A CONVERGENT ADAPTIVE WAVELET-ROTHE METHOD FOR ELASTOPLASTIC HARDENING *

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Abstract

This paper is concerned with the development of adaptive wavelet methods for the hardening problem in elastoplasticity. We propose a Rothe method using some implicit scheme in time. Then, we consider a standard elastic predictor-plastic corrector method. The (non-linear) correction is performed by some convergent scheme such as a Newton-Raphson iteration or suitable modifications of it. Thus, it remains to solve Helmholtz-type problems with varying right-hand sides. These are solved by the convergent adaptive wavelet method introduced recently by Cohen, Dahmen and DeVore.

In the plastic correction, the trial strain might have to be corrected according to pointwise formulated hardening conditions. We propose an adaptive corrector method based on biorthogonal B-splines. This allows a fully adaptive stress correction. We obtain an overall convergent method. Some preliminary numerical results are presented.

Keywords: elastoplasticity, wavelets, adaptive methods
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1 INTRODUCTION

This paper draws on two sources of motivation. Firstly, we have investigated Wavelet- and Multiresolution-Galerkin methods for elastoplastic hardening in the past, [1, 2, 3]. These investigations were based on a fixed wavelet (or multiresolution) discretization in space combined with a finite difference method in time. For the plastic correction, we used interpolatory wavelets and suitable change of bases. It turned out that the solution of such problems exhibit highly localized singularities which calls for some kind of adaptation. To our knowledge there is no proof that adaptive methods converge faster then linear methods for these kind of non-linear problems. Such results are e.g. known for elliptic problems on polygonal domains [4] and 1D hyperbolic conservation laws [5].

The second motivation comes from recent progress in the development of adaptive wavelet methods, [6, 7]. There, an adaptive wavelet method has been constructed that was proven to converge at an optimal rate for a large class of operator equations. Note that even the convergence is by no means automatic. Corresponding results for adaptive finite elements are quite recent and restricted to particular families of finite elements, [8, 9, 10]. Moreover, it was shown in [7] that this adaptive wavelet method is (asymptotically) optimal efficient.

As already mentioned, the investigations in [1, 2, 3] did not treat any kind of adaptivity even though the numerical experiments showed a great potential. The main problem was that the corrections of the stress within a standard elastic predictor-plastic corrector method have to be performed *pointwise*

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due to the physical formulation of the problem. This was done by interpolatory wavelets as also in [11, 12]. This is somehow unsatisfactory since all corrections can only be done on the same level as the discretization level. Obviously, a pointwise local correction should influence the approximation in particular on *higher* levels which cannot be taken into account using a fixed discretization.

The idea followed in this paper is to use a Rothe method, i.e., to discretize in time first and leave the problem in space on the continuous level. Thus the elastic predictor-plastic corrector method is still formulated on the continuous level, i.e., without any discretization in space. It turns out that this algorithm in particular requires the solution of several Helmholtz-type problems. Then, we also formulate the corrector step as a continuous problem. Thus, we are left with two kind of problems, namely

- solve the arising Helmholtz-type problems by a convergent adaptive scheme;
- construct an adaptive stress correction method.

For the first issue we use the adaptive wavelet method from [6, 7]. In view of the second issue we restrict ourselves to biorthogonal B-spline wavelets [13, 14] since they easily allow pointwise evaluation and –as we will show– also correction. Thus we completely avoid changing to interpolatory wavelets. By using this strategy, we obtain a convergent adaptive method.

We would like to mention also recent progress in the development of adaptive wavelet methods for non-linear variational problems [15]. In principle, one could try to adapt the method in [15] for the hardening problem. One should note however, that the hardening problem offers a variety of difficulties. Just to mention one of them, the yielding stresses are not known beforehand, but they have to be computed. This means that the domain in which the material is elastic and plastic, respectively, is changing in time. It is not clear to us how to handle such a problem with the method in [15]. Finally, our method also offers the possibility of using it within a known and widely accepted framework.

The paper is organized as follows: In Section 2, we recall the hardening problem under investigation. Section 3 is devoted to a review of adaptive wavelet methods and Section 4 contains the description and analysis of the new adaptive Wavelet-Rothe method. Finally, some numerical results in Section 5 show the potential of the new method.

2 THE HARDENING PROBLEM

We study the dynamic response of a straight elastoplastic rod. In this section, we briefly review the governing equations.

Referring to Fig. 2, the space variable is denoted by x, the time variable by t and $u(x,t)$ is the axial displacement of the rod. The physical problem is governed by classical relations expressing equilibrium between applied forces and induced internal stresses, compatibility between displacements and strains and the nonlinear constitutive law that relates stresses to strains. As to equilibrium one may write

$$(A\sigma)' + f = \rho\ddot{u}, \tag{2.1}$$

where $A(x)$ and $\rho(x)$ are the cross section and the mass density, $f(x,t)$ is the axial force and σ is the axial stress. Furthermore, space and time differentiation are indicated by a superposed prime and dot, respectively. The hypothesis of small displacement gradients will be made under which the compatibility condition may be written as

$$\varepsilon = u', \tag{2.2}$$

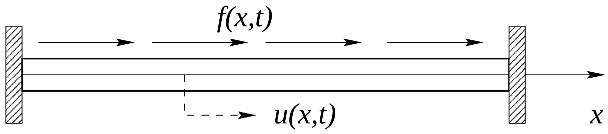

Figure 1: The rod under study

where $\varepsilon(x,t)$ is the axial strain of the rod. The elastoplastic constitutive law classically needs to be introduced in incremental form since the stress does not only depend on the current strain as it happens in the purely elastic case, but also on the entire past history of it. For clarity sake, we first remark that the purely *linear*-elastic problem is governed by a constitutive law that reads

$$\sigma = E_Y \varepsilon, \tag{2.3}$$

in which E_Y is the Young modulus. Therefore, by eliminating stress and strain in (2.1), (2.2) and (2.3), one ends up with a wave equation having the displacement u as unknown, i.e.

