\nabla\times\boldsymbol{E},\nabla_h\times\boldsymbol{v}_h)_\Omega-\omega^2(\varepsilon\boldsymbol{E},\boldsymbol{v}_h)_\Omega-i\omega(\sigma\boldsymbol{E},\boldsymbol{v}_h)_\Omega\\
&\quad-(\mu^{-1}\nabla\times\boldsymbol{E},\mathcal{L}_\mu(\boldsymbol{v}_h))_\Omega-i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\langle\boldsymbol{E}_T,(\boldsymbol{v}_h)_T\rangle_\Sigma\\
&\quad-(\mathcal{L}_\mu(\boldsymbol{E}),\mu^{-1}\nabla_h\times\boldsymbol{v}_h)_\Omega+\langle\tau\mathtt{h}^{-1}[\![\boldsymbol{E}]\!],[\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}}-f(\boldsymbol{v}_h;\,\omega).
\end{aligned}$$

The first two expressions in the last line vanish since $[\![\boldsymbol{E}]\!]=0$ on $\mathcal{F}_h^{I\cup\Gamma}$ and, hence, $\mathcal{L}_\mu(\boldsymbol{E})=0$. Applying integration by parts for the integral $(\mu^{-1}\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{v}_h)_T$ on every element $T\in\mathcal{T}_h$ and afterwards the dG formula (3.3) yields

$$\begin{aligned}
&=(\nabla\times(\mu^{-1}\nabla\times\boldsymbol{E}),\boldsymbol{v}_h)_\Omega-((\omega^2\varepsilon+i\omega\sigma)\boldsymbol{E},\boldsymbol{v}_h)_\Omega-(\mu^{-1}\Pi_\mu(\nabla\times\boldsymbol{E}),\mathcal{L}_\mu(\boldsymbol{v}_h))_\Omega\\
&\quad-\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{n}_T\times(\mu^{-1}\nabla\times\boldsymbol{E}),\boldsymbol{v}_h\rangle_{\partial T}-i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\langle\boldsymbol{E}_T,(\boldsymbol{v}_h)_T\rangle_\Sigma-\mu_0^{-1}\langle\boldsymbol{g},(\boldsymbol{v}_h)_T\rangle_\Sigma\\
&\quad-i\omega\sqrt{\varepsilon_0}\,(\boldsymbol{J_a},\boldsymbol{v}_h)_\Omega\\
&=-\langle[\![\mu^{-1}\nabla\times\boldsymbol{E}]\!],\{\!\{\boldsymbol{v}_h\}\!\}\rangle_{\mathcal{F}_h^I}+\langle\{\!\{\mu^{-1}\nabla\times\boldsymbol{E}\}\!\},[\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^I}+\langle\mu^{-1}\nabla\times\boldsymbol{E},\boldsymbol{n}\times\boldsymbol{v}_h\rangle_{\mathcal{F}_h^\Gamma}\\
&\quad-\langle\boldsymbol{n}\times(\mu^{-1}\nabla\times\boldsymbol{E}),\boldsymbol{v}_h\rangle_{\mathcal{F}_h^\Sigma}-\langle\{\!\{\mu^{-1}\Pi_\mu(\nabla\times\boldsymbol{E})\}\!\},[\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}}-\mu_0^{-1}\langle\boldsymbol{g},(\boldsymbol{v}_h)_T\rangle_\Sigma\\
&\quad-i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\langle\boldsymbol{E}_T,(\boldsymbol{v}_h)_T\rangle_\Sigma\\
&=\langle\{\!\{\mu^{-1}\nabla\times\boldsymbol{E}-\mu^{-1}\Pi_\mu(\nabla\times\boldsymbol{E})\}\!\},[\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}}-\langle\boldsymbol{n}\times(\mu^{-1}\nabla\times\boldsymbol{E})\\
&\quad+i\omega\lambda\sqrt{\varepsilon_0\mu_0^{-1}}\boldsymbol{E}_T+\mu_0^{-1}\boldsymbol{g},(\boldsymbol{v}_h)_T\rangle_\Sigma=\langle\{\!\{\mu^{-1}(\nabla\times\boldsymbol{E}-\Pi_\mu(\nabla\times\boldsymbol{E}))\}\!\},[\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}},
\end{aligned}$$

since $[\![\mu^{-1}\nabla\times\boldsymbol{E}]\!]=0$ on $\mathcal{F}_h^I$, $\langle\boldsymbol{n}\times(\mu^{-1}\nabla\times\boldsymbol{E}),\boldsymbol{v}_h\rangle_{\mathcal{F}_h^\Sigma}=\langle\boldsymbol{n}\times(\mu^{-1}\nabla\times\boldsymbol{E}),(\boldsymbol{v}_h)_T\rangle_\Sigma$ and $\boldsymbol{E}$ satisfies (2.1c). Using this representation, we can estimate as follows

