Domination and Total Domination in Cubic Graphs of Large Girth

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Abstract

The domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$ of a graph G without an isolated vertex are among the most well studied parameters in graph theory. While the inequality $\gamma_t(G) \leq 2\gamma(G)$ is an almost immediate consequence of the definition, the extremal graphs for this inequality are not well understood. Furthermore, even very strong additional assumptions do not allow to improve the inequality by much.

In the present paper we consider the relation of $\gamma(G)$ and $\gamma_t(G)$ for cubic graphs G of large girth. Clearly, in this case $\gamma(G)$ is at least $n(G)/4$ where $n(G)$ is the order of G. If $\gamma(G)$ is close to $n(G)/4$, then this forces a certain structure within G. We exploit this structure and prove an upper bound on $\gamma_t(G)$, which depends on the value of $\gamma(G)$. As a consequence, we can considerably improve the inequality $\gamma_t(G) \leq 2\gamma(G)$ for cubic graphs of large girth.

Keywords: Domination; total domination; cubic graph; girth

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1 Introduction

For a finite, simple, and undirected graph G, a set D of vertices of G is a *dominating set of* G if every vertex in $V(G) \setminus D$ has a neighbor in D. Similarly, a set T of vertices of G is a *total dominating set of* G if every vertex in $V(G)$ has a neighbor in T. Note that a graph has a total dominating set exactly if it has no isolated vertex. The minimum cardinalities of a dominating and a total dominating set of G are known as the *domination number* $\gamma(G)$ *of* G and the *total domination number* $\gamma_t(G)$ *of* G, respectively. These two parameters are among the most fundamental and well studied parameters in graph theory [5, 6, 8]. In view of their computational hardness especially upper bounds were investigated in great detail.

The two parameters are related by some very simple inequalities. Let G be a graph without isolated vertices. Since every total dominating set of G is also a dominating set of G, we immediately obtain

$$\gamma_t(G) \geq \gamma(G). \tag{1}$$

Similarly, if D is a dominating set of G, then adding, for every isolated vertex u of the subgraph $G[D]$ of G induced by D, a neighbor of u in G to the set D, results in a total dominating set of G, which implies

$$\gamma_t(G) \leq 2\gamma(G). \tag{2}$$

The complete bipartite graph $K_{n/2,n/2}$ and the complete graph K_n show that (1) and (2) are sharp, respectively. In [4, 7] the trees that satisfy (1) or (2) with equality are characterized constructively.

While numerous very deep results concerning bounds on the domination number and the total domination number under various conditions have been obtained, the relation of these two parameters is not really well understood. The characterization of the extremal graphs for (1) and (2) and/or improvements of (1) and (2) even under strong additional assumptions appear to be very difficult. If the graph G arises, for instance, by subdividing every edge of the complete graph K_n with $n \geq 3$ twice, then $\gamma(G) = n$ and $\gamma_t(G) = 2n-1$, that is, forbidding cycles of length up to 8 does not allow to improve (2) by much. For a positive integer k, let $[k]$ denote the set $\{1, 2, \ldots, k\}$. If the graph G has vertex set $\bigcup_{i\in[k]}(A_i \cup B_i \cup C_i)$, where

- the sets A_i, B_i, and C_i for all $i \in [k]$ are disjoint,
- $|A_i| = a$, $|B_i| = a + 1$, and $|C_i| = ka$ for every $i \in [k]$ and some $a \in \mathbb{N}$,
- the closed neighborhood $N_G[u]$ of a vertex u in A_j for $j \in [k]$ is $B_j \cup \bigcup_{i\in[k]} A_i$,
- the closed neighborhood $N_G[v]$ of a vertex v in B_j for $j \in [k]$ is $A_j \cup \{v\} \cup C_j$, and
- the closed neighborhood $N_G[w]$ of a vertex w in C_j for $j \in [k]$ is $B_j \cup C_j$,

then G is regular of degree $(k + 1)a$, has connectivity a, diameter 5, $\gamma(G) = k + 1$, and $\gamma_t(G) = 2k$, that is, a large minimum degree, a large degree of regularity, a large connectivity, a small diameter, and a large value of the domination number do not force any serious improvement of (2).

