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Wavelets in Image Compression and Numerical Simulation
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Exercise Sheet 5

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In the theory and construction of MRAs a frequently used tool are Fourier transforms and trigonometric series. Analyzing FTs of scaling functions yields both sufficient and necessary conditions for existence, uniqueness, stability and provides a tool for constructing (bi)orthogonal MRAs. In the following exercises we will see examples for the application of FTs to MRAs.

Problem 1 (Stability) (15 Points)

Let $\boldsymbol{c} \in \ell_2(\mathbb{Z})$ be a sequence. We define its *symbol* as

$$c(\xi) := \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi} \in \mathbb{C}, \quad i = \sqrt{-1},\ 0 \le \xi \le 2\pi.$$

A function $\varphi \in L_2(\mathbb{R})$ is said to have *stable shifts*, if

$$c \sum_k |c_k|^2 \le \| \sum_k c_k \varphi(\cdot - k)\|_0^2 \le C \sum_k |c_k|^2,$$

for some constants $0 < c \le C < \infty$ and any $\boldsymbol{c} \in \ell_2(\mathbb{Z})$. Finally, recall the definition of the Fourier transform for a function $f \in L_2(\mathbb{R})$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx,\ \hat{f} \in L_2(\mathbb{R}).$$

Show that

(a) $\varphi \in L_2(\mathbb{R})$ has stable shifts if and only if $\exists\ 0 < c \le C < \infty$

$$c \le \sum_k |\hat{\varphi}(\xi + 2k\pi)|^2 \le C, \quad 0 \le \xi \le 2\pi.$$

Hint: For a given $\boldsymbol{c} \in \ell_2(\mathbb{Z})$, consider its symbol

$$c(\xi) = \sum_k c_k e^{-ik\xi}.$$

Consider $\int_0^{2\pi} |c(\xi)|^2 d\xi$. How is $\sum_k c_k\varphi(x-k)$ related to $c(\xi)$? To show the "only if" part, start by showing that $\sum_k |\hat{\varphi}(\xi + 2k\pi)|^2$ is a trigonometric series (symbol) and deduce a contradiction (or contraposition). To that end consider the autocorellation function from (b) and use the Poisson summation formula.

(b) If $\varphi, \tilde{\varphi} \in L_2(\mathbb{R})$ are biorthogonal, they have stable shifts.

Hint: Consider the function

$$a(x) = (\varphi, \tilde{\varphi}(\cdot - x))_0,$$

and accordingly the *autocorrelation function*

$$\omega(\xi) := \sum_k a(k) e^{-ik\xi}.$$

You will need the Poisson summation formula to show $\omega(\xi) = \ldots$, then use biorthogonality and (a).

Problem 2 (Moments) **(10 Points)**

Let $\varphi \in L_2(\mathbb{R})$ be a compactly supported refinable function with finite mask $\boldsymbol{a}$ and let $a : \mathbb{R} \to \mathbb{C}$ be defined by

$$a(\xi) := \frac{1}{2} \sum_k a_k e^{-ik\xi}.$$

The k-*th discrete moment* is defined via

$$m_k := \frac{1}{2} \sum_l l^k a_l.$$

The k-*th continuous moment* is defined via

$$\mu_k := \int_{\mathbb{R}} x^k \varphi(x) dx.$$

Show

(a) $m_k = i^k D^k a(0)$ and $\mu_k = \sqrt{2\pi} i^k D^k \hat{\varphi}(0)$.

(b) The continuous and discrete moments are related by

$$\mu_k = 2^{-k} \sum_{t=0}^{k} \binom{k}{t} m_{k-t} \mu_t.$$

Hint: Apply the FT to the refinement equation and use (a).

(c) Define

$$\mu^j_{r,k} = \int_{\mathbb{R}} x^r \varphi_{[j,k]}(x) dx.$$

The refinable function φ is given only implicitly by its finite mask $\boldsymbol{a}$. Your task is to compute the moments $\mu^j_{r,k}$ for any $r, k, j \in \mathbb{N}_0$.

Hint: In the end you should obtain a recursive formula. I.e., start by considering $r = 0$. Then arbitrary r but $j = k = 0$. And finally arbitrary r, j, k.