$$\begin{aligned}
|r_h(\boldsymbol{v}_h;\,\omega,\tau)|&=\left|\langle\{\!\{\mu^{-1}(\nabla\times\boldsymbol{E}-\Pi_\mu(\nabla\times\boldsymbol{E}))\}\!\},[\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^{I\cup\Gamma}}\right|\\
&\leq\|\mathtt{h}^{-\frac{1}{2}}[\![\boldsymbol{v}_h]\!]\|_{\mathcal{F}_h^{I\cup\Gamma}}\|\mathtt{h}^{\frac{1}{2}}\mu^{-1}\{\!\{\nabla\times\boldsymbol{E}-\Pi_\mu(\nabla\times\boldsymbol{E})\}\!\}\|_{\mathcal{F}_h^{I\cup\Gamma}}
\end{aligned}$$

$$
\begin{aligned}
&\le \mu_-^{-1}\|\boldsymbol{v}_h\|_{\mathrm{DG}}\left(\sum_{F\in\mathcal{F}_h} h_F\|\{\!\{\nabla\times\boldsymbol{E}-\Pi_\mu(\nabla\times\boldsymbol{E})\}\!\}\|^2_{L^2(F)^3}\right)^{\frac12}\\
&\le C\,\|\boldsymbol{v}_h\|_{\mathrm{DG}}\left(\sum_{T\in\mathcal{T}_h} h_T\|\nabla\times\boldsymbol{E}-\Pi_\mu(\nabla\times\boldsymbol{E})\|^2_{L^2(\partial T)^3}\right)^{\frac12}\\
&\le Ch^t\,\|\boldsymbol{v}_h\|_{\mathrm{DG}}\|\nabla\times\boldsymbol{E}\|_{H^t(\Omega)^3},
\end{aligned}
$$

where we used estimate (3.21) for Π_μ in the last step. □

Next, we refer to an error estimate for the Nédélec interpolant with respect to the $\|\cdot\|_X$-norm for the case $p=1$, i.e., for piecewise affine functions.

Lemma 3.15 (Nédélec interpolant)**.** *Let $\boldsymbol{v}\in H^t(\Omega;\mathbb{C}^3)$ with $\nabla\times\boldsymbol{v}\in H^t(\Omega;\mathbb{C}^3)$ and $\boldsymbol{v}_T\in H^t_{\|}(\Sigma)^3\cap H(\mathrm{curl}_\Sigma,\Sigma)$ for some $t\in\left(\frac12,1\right]$, where*

$$
H^t_{\|}(\Sigma)^3 := \left\{\boldsymbol{w}\in L^2_t(\Sigma;\mathbb{C}^3)\,|\,\exists\,\boldsymbol{\xi}\in H^{t+\frac12}(\Omega;\mathbb{C}^3):\boldsymbol{w}=\gamma_T(\boldsymbol{\xi})=(\boldsymbol{n}\times\boldsymbol{\xi}|_\Sigma)\times\boldsymbol{n}\right\}
$$

and $H(\mathrm{curl}_\Sigma,\Sigma):=\left\{\boldsymbol{w}\in L^2_t(\Sigma;\mathbb{C}^3)\,|\,\nabla_\Sigma\times\boldsymbol{w}\in L^2(\Sigma;\mathbb{C})\right\}$, where $\nabla_\Sigma\times$ denotes the surface curl operator on the surface Σ, cf. [7, Prop. 3.6]*. Then there exists a function $\Pi_N\boldsymbol{v}\in W_h\cap X$, where $W_h:=\left\{\boldsymbol{v}\in H(\mathrm{curl},\Omega)\,|\,\boldsymbol{v}|_T\in R^1(T)\quad\forall T\in\mathcal{T}_h\right\}\subset V_h$ and $R^1(T):=\left\{\boldsymbol{v}:T\to\mathbb{C}^3\,|\,\exists\,\boldsymbol{a},\boldsymbol{b}\in\mathbb{C}^3:\boldsymbol{v}(\boldsymbol{x})=\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}\quad\forall\boldsymbol{x}\in T\right\}$ satisfying*

$$
\begin{aligned}
\|\boldsymbol{v}-\Pi_N\boldsymbol{v}\|_X &= \left[\|\boldsymbol{v}-\Pi_N\boldsymbol{v}\|^2_{H(\mathrm{curl},\Omega)}+\|(\boldsymbol{v}-\Pi_N\boldsymbol{v})_T\|^2_{L^2(\Sigma)^3}\right]^{\frac12}\\
&\le Ch^t\left[\|\boldsymbol{v}\|_{H^t(\Omega)^3}+\|\nabla\times\boldsymbol{v}\|_{H^t(\Omega)^3}+\|\boldsymbol{v}_T\|_{H^t_{\|}(\Sigma)^3}+\|\nabla_\Sigma\times(\boldsymbol{v}_T)\|_{L^2(\Sigma)}\right],
\end{aligned}
$$

where $\|\boldsymbol{w}\|_{H^t_{\|}(\Sigma)^3}:=\inf_{\boldsymbol{\xi}\in H^{t+\frac12}(\Omega)^3}\left\{\|\boldsymbol{\xi}\|_{H^{t+\frac12}(\Omega)^3}\,\middle|\,\gamma_T(\boldsymbol{\xi})=\boldsymbol{w}\right\}$.