In the present paper we consider the relation between the domination number and the total domination number for cubic graphs of large girth.

Let G be a cubic graph of order n and girth at least g, that is, G has no cycles of length less than g. Clearly, $\gamma(G) \geq \frac{1}{4}n$ and $\gamma_t(G) \geq \frac{1}{3}n$. The best published upper bound on the domination number of G, improving earlier results from [13, 14], is due to Král' et al. [12], who show

$$\gamma(G) \leq 0.299871n + O\left(\frac{n}{g}\right). \tag{3}$$

Combining this with $\gamma_t(G) \geq \frac{1}{3}n$, we obtain the following improvement of (1).

Corollary 1 *If G is a cubic graph of order n and girth at least g, then*

$$\frac{\gamma_t(G)}{\gamma(G)} \geq 1.111589 - O\left(\frac{1}{g}\right).$$

In a recent preprint [11] Hoppen and Wormald improve (3) further to $\gamma(G) \leq 0.27942n + O\left(\frac{n}{g}\right)$, which improves the bound in Corollary 1 to $\frac{\gamma_t(G)}{\gamma(G)} \geq 1.192947 - O\left(\frac{1}{g}\right)$.

For a graph G of order n, minimum degree at least 2, and girth at least g, Henning and Yeo [9,10] show $\gamma_t(G) \leq \frac{1}{2}n + O\left(\frac{n}{g}\right)$. Applying a trick from [13], this result leads to the following corollary. Recall that the line graph of a graph G has vertex set $E(G)$ and edge set $\{ef : e, f \in E(G) \text{ and } |e \cap f| = 1\}$. Furthermore, the kth power of a graph G has vertex set $V(G)$ and edge set $\{uv : u, v \in V(G) \text{ and } 0 < \text{dist}_G(u, v) \leq k\}$.

Corollary 2 *If G is a cubic graph of order n and girth at least g, then*

$$\gamma_t(G) \leq \frac{121}{248}n + O\left(\frac{n}{g}\right) \leq 0.488n + O\left(\frac{n}{g}\right). \tag{4}$$

Proof: Let G be as in the statement. In view of the desired bound, we may assume that g is sufficiently large. Since the 5th power of the line graph of G is neither an odd cycle nor complete, has order $\frac{3}{2}n$, and maximum degree 124, the theorem of Brooks [3] implies that there is a set M of at least $\frac{3}{248}n$ edges of G such that for every two vertices u and v that are incident with distinct edges in M, we have $\text{dist}_G(u, v) \geq 5$. Let T_0 denote the set of $2|M|$ vertices incident with the edges in M and let $G_1 = G \setminus N_G[T_0]$. By construction, the graph G_1 has order $n - 6|M|$, minimum degree at least 2, and girth at least g. By the above result of Henning and Yeo, the graph G_1 has a total dominating set T_1 of order at most $\frac{1}{2}(n - 6|M|) + O\left(\frac{n}{g}\right)$. Since $T_0 \cup T_1$ is a total dominating set of G, we obtain

$$\begin{aligned}
\gamma_t(G) &\leq 2|M| + \frac{1}{2}(n - 6|M|) + O\left(\frac{n}{g}\right) \\
&= \frac{1}{2}n - |M| + O\left(\frac{n}{g}\right) \\
&\leq \frac{1}{2}n - \frac{3}{248}n + O\left(\frac{n}{g}\right) \\
&= \frac{121}{248}n + O\left(\frac{n}{g}\right),
\end{aligned}$$

which completes the proof. $\square$

Combining Corollary 2 with $\gamma(G) \geq \frac{1}{4}n$, we obtain the following improvement of (2).

Corollary 3 *If G is a cubic graph of order n and girth at least g, then*

$$\frac{\gamma_t(G)}{\gamma(G)} \leq \frac{121}{62} + O\left(\frac{1}{g}\right) \leq 1.952 + O\left(\frac{1}{g}\right).$$

Note that Corollary 3 can only be close to the truth if the domination number is close to $\frac{1}{4}n$. Our main result shows that in this case, the total domination number is smaller than guaranteed by (4). Specifically, we prove the following result.