$$\big(E_Y A u'\big)' + f = \rho \ddot{u}. \tag{2.4}$$

We now introduce the basic incremental relationships defining the elastoplastic behavior of the rod. The main hypothesis, widely used and accepted in the literature, [16, 17], is the additive decomposition of the total strain rate $\dot{\varepsilon}$ into its elastic and plastic contributions, i.e.,

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p. \tag{2.5}$$

The stress rate $\dot{\sigma}$ may then be written in terms of any of its above described contributions by introducing the tangent modulus E_t and the plastic one E_p. These are defined by the following relations

$$\dot{\sigma} = E_t \dot{\varepsilon} = E_Y \dot{\varepsilon}^e - E_p \dot{\varepsilon}^p, \tag{2.6}$$

where Fig. 2 visualizes the introduced quantities. Notice that in Fig. 2, σ_{Y_1} is the so called (tension)

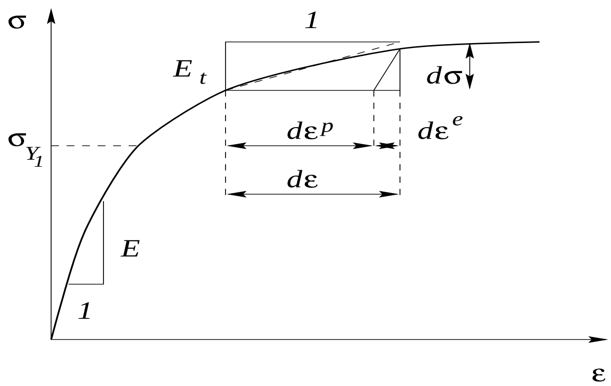

Figure 2: Stress and strain increments

yielding stress, i.e., the stress above which the material is no longer elastic but undergoes permanent, unreversible deformations. Furthermore, σ_{Y_1} is a variable itself and is to be updated at each time instant according to the so called *hardening rule*, see, e.g. [1]. From a computational point of view, the difficulty is that one does not know in advance whether a stress or strain increment will cause plastic loading or elastic unloading. We assume the rod to be stress-free for $t = 0$ and to behave elastically as long as $(t,x) \in I$ where I is the instantaneous elastic domain defined as

$$I := \{(t,x) \in [0,T] \times (0,1) : -\sigma_{Y_2}(t,x) \le \sigma(t,x) \le \sigma_{Y_1}(t,x)\} \tag{2.7}$$

In (2.7), σ_{Y_1} and σ_{Y_2} are the yielding stresses in tension and compression, respectively, which we group into the vector $\sigma_Y = (\sigma_{Y_1}, \sigma_{Y_2})$. For $t = 0$ the yielding stresses are known from experimental tests and they evolve with time following some hardening rule. Then we obtain our problem under consideration as

Problem 2.1 *For given ρ, E_Y, E_t, E_p, A and f we seek for u, ε and σ such that:*

$$\rho(x)\,\ddot{u}(t,x) - (E_Y(x)\,A(x)\,u'(t,x))' = f(t,x), \qquad (t,x) \in I, \tag{E}$$

and

$$\begin{aligned} \rho(x)\,\ddot{u}(t,x) - (A(x)\,\sigma(t,x))' &= f(t,x), \quad (t,x) \notin I, \\ u'(t,x) - \varepsilon(t,x) &= 0, \qquad\quad (t,x) \notin I, \\ \dot{\sigma}(t,x) - E_t(\sigma(t,x))\,\dot{\varepsilon}(t,x) &= 0, \qquad\quad (t,x) \notin I, \end{aligned} \tag{P}$$

Finally, we pose the following initial and boundary conditions

$$u(0,x) = u_0(x), \quad u'(0,x) = u_1(x), \qquad x \in [0,1], \tag{B_1}$$

for some given functions u_0 and u_1, as well as

$$u(t,0) = u(t,1) = 0, \qquad t \in [0,T]. \tag{B_2}$$

3 ADAPTIVE WAVELET METHODS FOR ELLIPTIC PROBLEMS

In this section, we briefly review the main ingredients from [6, 7] of adaptive wavelet methods for solving (second order) elliptic partial differential equations. Let

$$u \in H_0^1(\Omega): \qquad a(u,v) = (f,v)_{0;\Omega}, \quad v \in H_0^1(\Omega) \tag{3.1}$$

be the variational formulation of a second order elliptic boundary value problem on a bounded domain $\Omega \subset \mathbb{R}^n$, where $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is the (bounded and coercive) bilinear form associated to the pde. As usual, we associate the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$\langle Au, v \rangle := a(u,v), \qquad u,v \in H_0^1(\Omega), \tag{3.2}$$

where $\langle \cdot,\cdot, \rangle$ is the duality pairing of $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Because of the ellipticity, we have

$$\|Av\|_{-1;\Omega} \sim \|v\|_{1;\Omega}, \qquad v \in H_0^1(\Omega). \tag{3.3}$$

Here and in the sequel we use the abbreviation $A \lesssim B$ if there exists a constant $c > 0$ such that $A \le c\,B$. Moreover, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

Wavelet Bases. The next ingredient is a wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \mathcal{J}\}$ of $H_0^1(\Omega)$. Here $\mathcal{J}$ denotes an infinite set of indices of the form

$$\lambda = (j,k), \qquad |\lambda| := j \in \mathbb{N}, \tag{3.4}$$

where j denotes the *level* or *scale* of ψ_λ and k encodes information like location in space, type of wavelet etc. The wavelet system is assumed to be *localized*, i.e.