Proof. See [16, Lem. 5.2] and [8, Lem. 15]. □

Proposition 3.16 (Partition of the error)**.** *Let Assumption 2.1 be satisfied, and $\boldsymbol{E}$ solve* (2.1a)–(2.1c) *for $\omega>0$. If $\boldsymbol{E}_h$ is the dG approximation in* (3.10) *for $\tau>\tau^*$, then there exists a constant $C>0$, independent of h, such that*

$$
\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\mathrm{DG}}\le C\left[\inf_{\boldsymbol{v}_h\in V_h}\|\boldsymbol{E}-\boldsymbol{v}_h\|_{\mathrm{DG}}+\sup_{\boldsymbol{w}_h\in V_h\setminus\{0\}}\frac{|r_h(\boldsymbol{w}_h;\,\omega,\tau)|}{\|\boldsymbol{w}_h\|_{\mathrm{DG}}}\right].
$$

Proof. Let $\boldsymbol{v}_h\in V_h$. Then, $\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\mathrm{DG}}\le\|\boldsymbol{E}-\boldsymbol{v}_h\|_{\mathrm{DG}}+\|\boldsymbol{v}_h-\boldsymbol{E}_h\|_{\mathrm{DG}}$, and using coercivity and continuity of a_h and $\widetilde{a}_h$, respectively,

$$
\begin{aligned}
\alpha\,\|\boldsymbol{v}_h-\boldsymbol{E}_h\|^2_{\mathrm{DG}} &\le |a_h(\boldsymbol{v}_h-\boldsymbol{E}_h,\boldsymbol{v}_h-\boldsymbol{E}_h;\,\omega,\tau)| = |\widetilde{a}_h(\boldsymbol{v}_h-\boldsymbol{E}_h,\boldsymbol{v}_h-\boldsymbol{E}_h;\,\omega,\tau)|\\
&\le |\widetilde{a}_h(\boldsymbol{v}_h-\boldsymbol{E},\boldsymbol{v}_h-\boldsymbol{E}_h;\,\omega,\tau)|+|\widetilde{a}_h(\boldsymbol{E}-\boldsymbol{E}_h,\boldsymbol{v}_h-\boldsymbol{E}_h;\,\omega,\tau)|\\
&\le \gamma\,\|\boldsymbol{v}_h-\boldsymbol{E}\|_{\mathrm{DG}}\|\boldsymbol{v}_h-\boldsymbol{E}_h\|_{\mathrm{DG}}+|r_h(\boldsymbol{v}_h-\boldsymbol{E}_h;\,\omega,\tau)|\,,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|\boldsymbol{v}_h-\boldsymbol{E}_h\|_{\mathrm{DG}} &\le C\left[\|\boldsymbol{v}_h-\boldsymbol{E}\|_{\mathrm{DG}}+\frac{|r_h(\boldsymbol{v}_h-\boldsymbol{E}_h;\,\omega,\tau)|}{\|\boldsymbol{v}_h-\boldsymbol{E}_h\|_{\mathrm{DG}}}\right]\\
&\le C\left[\|\boldsymbol{v}_h-\boldsymbol{E}\|_{\mathrm{DG}}+\sup_{\boldsymbol{w}_h\in V_h}\frac{|r_h(\boldsymbol{w}_h;\,\omega,\tau)|}{\|\boldsymbol{w}_h\|_{\mathrm{DG}}}\right].
\end{aligned}
$$

The assertion follows now by taking the infimum over $\boldsymbol{v}_h$ in V_h. □