Theorem 4 *If G is a cubic graph of order n, girth at least g, and domination number $\left(\frac{1}{4} + \epsilon\right)n$ for some $\epsilon \geq 0$, then*

$$\gamma_t(G) \leq \frac{13}{32}n + \frac{3n}{4(g-2)} + \frac{91}{8}\epsilon n \leq 0.40625n + O\left(\frac{n}{g}\right) + O(\epsilon n).$$

This result allows to improve Corollary 3 as follows.

Corollary 5 *If G is a cubic graph of order n and girth at least g, then*

$$\frac{\gamma_t(G)}{\gamma(G)} \leq \frac{11011}{5804} + O\left(\frac{1}{g}\right) \leq 1.89714 + O\left(\frac{1}{g}\right).$$

Proof: Let G be as in the statement and let $\gamma(G) = \left(\frac{1}{4} + \epsilon\right) n$ for some $\epsilon \geq 0$. By Corollary 2 and Theorem 4, we obtain

$$\frac{\gamma_t(G)}{\gamma(G)} \leq \frac{\min\left\{\frac{13}{32} + \frac{91}{8}\epsilon, \ \frac{121}{248}\right\}}{\frac{1}{4} + \epsilon} + O\left(\frac{1}{g}\right).$$

Since $\left(\frac{13}{32} + \frac{91}{8}\epsilon\right) / \left(\frac{1}{4} + \epsilon\right)$ is increasing as a function of $\epsilon \geq 0$ and $\frac{13}{32} + \frac{91}{8}\epsilon = \frac{121}{248}$ for $\epsilon = \frac{81}{11284}$, the desired result follows. $\Box$

The rest of the paper is devoted to the proof of Theorem 4.

2 Proof of Theorem 4

Let G be a cubic graph of order n, girth at least g, and domination number $\left(\frac{1}{4} + \epsilon\right) n$. Let $\gamma = \gamma(G)$ and $\gamma_t = \gamma_t(G)$.

Let D be a minimum dominating set of G. Assign each vertex in $V(G) \setminus D$ arbitrarily to some neighbor in D. By a result of Bollobás and Cockayne [2], we may assume that for every vertex u in D, at least one vertex in $V(G) \setminus D$ is assigned to u. Let D_0 be the set of vertices in D to which three vertices in $V(G) \setminus D$ have been assigned. Let $D_1 = D \setminus D_0$, $\gamma_0 = |D_0|$, and $\gamma_1 = |D_1|$. Since the closed neighborhoods of the vertices in D_0 are disjoint and, to every vertex in D_1, at least one vertex was assigned, we conclude $4\gamma_0 + 2(\gamma - \gamma_0) \leq n$, which implies

$$\gamma_0 \leq \left(\frac{1}{4} - \epsilon\right) n$$

and hence

$$\gamma_1 \geq 2\epsilon n.$$

Since D is dominating, we have $n \leq 4(\gamma - \gamma_1) + 3\gamma_1 = 4\gamma - \gamma_1 = (1 + 4\epsilon) n - \gamma_1$, which implies

$$\gamma_1 \leq 4\epsilon n.$$

If U is the set of vertices in $V(G) \setminus D$ assigned to vertices in D_0 and $n_U = |U|$, then, since at least one vertex was assigned to every vertex in D_1,

$$n_U \leq n - \gamma_0 - 2\gamma_1 = n - \gamma - \gamma_1 \leq \left(\frac{3}{4} - 3\epsilon\right) n.$$