$$|\operatorname{supp} \psi_\lambda| \sim 2^{-|\lambda|} \tag{3.5}$$

and that it gives rise to a norm equivalence of the form

$$\|\boldsymbol{d}^T\Psi\|_{1;\Omega} := \Big\| \sum_{\lambda\in\mathcal{J}} d_\lambda\psi_\lambda \Big\|_{1;\Omega} \sim \left(\sum_{\lambda\in\mathcal{J}} 2^{2|\lambda|}|d_\lambda|^2 \right)^{1/2} =: \|\boldsymbol{D}\boldsymbol{d}\|_{\ell_2(\mathcal{J})}, \tag{3.6}$$

for all $\boldsymbol{d} \in \ell_2(\mathcal{J})$, where $\boldsymbol{D} := \operatorname{diag}(2^{|\lambda|})$. Finally, we assume that Ψ has *vanishing moments* of order d, i.e.,

$$\int_\Omega x^\alpha \psi_\lambda(x)\, dx = 0, \qquad 0 \le |\alpha| < d, \tag{3.7}$$

i.e., Ψ is orthogonal to algebraic polynomials up to degree $d-1$. Note that various examples of such bases even on fairly complex domains are nowadays available.

An Equivalent ℓ_2-Problem. Due to (3.3) and (3.6), it can be shown that (3.1) is equivalent to the problem

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}, \qquad \boldsymbol{A} := \boldsymbol{D}^{-1}\langle A\Psi, \Psi\rangle \boldsymbol{D}^{-1}, \qquad \boldsymbol{f} := \boldsymbol{D}^{-1}(f, \Psi)_{0;\Omega}, \qquad \boldsymbol{u} := \boldsymbol{D}\boldsymbol{d}, \tag{3.8}$$

where $u = \boldsymbol{d}^T\Psi$ is the solution of (3.1) and

$$\operatorname{cond}(\boldsymbol{A}) < \infty, \tag{3.9}$$

i.e., the original problem (3.1) is transformed into a well-conditioned problem in $\ell_2(\mathcal{J})$. Moreover, due to the symmetry of $a(\cdot,\cdot)$, also $\boldsymbol{A}$ is symmetric.

Infinite Dimensional Iterations. Ignoring for a moment that (3.8) is an infinite-dimensional problem and that $\boldsymbol{A}$ is an infinite operator, (3.9) can be solved by a Richardson-type iteration. Starting by some $\boldsymbol{d}^{(0)}$, the iteration

$$\boldsymbol{d}^{(k+1)} = \boldsymbol{d}^{(k)} + \alpha(\boldsymbol{f} - \boldsymbol{A}\boldsymbol{d}^{(k)}) \tag{3.10}$$

converges to $\boldsymbol{d}$ for appropriately chosen damping parameter $\alpha \in \mathbb{R}^+$. Moreover, (3.10) yields a fixed reduction of the error in each step.

Approximate Applications. As already said, in the present form (3.10) cannot be executed on a computer since it involves two possibly infinite quantities, namely the vector $\boldsymbol{f}$ and $\boldsymbol{A}\boldsymbol{d}^{(k)}$ which in general is infinite even if the input vector $\boldsymbol{d}^{(k)}$ is finite. In [6, 7] two routines have been introduced to solve this problem.

Algorithm 3.1 *The routine* **RHS** $[\boldsymbol{f}, \varepsilon] \to [\boldsymbol{f}_\varepsilon]$ *determines for any given desired tolerance* $\varepsilon > 0$ *an approximation* $\boldsymbol{f}_\varepsilon$ *to* $\boldsymbol{f}$ *of compact support such that* $\|\boldsymbol{f} - \boldsymbol{f}_\Lambda\| < \varepsilon$.

For a sequence (or vector) $\boldsymbol{c} \in \ell(\mathcal{J})$ the *support* is the set of indices corresponding to non-vanishing coefficients, i.e.,

$$\operatorname{supp} \boldsymbol{c} := \{k \in \mathcal{J} : c_k \neq 0\}.$$

Algorithm 3.2 *The routine* **APPLY** $[\boldsymbol{A}, \varepsilon, \boldsymbol{v}] \to [\boldsymbol{w}_\varepsilon]$ *determines for any given desired tolerance* $\varepsilon > 0$ *and any* compactly supported *input* $\boldsymbol{v}$ *a compactly supported* $\boldsymbol{w}_\varepsilon$ *such that* $\|\boldsymbol{A}\boldsymbol{v} - \boldsymbol{w}_\varepsilon\| \leq \varepsilon$.

Replacing in (3.10) $\boldsymbol{f}$ by **RHS** $[\boldsymbol{f}, \varepsilon] \to [\boldsymbol{f}_\varepsilon]$ and $\boldsymbol{A}\boldsymbol{d}^{(k)}$ by the result of **APPLY** $[\boldsymbol{A}, \varepsilon, \boldsymbol{v}]$ for appropriately chosen tolerances ε leads to a convergent algorithm, [7]. Again, it can be shown that the error decreases in each step by a fixed amount. However, the size of the supports, i.e., the number of degrees of freedom is not yet under control. One would like to obtain an 'optimal' method, where optimal means an optimal balance of error and degrees of freedom. It turns out to be necessary to do a 'clean up' step from time to time which is realized by the following routine.

Algorithm 3.3 *The routine* **COARSE** $[\boldsymbol{v}, \varepsilon] \to [\boldsymbol{w}_\varepsilon]$ *determines for any given tolerance* ε *and any finitely supported input vector* $\boldsymbol{v}$ *such that* $\|\boldsymbol{u} - \boldsymbol{v}\| \leq \varepsilon$ *(where* $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$*) a vector* $\boldsymbol{w}_\varepsilon$ *of (almost) shortest support such that* $\|\boldsymbol{u} - \boldsymbol{w}_\varepsilon\| \leq 4\varepsilon$.