Theorem 3.17 (Error in the energy norm)**.** *Let Assumption 2.1 be satisfied and assume that the solution* $\boldsymbol{E}$ *of* (2.1a)–(2.1c) *for* $\omega > 0$ *satisfies* $\boldsymbol{E} \in H^t(\Omega;\mathbb{C}^3)$ *with* $\nabla \times \boldsymbol{E} \in H^t(\Omega;\mathbb{C}^3)$ *and* $\boldsymbol{E}_T \in H^t_{\|}(\Sigma)^3 \cap H(\mathrm{curl}_\Sigma, \Sigma)$ *on* Σ *for some* $t \in \left(\frac{1}{2}, 1\right]$*. Let* $\boldsymbol{E}_h \in V_h$ *be the corresponding dG approximation solving* (3.10) *for* $\tau > \tau^*$*. Then there exists a constant, independent of* h*, such that*

$$\begin{aligned} \|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathrm{DG}} \le Ch^t \Big[& \|\boldsymbol{E}\|_{H^t(\Omega)^3} + \|\nabla \times \boldsymbol{E}\|_{H^t(\Omega)^3} \\ & + \|\boldsymbol{E}_T\|_{H^t_{\|}(\Sigma)^3} + \|\nabla_\Sigma \times (\boldsymbol{E}_T)\|_{L^2(\Sigma)} \Big]. \end{aligned} \tag{3.24}$$

Proof. This result follows from Proposition 3.14, Lemma 3.15 and Proposition 3.16:

$$\begin{aligned} \|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathrm{DG}} &\le C \left[\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{E} - \boldsymbol{v}_h\|_{\mathrm{DG}} + \sup_{\boldsymbol{w}_h \in V_h} \frac{|r_h(\boldsymbol{w}_h;\, \omega, \tau)|}{\|\boldsymbol{w}_h\|_{\mathrm{DG}}} \right] \\ &\le C \left[\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{E} - \boldsymbol{v}_h\|_{\mathrm{DG}} + h^t \|\nabla \times \boldsymbol{E}\|_{H^t(\Omega)^3} \right] \\ &\le C \left[\|\boldsymbol{E} - \Pi_N \boldsymbol{E}\|_{\mathrm{DG}} + h^t \|\nabla \times \boldsymbol{E}\|_{H^t(\Omega)^3} \right] \\ &\le Ch^t \Big[\|\boldsymbol{E}\|_{H^t(\Omega)^3} + \|\nabla \times \boldsymbol{E}\|_{H^t(\Omega)^3} + \|\boldsymbol{E}_T\|_{H^t_{\|}(\Sigma)^3} + \|\nabla_\Sigma \times (\boldsymbol{E}_T)\|_{L^2(\Sigma)} \Big], \end{aligned}$$

which proves the theorem. □

4. Reduced Basis Method

The aim of this section is to present an approach which allows to investigate electromagnetic wave propagation by computing reliable approximations of the electric field density $\boldsymbol{E}$ as a solution to the model problem presented in Section 2 for many different values of the frequency ω – a so-called *multi-query* problem – in a reasonable time.

Our starting point hereby is the dG formulation (3.10). In the context of the Reduced Basis Method (RBM) the detailed, i.e., high dimensional approximation $\boldsymbol{E}_h$ to the exact solution $\boldsymbol{E}$ of (2.5) is called *truth approximation.* The key to set up the formulation for a lower dimensional reduced basis approximation is the idea that the sets of all possible truth approximations $\mathcal{M}^{\mathcal{N}} := \{\boldsymbol{E}_h(\omega) \text{ solves (3.10)} \,|\, \omega \in \mathcal{D}\}$ lie on a low-dimensional manifold in V_h, [29]. Instead of computing the –expensive– truth approximations $\boldsymbol{E}_h(\omega)$ for all frequencies ω in a given parameter domain $\mathcal{D}$, the RBM amounts to finding a suitable approximation space $X_N \subset V_h$ of $\mathcal{M}^{\mathcal{N}}$ with lower dimension $N := \dim(X_N) \ll \dim(V_h) =: \mathcal{N} = \mathcal{N}_h$ and then computing –cheap– approximations $\boldsymbol{E}_N(\omega) \in X_N$. This is done using *snapshots*, i.e., truth approximations for N different values of the parameter, cf. [29],

$$X_N := \mathrm{span}\left\{\boldsymbol{E}_h(\omega) \text{ solves (3.10)} \,|\, \omega \in S_N\right\}, \qquad S_N := \{\omega_1, \dots, \omega_N\} \subset \mathcal{D}.$$

Afterwards one computes the Galerkin approximation in X_N, i.e.,[b]

$$a_h(\boldsymbol{E}_N(\omega), \boldsymbol{v}_N;\, \omega) = f(\boldsymbol{v}_N;\, \omega) \quad \forall \boldsymbol{v}_N \in X_N.$$

The spaces X_N are called *reduced basis spaces*, a standard procedure to construct them is the *Greedy algorithm*, see [6, 29]. This algorithm builds up the space X_N iteratively, enriching it with one new basis function in each iteration. The particular choice of the basis function is based upon an error indicator, the algorithm therefore depends on efficiently computable a posteriori error bounds $\|\boldsymbol{E}_h(\omega) - \boldsymbol{E}_N(\omega)\|_{\mathrm{DG}} \le$

[b]To shorten notation we omit τ since it is chosen constant in our numerical experiments.