Let $R = V(G) \setminus (D_0 \cup U)$. If $u \in D_1$ is such that exactly one vertex, say v, was assigned to u, then there are at most 2 edges between u and U and at most 2 edges between v and U. If $u \in D_1$ is such that exactly two vertices, say v_1 and v_2, were assigned to u, then there are at most 1 edge between u and U, at most 2 edges between v_1 and U, and at most 2 edges between v_2 and U. Altogether, there are at most $5\gamma_1 \leq 20\epsilon n$ edges between U and R. Since every vertex in U has exactly one neighbor in D_0, the graph $G[U]$ is the disjoint union of r cycles and s paths such that $s \leq 10\epsilon n$. Since G has girth at least g, we obtain $r \leq \frac{n_U}{g}$. If H is a cycle or a path of order ℓ, then it is possible to partition $V(H)$ into at most $\frac{\ell}{g-2} + 1$ sets each of which induces a path of order at most $g - 2$. Therefore, it is possible to partition U into k sets that induce k paths $P_1, \dots, P_k$ of order at most $g-2$ such that

$$k \leq \frac{n_U}{g-2} + r + s \leq \frac{n_U}{g-2} + \frac{n_U}{g} + s \leq \frac{2n_U}{g-2} + s \leq \frac{3n}{2(g-2)} + 10\epsilon n.$$

Note that, by the girth condition, no vertex in $V(G) \setminus V(P_i)$ has more than one neighbor in $V(P_i)$ for every $i \in [k]$.

We now construct a random total dominating set T of G starting with the empty set.

- Add all vertices in D_1 to T.
- For every vertex u in D_1, choose one of the vertices assigned to u uniformly and independently at random and add it to a set T'.
- Add all vertices in T' to T.

Since to every vertex in D_1 either one or two vertices were assigned, every vertex in R has a neighbor in T and every vertex in R belongs to T with probability at least $\frac{1}{2}$.

- For every $i \in [k]$, we proceed as follows.
  - Let $P_i = u_1 u_2 \dots u_\ell$.
  - We choose $x_i \in \{0, 2\}$ independently and uniformly at random.
  - Add to T all vertices in the set T_i with

  $$T_i = \{u_j : (j \in [\ell]) \wedge ((j \equiv x_i \bmod 4) \vee ((j-1) \equiv x_i \bmod 4))\}.$$

Note that every vertex in U belongs to T with probability $\frac{1}{2}$. This implies that the expected value of the cardinality of $\bigcup_{i \in [k]} T_i$ satisfies $\mathbf{E}\left[\sum_{i \in [k]} |T_i|\right] = \frac{n_U}{2}$. By now, all internal vertices of the paths $P_1, \dots, P_k$ have a neighbor in T. Furthermore, every end vertex of $P_1, \dots, P_k$ has no neighbor in T with probability at most $\frac{1}{4}$.

- For each vertex u in U that has no neighbor in T so far, add a neighbor of u to a set T''.
- Add all vertices in T'' to T.

Note that $\mathbf{E}[|T''|] \leq \frac{1}{4} \cdot 2k = \frac{k}{2}$.

- For each vertex u in D_0 that has no neighbor in T so far, add a neighbor of u to a set T'''.
- Add all vertices in T''' to T.

Since every vertex in D_0 has three neighbors in U, no two of which lie in a single path P_i, we obtain $\mathbf{E}[|T'''|] \leq \left(\frac{1}{2}\right)^3 \gamma_0$. Now T is a total dominating set of G and, by the first moment method [1], we obtain

$$\begin{aligned}
\gamma_t(G) &\leq \mathbf{E}[|T|] \\
&= |D_1| + \mathbf{E}[|T'|] + \mathbf{E}\left[\sum_{i \in [k]} |T_i|\right] + \mathbf{E}[|T''|] + \mathbf{E}[|T'''|] \\
&\leq \gamma_1 + \gamma_1 + \frac{n_U}{2} + \frac{k}{2} + \frac{1}{8}\gamma_0 \\
&\leq \frac{13}{32}n + \frac{3n}{4(g-2)} + \frac{91}{8}\epsilon n,
\end{aligned}$$

which completes the proof of Theorem 4. $\Box$

3 Conclusion

While the constants in our results improve previous estimates, we believe that they can still be improved. Suitably modifying the proof strategy of Theorem 4, it is possible to show an upper bound on the domination number of a cubic graph of order n and girth at least g, for which the total domination number is close to $\frac{1}{3}n$. Unfortunately, this bound is weaker than the result of Král' et al. [12].

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