The idea is to perform a **COARSE** after a fixed number of iterations of the Richardson iteration. This will increase the error by a small amount but will reduce the number of unknowns. The corresponding routine is called **SOLVE** $[\boldsymbol{A}, \boldsymbol{f}, \varepsilon] \to [u_\varepsilon]$. Its optimality is reflected by the following result, [7].

Theorem 3.4 *The routine* **SOLVE** $[\boldsymbol{A}, \boldsymbol{f}, \varepsilon] \to [\boldsymbol{u}_\varepsilon]$ *determines for any given target accuracy* ε *a finitely supported approximation* $\boldsymbol{u}_\varepsilon$ *in a fixed number of steps such that* $\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\| \leq \varepsilon$. *Moreover, if for the error of the* best N-term approximation *we have*

$$\varrho_N(\boldsymbol{u}) := \inf\{\|\boldsymbol{u} - \boldsymbol{v}_N\| : \#\operatorname{supp}(\boldsymbol{v}_N) = N\} \lesssim N^{-s},$$

then we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_N\| \lesssim N^{-s},$$

where $N = \varepsilon^{-1/s}$.

4 THE ADAPTIVE WAVELET-ROTHE METHOD

In this section, we are going to describe the numerical treatment of Problem 2.1 in terms of the proposed adaptive Wavelet-Rothe Method.

4.1 Elastic Prediction and Plastic Correction

Let us start by the elastic predictor-plastic corrector strategy. There exist several cases to be handled, all of which are particular cases of the complementarity rule. For clarity sake we hereafter focus on one of them, i.e., the case of plastic loading or elastic unloading in tension. Let the stress $\sigma(t, x)$ be on the boundary of the instantaneous elastic domain. Given is also $u(t, x)$ for some time t. Then, for a given $\Delta t > 0$, we compute the *elastic predictor* $u^*(t + \Delta t, x)$ by solving the problem (E) (since we always consider the initial and boundary conditions, we will omit referring to them all the time).

Then, by using the second equation in (P) (see also (2.2)), we compute ε^* and then, by using (2.3), we obtain the elastic trial stress σ^*. If $\sigma^* < \sigma_{Y_1}$ then an elastic unloading has taken place and therefore $\sigma(t + \Delta t, x) = \sigma^*$ and no correction is required. If conversely $\sigma^* > \sigma_{Y_1}$, a strain-driven correction scheme is used. In mathematical terms this requires the solution of a non-linear problem. A Newton method is often too costly so that one typically choses a method that only requires the evaluation of the function (and not its derivative). The particular choice of the method is not important for what follows. However, we have to assume that the method of choice converges. One frequently used example is a modified *Newton-Raphson* iteration which is illustrated in Fig. 3. Such a method is also known as *parallel modified Newton method.*

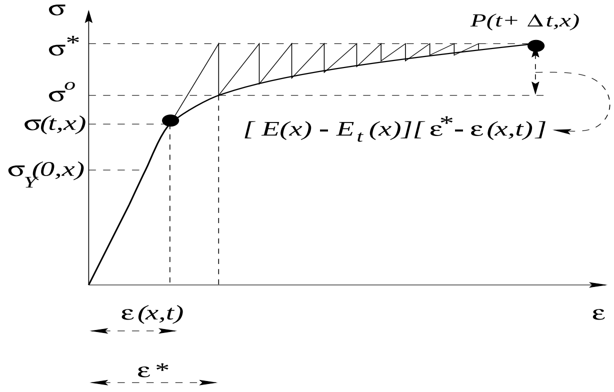

Figure 3: The elastic-predictor plastic-corrector procedure

Independent of the particular choice of the correction scheme, in Fig. 3, the stress $\sigma^* = \sigma(t, x) + E_Y(x)[\varepsilon^*(t, x) - \varepsilon(t, x)]$ is *not* updated and will therefore coincide with the value $\sigma(t + \Delta t, x)$. A lack of consistency may however be noticed between the so computed stress σ^* and the one which is compatible with the actual stress-strain curve, i.e., $\sigma^o(t, x) = \sigma(t, x) + E_t(x)\varepsilon^*(t, x)$. Thus, the quantity $A(x)[\sigma^* - \sigma^o](t, x)$ becomes a virtual, un-equilibrated force that is brought to the right-hand side of eq. (E) so as to allow the computation of a further update for the displacement and for the strain ε. The procedure ends when the actual stress-strain curve joins the plateau σ^* where the solution in terms of stress, strain and displacement is attained (point $P(t + \Delta t, x)$ in Fig. 3). Notice that σ^* is not only the stress $\sigma(t + \Delta t, x)$ but also the new value of the yielding stress σ_{Y_1} to be used for the subsequent stress computation.

When linear isotropic hardening is adopted, the whole stress correction procedure is governed by eight alternative cases. We will hereafter describe the tension cases in detail. The remaining (compression) cases can easily be obtained by symmetry arguments and by replacing σ_{Y_1} by σ_{Y_2}. We will be using discrete-time notations and denote by σ_n the computed stress value at time t_n, by σ^*_{n+1} the trial stress value at time t_{n+1} and by σ_{Y_1}, σ_{Y_2} the current values of the yielding stress. The value of the corrected stress is denoted by σ°.

1. $0 \leq \sigma_n \leq \sigma^*_{n+1} \leq \sigma_{Y_1}$:
   If σ_n and the trial stress σ^*_{n+1} are both below the yielding stress σ_{Y_1}, we are still in the elastic range, i.e., no correction has to be performed. This implies $\sigma^\circ := \sigma^*_{n+1}$ and the yielding stress is not changed.