$\Delta_N(\omega)$. The Greedy procedure then maximizes the efficiently computable Δ_N w.r.t. an appropriate training set $\Xi_{\text{train}} \subset \mathcal{D}$ to define the snapshots.

The key to these efficiently evaluable error bounds as well as the efficiency of the calculation of the RBM solution $\boldsymbol{E}_N(\omega)$ is an affine dependency of a_h and f w.r.t. the parameter ω, i.e., they must be of the following form:

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h;\, \omega) = \sum_{q=1}^{Q_a} \Theta_q^a(\omega) a_h^q(\boldsymbol{u}_h, \boldsymbol{v}_h), \qquad f(\boldsymbol{v}_h;\, \omega) = \sum_{q=1}^{Q_f} \Theta_q^f(\omega) f^q(\boldsymbol{v}_h)$$

for all $\boldsymbol{u}_h, \boldsymbol{v}_h \in V_h$ and all $\omega \in \mathcal{D}$ with parameter-dependent functions $\Theta_q^a(\omega)$, $\Theta_q^f(\omega)\colon \mathcal{D} \to \mathbb{R}$, as well as sesquilinear forms $a_h^q(\cdot,\cdot)$ and antilinear forms $f_q(\cdot)$ which are independent of the parameter ω.
The online-efficiency is obtained via precomputing the values $a_h^q(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j)$, $i, j \in \{1, \dots, N\}$, $q \in \{1, \dots, Q_a\}$ and $f^q(\boldsymbol{\xi}_i)$, $i \in \{1, \dots, N\}$, $q \in \{1, \dots, Q_f\}$ in a precedent, possibly time-consuming offline phase, where $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N\}$ denotes a basis of the RB space X_N. The mentioned error estimator $\Delta_N(\omega)$ can, e.g., be based upon the dual norm of the RB residual $r_N(\cdot;\, \omega) = f(\cdot;\, \omega) - a_h(\boldsymbol{E}_N, \cdot;\, \omega)\colon V_h \to \mathbb{C}$ and the coercivity constant $\alpha(\omega)$ of $a_h(\cdot,\cdot;\, \omega)$. It takes the following form, cf. [29]:

$$\Delta_N(\omega) = \frac{\|r_N(\cdot;\, \omega)\|_{\mathrm{DG}^*}}{\alpha(\omega)} = \frac{1}{\alpha(\omega)} \sup_{\boldsymbol{v}_h \in V_h} \frac{r_N(\boldsymbol{v}_h;\, \omega)}{\|\boldsymbol{v}_h\|_{\mathrm{DG}}}.$$

For $\omega \in \mathcal{D}$, the dual norm of $r_N(\cdot;\, \omega)$ can be evaluated via the dG-norm of its Riesz representative $\boldsymbol{v}_{r_N}(\omega) \in V_h$ which satisfies $\|\boldsymbol{v}_{r_N}(\omega)\|_{\mathrm{DG}} = \|r_N(\cdot;\, \omega)\|_{\mathrm{DG}^*}$. Based upon the affine decomposition of a_h and f, also the norm of the Riesz representative $\boldsymbol{v}_{r_N}(\omega)$ is offline-online-decomposable (e.g. [10, 18, 29]) and can therefore be evaluated efficiently in the online phase. The coercivity constant $\alpha(\omega)$ can be computed by an eigenvalue problem or it can be approximated via the Successive Constraint Method (SCM), [22].

5. Numerical results

In this section we present results of some numerical experiments for the investigated problem being treated with the RBM. As already mentioned, one crucial ingredient of the RBM is the affine decomposition of a_h and f w.r.t. the parameter. For our dG formulation (3.10), this affine form is readily given by:

$$\begin{aligned}
\Theta_1^a(\omega) &:\equiv 1, \quad a_1(\boldsymbol{u}_h, \boldsymbol{v}_h) := (\mu^{-1}\nabla_h \times \boldsymbol{u}_h, \nabla_h \times \boldsymbol{v}_h)_\Omega - \langle [\![\boldsymbol{u}_h]\!], \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{v}_h\}\!\}\rangle_{\mathcal{F}_h^I \cup \Gamma} \\
&\qquad - \langle \{\!\{\mu^{-1}\nabla_h \times \boldsymbol{u}_h\}\!\}, [\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^I \cup \Gamma} + \langle \tau \mathtt{h}^{-1} [\![\boldsymbol{u}_h]\!], [\![\boldsymbol{v}_h]\!]\rangle_{\mathcal{F}_h^I \cup \Gamma}, \\
\Theta_2^a(\omega) &:= \omega^2, \; a_2(\boldsymbol{u}_h, \boldsymbol{v}_h) := -(\varepsilon \boldsymbol{u}_h, \boldsymbol{v}_h)_\Omega, \\
\Theta_3^a(\omega) &:= \omega, \quad a_3(\boldsymbol{u}_h, \boldsymbol{v}_h) := -i(\sigma \boldsymbol{u}_h, \boldsymbol{v}_h)_\Omega - i\lambda \sqrt{\varepsilon_0 \mu_0^{-1}}\, \langle (\boldsymbol{u}_h)_T, (\boldsymbol{v}_h)_T \rangle_\Sigma, \\
\Theta_1^f(\omega) &:= \omega, \quad f_1(\boldsymbol{v}_h) := i\sqrt{\varepsilon_0} (\boldsymbol{J_a}, \boldsymbol{v}_h)_\Omega, \\
\Theta_2^f(\omega) &:\equiv 1, \quad f_2(\boldsymbol{v}_h) := \langle \mu_0^{-1} \boldsymbol{g}, (\boldsymbol{v}_h)_T \rangle_\Sigma,
\end{aligned}$$

and, therefore, $Q_a = 3$, $Q_f = 2$.

The models we use for our numerical tests were created using COMSOL Multiphysics 4.2a. Details of the implementation are given in Appendix A. All RB calculations were implemented in RBmatlab, see `http://www.morepas.org`.

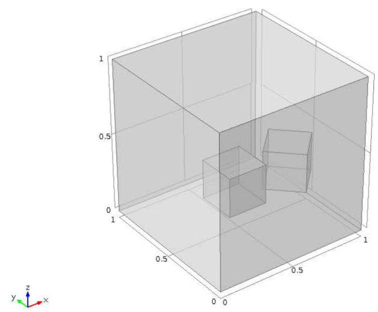


FIGURE 2. Geometry

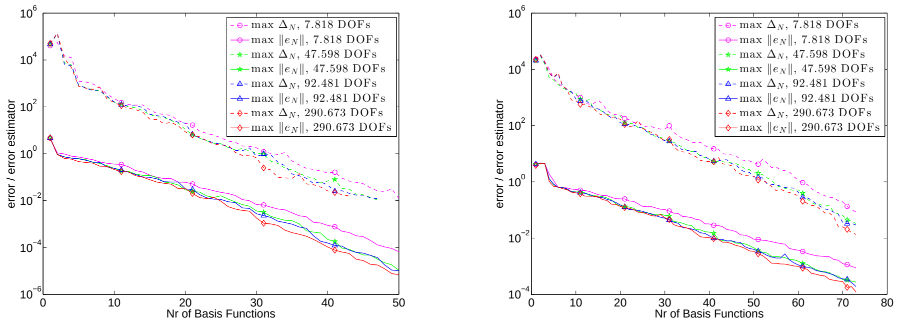
As geometry we use a 3d-Block of side length 1 where we cut out two smaller blocks, each of side length 1/4. One of the smaller blocks is placed parallel to the large block, one is rotated by 45 degrees about the z-axis, see Figure 2. The two interior blocks are supposed to be perfectly conducting (2.1b), whereas an impedance boundary condition (2.1c) is imposed on the exterior boundary of the block. We implemented two versions of the model: Model 1 has constant coefficients $\mu \equiv \mu_0 = 4\pi \cdot 10^{-7}, \varepsilon \equiv \varepsilon_0 = 8.854 \cdot 10^{-12}$, and $\sigma \equiv 0.01$. In Model 2 the coefficients are given by $\mu(\boldsymbol{x}) = \mu_0(1 + \|\boldsymbol{x} - (0.5, 0.5, 0.5)^T\|), \varepsilon(\boldsymbol{x}) = \varepsilon_0(1 + \|\boldsymbol{x}\|)$, and $\sigma(\boldsymbol{x}) = 0.01(1 + 0.5\,x_1^2)$. Referring to [21], we chose $\tau = 1000/\mu_0$ for both models. With these two models we performed a Greedy sampling with a parameter domain $\mathcal{D} = [1, 50]$ GHz which was discretized into 97 equidistant sampling points to obtain Ξ_{train}.

Note also that we are dealing with *complex-valued* degrees of freedom (DOFs), which has to be taken into account, when choosing a solver. Since the MATLAB backslash-solver (at least for finer discretizations) was not capable of an efficient numerical solution, we used MUMPS [2, 3], which allows us to use about 300,000 DOFs on an iMac, 3.2 GHz Intel Core i3 with 8GB RAM.