2. $0 < \sigma_n < \sigma_{Y_1} < \sigma^*_{n+1}$:
   In the case that σ_n is below σ_{Y_1}, but the elastic trial σ^*_{n+1} is above, then there is a transition from

the elastic to the plastic regime and the following correction has to be made. Using the third equation in (P), one projects the non-equilibrated part of the stress, namely $\sigma^*_{n+1} - \sigma_{Y_1}$ to the stress-strain curve as can be seen in Fig. 4. To be precise, setting $r = (\sigma^*_{n+1} - \sigma_{Y_1})\,(\sigma^*_{n+1} - \sigma_n)^{-1}$, we obtain $\sigma^\circ = \sigma_{Y_1} + rE_t\varepsilon_n$. Note that the stress difference $\sigma_{Y_1} - \sigma_n$ (and the corresponding strain) still belongs to the elastic range and needs not to be corrected. The yielding stress is updated to σ°.

3. $\sigma^*_{n+1} \geq \sigma_n > \sigma_{Y_1}$:
In this case, σ_n is in the plastic range and σ^*_{n+1} remains plastic. As in the latter case, the non-equilibrated part of the stress needs to be corrected as indicated by Fig. 4, and we obtain in this case $\sigma^\circ = \sigma_n + E_t(\varepsilon^p_{n+1} - \varepsilon^p_n)$ which is also the new yielding stress.

4. $-\sigma_{Y_2} \leq 0 \leq \sigma^*_{n+1} < \sigma_n$:
In this case, unloading is performed and we reenter (or stay in) the elastic regime. Hence, we have to check that σ^*_{n+1} is larger than the compression yielding stress (recall, that $\sigma_{Y_2} > 0$) and no correction is made, $\sigma^\circ = \sigma^*_{n+1}$.

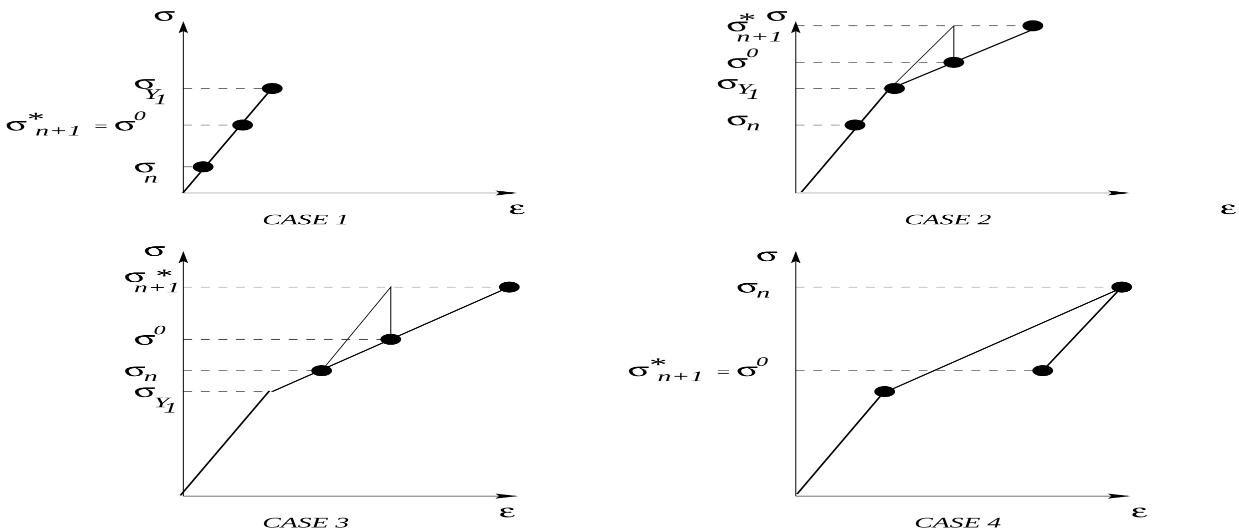

Figure 4: Hardening cases.

We may summarize the above discussion in the following

Algorithm 4.1 ***(Abstract Stress Correction)*** *For a given elastic trial* $(\sigma_{\text{trial}}, \varepsilon_{\text{trial}}, u_{\text{trial}})$*, set*

$$\sigma = \sigma^* := \sigma_{\text{trial}}, \quad \varepsilon_0 := \varepsilon_{\text{trial}}, \quad u_0 := u_{\text{trial}}.$$

For $k = 0, 1, 2, \ldots$ *do*

a.) *Compute the un-equilibrated stress* σ^0_k *(pointwise) corresponding to the 8 different yielding cases;*

b.) *Compute* Δu_k *by solving* (E) *with the right-hand side*

$$A(x)[\sigma - \sigma^0_k];$$

c.) *Set* $\Delta\varepsilon_k := \frac{d}{dx}\Delta u_k$, $\varepsilon_{k+1} := \varepsilon_k + \Delta\varepsilon_k$; $\sigma_{k+1} := E_Y \varepsilon_{k+1}$
If $\sigma^* - \sigma_{k+1}$ *'sufficiently small', set* $\varepsilon^* = \varepsilon := \varepsilon_{k+1}$, $u := u_k + \Delta u_k$, *STOP.*

Note that this algorithm is still abstract in the sense that we did not specify so far how to compute the un-equilibrated stress in step a.).

4.2 A Rothe-type Method

Introducing any implicit discretization in time e.g. a Newmark scheme as used in [1, 2, 3], solving the elastic problem (E) requires to solve a Helmholtz-type problem of the form

$$-\mu\, u^n_{xx}(x) + \nu\, u^n(x) = h^n(x), \qquad x \in [0,1], \tag{E_t}$$

where $u^n(x)$ is an approximation for $u(t_n, x)$ (to be determined), $0 = t_0 < t_1 < \cdots t_n < T$ are the time steps and h^n is a known function (maybe also depending on approximations on previous time steps). The constants $\mu, \nu \in \mathbb{R}$ are known and depend on the particular choice of the discretization in time. Note that (E_t) is still a continuous problem in space, i.e., it is posed for all $x \in [0,1]$. We will keep this in order to introduce an adaptive scheme.

Now two remaining issues have to be solved, namely

- construct a convergent adaptive solver for the elastic trial and for the stress correction;
- construct a concrete scheme for the stress correction in Algorithm 4.1.

These issues will be discussed in the following two subsections.

4.3 The Adaptive Wavelet Method for the Elastic Problem

We have seen that we have to solve problems of the form (E_t). The idea is to use the framework presented in Section 3 for this purpose. In fact, choosing an appropriate wavelet bases $\Psi = \{\psi_\lambda : \lambda \in \mathcal{J}$ of $H^1_0(0,1)$ (see e.g. [14]), we can in fact transform (E_t) into a well-conditioned problem in $\ell_2(\mathcal{J})$ (see (3.8)):

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}, \qquad \boldsymbol{A} := \boldsymbol{D}^{-1} a(\Psi,\Psi)\boldsymbol{D}^{-1}, \boldsymbol{f} = \boldsymbol{D}^{-1}(h^n, \Psi)_{0;(0,1)},$$

where $a(u,v) := \mu(u', v')_{0;(0,1)} + \nu(u,v)_{0;(0,1)}$ is the bilinear form induced by (E_t). Thus invoking the above mentioned routines **APPLY** and **RHS** into (3.10) gives an optimal convergent adaptive solution method, see Theorem 3.4. Note that a corresponding implementation and numerical tests are documented in [18].

4.4 Adaptive Stress Correction and B-Spline Wavelet Bases

So far, we did not specify which kind of wavelet bases we are going to use. Since we need pointwise calculation and correction for the stress correction, we restrict ourselves for the remainder of this paper to biorthogonal B-spline wavelets (see [13] and [14] for an adaptation to bounded intervals). Due to the relation between displacement and stress/strain via a derivative we use splines of order $d+1$ for the displacement and of order d for the strain and stress.

We denote by $\Theta = \{\vartheta_\mu : \mu \in \mathcal{K}\}$ the biorthogonal B-spline system of order $d+1$ for the displacement and by $\Psi = \{\psi_\lambda : \lambda \in \mathcal{J}\}$ the biorthogonal B-spline system of order d for strain and stress. Let us assume that we are given an adaptive solution of (E_t) produced by **SOLVE** of the following form

$$u_{\tilde{\Lambda}} = \sum_{\mu \in \tilde{\Lambda}} u_\mu \vartheta_\mu,$$

then, we can easily determine the corresponding strain

$$\varepsilon_\Lambda = u'_{\tilde{\Lambda}} = \sum_{\lambda \in \Lambda} e_\lambda \psi_\lambda,$$

where the relation between the coefficients u_μ and e_λ is explicitly known, [19, 3]. Note that in general $\Lambda \neq \tilde{\Lambda}$, but the sizes are comparable. Next, we compute

$$\sigma_\Lambda = E_Y \varepsilon_\Lambda = \sum_{\lambda \in \Lambda} s_\lambda \psi_\lambda. \tag{4.1}$$

In order to perform the computation of the un-equilibrated stress, we perform for each wavelet in the linear combination in (4.1) *one* step of the fast wavelet transform, i.e., we represent each wavelet as a linear combination of B-spline of the next higher level. Doing this for all functions in (4.1), we end up with an expansion of the following form

$$\sigma_\Lambda = \sum_{(j,k) \in \mathcal{J}_\Lambda} c_{j,k}\, \varphi_{j,k}. \tag{4.2}$$

Since this is a fast wavelet transform applied to each wavelet, we have $|\mathcal{J}_\Lambda| \sim |\Lambda|$, i.e., (4.2) can be computed efficiently. Note that the representation in (4.2) is a multi-scale scaling function representation rather then a wavelet expansion. In particular, the involved functions $\varphi_{j,k}$ may not be linearly independent. This, however, is not important for our purpose. At this point, we use the fact that each $\varphi_{j,k}$ is a spline (either a scaled and shifted version of a cardinal B-spline φ or a fixed linear combination due to the adaptation to the boundary,[14]). In any case, there is a set of knots $\Xi_\Lambda = \{0 < \xi_1 < \ldots < \xi_{\#\mathcal{J}_\Lambda}\}$ corresponding to $\mathcal{J}_\Lambda$ such that σ_Λ is a polynomial of degree d in each subinterval $[\xi_i, \xi_{i+1}]$.

Note that the yielding function (which is an unknown itself) is also given in terms of Ψ, i.e., as a piecewise polynomial of the *same* degree. Hence we have reduced the problem now to the following question: Given a polynomial $p \in \mathcal{P}_d$ on an interval $[a,b]$, determine all $t \in [a,b]$ such that $p(x) < 0$ (by subtracting the yielding function). This however, is no big problem, since one can easily determine all roots of the polynomial and hence all subintervals in which the polynomial is negative. Hence we can determine for each σ_Λ a subset $I_\Lambda \subset [0,1]$ on which σ_Λ does not satisfy the yielding condition.

Now, the current stress might have to be corrected corresponding to the 8 different hardening cases. Again, this has to be performed pointwise or even interval-wise. Assume that we have detected an interval $[c,d] \subset [a,b]$ in which $p(t) < 0$. Then, we can construct a B-Spline of the *same* order d that coincides with p on $[c,d]$ and add it to p, so that $p_{|[c,d]} \equiv 0$ in order to respect the hardening rule. Hence, we obtain the un-equilibrated stress as

$$\sigma^0_\Lambda = \sum_{(j,k) \in \mathcal{I}_\Lambda} c^0_{j,k}\, \varphi_{j,k},$$

where in general $\mathcal{I}_\Lambda \supseteq \mathcal{J}_\Lambda$ and $c^0_{j,k} \neq c_{j,k}$, but determined by the hardening rules.