In absence of an analytic solution to (2.1a)–(2.1c), we investigate $\|\boldsymbol{E}^\star - \boldsymbol{E}_h\|_{\text{DG}}$ for decreasing mesh size h in order to validate our code w.r.t. (3.24). We start with $h = 1$ and refine the mesh uniformly until we end up at a mesh size of $h = 1/16$. Thus, we obtain conforming meshes. The solution $\boldsymbol{E}_{1/16}$ consisted of 1,413,120 DOFs and was used as reference solution $\boldsymbol{E}^\star$, whose computation took about 7 hours. The h-convergence results are shown in Table 1.

		Model 1		Model 2	
h	# DOFs	$\omega_1 = 1$	$\omega_2 = 10$	$\omega_1 = 1$	$\omega_2 = 10$
1	345	1.4637	2.5759	1.1366	4.3942
1/2	2,760	0.9465	0.6149	0.7506	1.1621
1/4	22,080	0.5653	0.2853	0.4521	0.7041
1/8	176,640	0.2886	0.1357	0.2286	0.2679

TABLE 1. h-convergence: $\|\boldsymbol{E}^\star - \boldsymbol{E}_h\|_{\text{DG}}$ for different h



(a) Model 1 (b) Model 2

FIGURE 3. Real error and error estimator during the Greedy algorithm

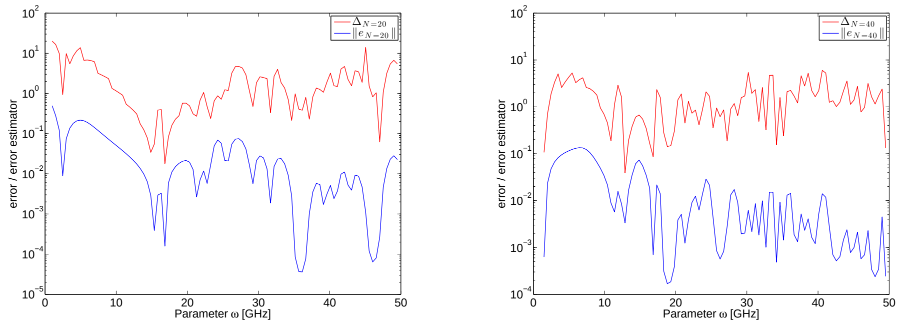
For the Greedy algorithm we used different mesh sizes that are pre-defined by COMSOL. The number of DOFs vary from 7,818 to 290,673. In Figure 3 we show the convergence of the error during the Greedy algorithm when using the real error (strong Greedy) as well as the error estimator as error indicator. There are several versions of the SCM for complex-valued problems, e.g. [10, 18, 19]. As proposed in [26], we instead obtained the coercivity constant via an interpolation method based on precalculated values of $\alpha(\omega)$.

We observe an exponential decay of the error. As expected, the decay is faster for Model 1 than for Model 2 which is the more sophisticated one. For Model 1 the reduced linear systems became unstable when creating reduced bases with more than 50 basis functions. At this point, the error measured in the dG-norm $\|\cdot\|_{\mathrm{DG}}$ was below 10^{-4}. For Model 2 the bases became unstable for $N > 73$ where the maximum error over the sampling set was at about 10^{-3}. The offline phase took between 3 minutes for the coarse meshes and 18 hours for the finer meshes. In Table 2 we show the online times needed for performing a detailed respectively reduced simulation. The online speedup factors vary from 112 to 75,859. One can also observe that for both models the error decays slightly faster for the finer discretized versions. This means that the physics of the problem can be represented better with a finer mesh and can therefore also be reproduced better with a reduced solution.

In order to verify the robustness of our RB approach, we finally investigate the dependency over the whole frequency range. In Figure 4, we show the error and error estimator over the whole parameter domain for the two models with 92,481 DOFs and reduced bases with dimension N=20 (Model 1) resp. N=40 (Model 2). As desired, the error estimator resembles the behavior of the true error and both stay in an acceptable range, which shows the robustness of our dG discretization as well as of the RBM.

APPENDIX A. DESCRIPTION OF THE COMSOL MODEL

In this section we provide details about the COMSOL model which we used for our numerical experiments. We use the *Weak Form PDE* from *Mathematics* $\rightarrow$ *PDE interfaces* and define three *Dependent variables* E1, E2, and E3. For reasons



(a) Model 1, N=20 (b) Model 2, N=40

FIGURE 4. Error and error estimator for Model 1 with basis length N=20 (left) and Model 2 with basis length N=40 (right)

	Model 1 ($N = 50$)				Model 2 ($N = 73$)			
# DOFs	290,673	92,481	47,598	7,818	290,673	92,481	47,598	7,818
t_{detailed}[sec]	158.767	14.872	4.923	0.258	157.087	14.710	4.831	0.270
t_{reduced}[sec]	0.002	0.002	0.001	0.001	0.002	0.002	0.002	0.002
speedup	67,449	9,487	3,648	219	75,859	7,280	2,534	112

TABLE 2. Runtimes

of compactness we only show the first entry here when dealing with vectors or multiple similar expressions.