The next step is again the adaptive solution of (E_t) using $A(x)(\sigma - \sigma^0_\Lambda)$ as a right-hand side. This amounts computing integrals of the form

$$\int_0^1 A(x)(\sigma(x) - \sigma^0_\Lambda(x))\psi_\lambda(x)\,dx.$$

This, however, is straightforward since all involved quantities are in fact B-splines (or –as for $A(x)$– can at least be approximated by B-splines) so that a routine like **RHS** (see Algorithm 3.1) can easily be realized.

4.5 Conclusions

Let us summarize the above algorithm and collect some statements concerning convergence and efficiency. As already mentioned, the routine **SOLVE** is proven to converge and even to be optimal (see Theorem 3.4). This means that our new adaptive elastic predictor-plastic corrector converges provided that the iteration for solving the non-linear stress correction converges.

We do not focus on the question of choosing an optimal iteration here but devote this to a forthcoming paper. We remark that the above modified Newton-Raphson process for the stress correction can also be reviewed in the framework of Kuhn-Tucker optimality conditions of a constrained convex optimization problem, [17]. The initial value for the iterarion is obtained by the elastic trial step. Then, the iterative correction method characterizes the solution as the closest point projection of the trial state onto the so called *yield surface* (which is related to the constrain of the optimization problem). The projection is considered with respect to some suitably chosen energy, which coincides with the functional to be optimized. In this framework, it is possible to proof the convergence of the stress correction algorithm, at least for the hardening case considered in this paper.

We do not have any result concerning the convergence rate so far. This is a delicate task indeed. First of all, one would need a precise estimate for the number of steps in the non-linear correction iteration which of course depends on the particular choice of the method. The main issue is however to estimate the number of terms that need to be added in a stress correction. This corresponds to local smoothness estimates for the current stress as well as the yielding stress. Since the physical formulation of the problem requires pointwise considerations, one would need estimates in L_∞. We are not aware of any results in this direction.

Finally, we do not have a complete implementation of the proposed method at hand at this time. This is also due to the fact that no optimized code for **SOLVE** is available so far (at least not to our knowledge). By 'optimized' we mean that all routines are implemented according to the optimality requirements. The main issue for the optimality is here the computation of the entries of the stiffness matrix and of the right-hand side. We will report on this somewhere else.

5 PRELIMINARY NUMERICAL RESULTS

In this section, we report on some preliminary numerical tests that show the *potential* of the method. The results have been obtained by a 'semi-adaptive' method which will be explained next.

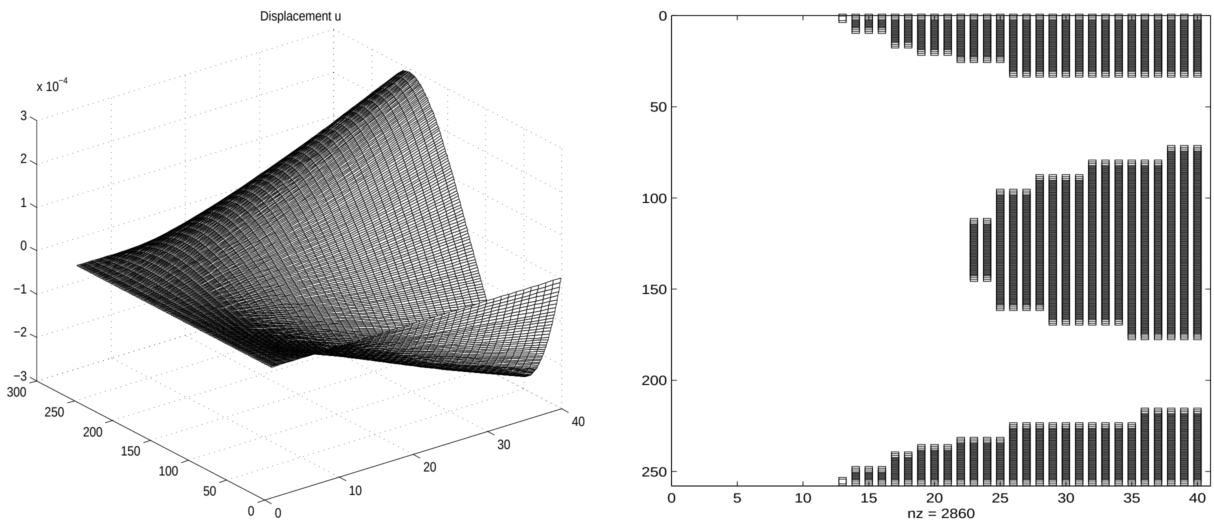

Figure 5: Displacement (left) and plastic indicators (right) for the numerical experiment.

Data. In order to separate effects due to the boundary conditions, we have chosen the input data in such a form that we obtain plastic zones starting at the boundaries and a clearly separated plastic zone in the interior of the interval. The test data have been chosen as

$$T = 0.25, \qquad \Delta t = 0.01, \qquad \sigma_{Y_1} = \sigma_{Y_2} = 1600,$$

$$A(x) \equiv 100, \qquad \rho(x) \equiv 7.85, \qquad E_Y = 2100000,$$

$$E_t = 200000, \qquad f(t,x) := a\tfrac{t}{T}(x - \tfrac{1}{2}), \qquad a = 5 \cdot 10^7.$$

The arising plastic indicators are shown in Fig. 5 (right) whereas the displacement is visualized on the left in Fig. 5. Note that all hardening cases mentioned above in fact appear.

By 'semi-adaptive' we mean that we fixed a highest level of resolution, say J and considered a Multiresolution-Galerkin method using multiresolution spaces up to that level. Then, we performed a threshold in order to determine the significant coefficients. The main goal of the numerical experiments is twofold. Firstly to show the adaptive potential, i.e., to show that only a few wavelets suffice to represent the solution up to a desired tolerance. Secondly, we would like to show that the transitions between elastic and plastic regions can in fact be detected from the wavelet coefficients. In order to so, referring to (3.6), we have considered the weightened sequence of wavelet coefficients

$$c_\lambda := 2^{2|\lambda|} d_\lambda \tag{5.1}$$

whose ℓ_2-norm is equivalent to the H^2-norm of the corresponding function (if the wavelets are sufficiently smooth). Hence, we investigate the size of these coefficients c_λ in order to resolve local effects. In Fig. 6, the scaled wavelet coefficients are shown for the case $d = 4$, $\tilde{d} = 8$, respectively. Starting from the upper left picture which corresponds to $t = 0.1$ we show column-wise the evolution until $t = T = 0.25$ in the lower right corner.

As we see, the scaled wavelet coefficients reflect the *interface* of the plastic region which gives rise to a sharp and sparse description of the hardening process. This shows that wavelet coefficients of high order discretizations give a sharp description of plastic zones which strongly indicates the potential for adaptive wavelet methods.

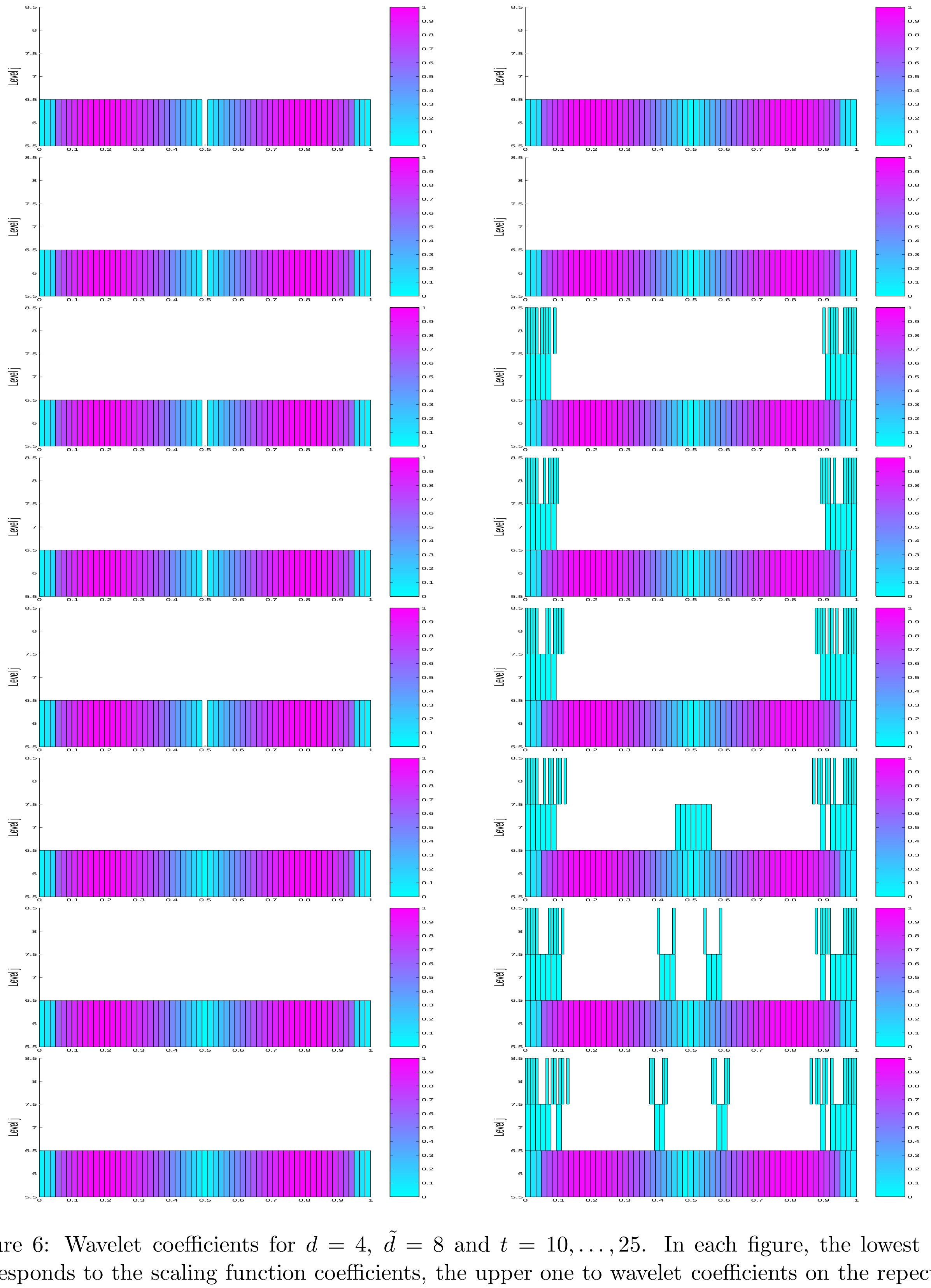

Figure 6: Wavelet coefficients for $d = 4$, $\tilde{d} = 8$ and $t = 10, \ldots, 25$. In each figure, the lowest row corresponds to the scaling function coefficients, the upper one to wavelet coefficients on the repective levels.

6 SUMMARY AND OUTLOOK

We have constructed a convergent adaptive Wavelet-Rothe method for the hardening problem in elastoplasticity. It is based upon a Rothe method for handling the time variable and a standard elastic predictor-plastic corrector formulation. The latter one has been modified here and was posed on a continuous level, i.e., without applying a discretization in space. For the arising linear problems, we use existing convergent adaptive wavelet methods and for the stress correction we propose an adaptive procedure based on biorthogonal B-spline wavelets.

We presented some preliminary numerical results indicating the potential of the method. A complete and optimized implementation of the proposed method is lacking so far. Also the optimal choice of the non-linear solver for the correction is subject to further research.

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