Table 3 shows the weak expressions (1), (2) and (3) defined by the *weak form PDE node* on the whole domain. The jump and average terms that are valid on mesh boundaries are given in Table 4. The boundary conditions on the two interior blocks are given in Table 5 whereas the exterior boundary condition is given in Table 6. We present the *Parameters* and *Variables* which we defined in Table 7, respectively, Table 8. A *physics-controlled mesh* was used, the *Element size* varied from *coarser* to *finer*. In all cases we used *Discontiuous Lagrange* as *shape function type*.

Weak Form	**Value**
weak (1)	1/mu *rotE1*rotv1 - omega*omega*epsilon *E1*test(E1) -i*omega*sigma*E1*test(E1)- i*omega*sqrt(epsilon0)*Ja1*test(E1)
weak (2),(3)	accordingly

TABLE 3. Weak Form PDE, defined on the whole domain

REFERENCES

[1] R.A. ADAMS AND J.J.F. FOURNIER, *Sobolev Spaces*, Academic Press, New York, 2003.

Weak Contributions on Mesh Boundaries	**Value**
Weak expression (1)	-jumpE1 * 1/mu * avrotv1 -1/mu *avrotE1 *jumpv1 +tau/h*jumpE1*jumpv1
Weak expression (2),(3)	accordingly

Table 4. Weak Contributions on Mesh Boundaries, defined on the whole domain.

Weak Contribution	**Value**
Weak expression (1)	-(ny*E3-nz*E2)*1/mu*rotv1 -1/mu*rotE1*(ny*test(E3)-nz*test(E2)) +tau/h*(ny*E3-nz*E2)*(ny*test(E3)-nz*test(E2))
Weak expression (2),(3)	accordingly

Table 5. Weak Contributions on Interior Block Boundaries, defined on the two interior blocks

Weak Contribution	**Value**
Weak expression (4)	- (g1*vT1+g2*vT2+g3*vT3) / mu0
Weak expression (5)	-i*omega*lambda *sqrt(epsilon0/mu0) * (ET1*vT1 + ET2*vT2 + ET3*vT3)

Table 6. Exterior Boundary Conditions, defined on the outer boundary

Global Parameters	**Expression**	**Description**
omega	1*1e9	frequency, $\omega \in [1 \cdot 1e9, 50 \cdot 1e9]$
epsilon0	8.854*1e-12	electric permittivity in vacuum
Ja1	1,00E+004	electric current density, first entry
Ja2	1,00E+004	electric current density, second entry
Ja3	1,00E+004	electric current density, third entry
mu0	4*pi*1e-7	magnetic permeability in vacuum
tau	1000/mu0	penalty parameter for dG formulation
lambda	1	$\lambda > 0$, intensity of impedance

Table 7. Parameters

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Variables	**Expression**	**Description**
rotE1	E3y-E2z	$\nabla_h \times E$, first entry
rotv1	test(E3y)-test(E2z)	$\nabla_h \times v$, first entry
jumpE1	dny*down(E3)-dnz*down(E2) +uny*up(E3)-unz*up(E2)	$[\![E]\!]$ on $\mathcal{F}_h^I$, first entry
jumpv1	dny*test(down(E3))-dnz*test(down(E2)) +uny*test(up(E3))-unz*test(up(E2))	$[\![v]\!]$ on $\mathcal{F}_h^I$, first entry
avrotE1	0.5*(up(E3y)-up(E2z) +down(E3y)-down(E2z))	$\{\!\{\nabla_h \times E\}\!\}$, first entry
avrotv1	0.5*(test(up(E3y))-test(up(E2z)) +test(down(E3y))-test(down(E2z)))	$\{\!\{\nabla_h \times v\}\!\}$, first entry
ET1	nz*E1*nz-nx*E3*nz -nx*E2*ny+ny*E1*ny	E_T, first entry
vT1	nz*test(E1)*nz-nx*test(E3)*nz -nx*test(E2)*ny+ny*test(E1)*ny	v_T, first entry
mu	mu0*(1+distance_midpoint)	magnetic permeability
sigma	0.01*(1+0.5*x^2)	electric conductivity
epsilon	epsilon0*(1+distance_000)	electric permittivity
distance _midpoint	sqrt((0.5-x)^2 + (0.5-y)^2 + (0.5-z)^2)	$\|\boldsymbol{x} - (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T\|$
distance_000	sqrt(x^2+y^2+z^2)	$\|\boldsymbol{x}\|$
g1, g2, g3	t1x, t1y, t1z	$g \in L_t^2(\Sigma)^3$

TABLE 8. Variables

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