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Mathematics of Games

Sample Exam - Total: 60 Points + 6 Bonus Points

1. Suppose there are n firms in the Stackelberg oligopoly model. Let q_i denote the quantity produced by firm i, and let $Q = q_1 + \ldots + q_n$ denote the aggregate quantity on the market. Let P denote the market-clearing price and assume that inverse demand is given by $P(Q) = a - Q$ (assuming $Q < a$, else $P = 0$). Assume that the total cost of firm i from producing quantity q_i is $C_i(q_i) = cq_i$. That is, there are no fixed costs and the marginal cost is constant at c, where we assume $c < a$. Suppose that: (1) firm 1 chooses q_1 and firm 2 chooses q_2 simultaneously; (2) firms $3 \leq i \leq n$ observe first both q_1 and q_2 and then simultaneously choose q_i. What is the subgame perfect Nash equilibrium? Develop an answer, only by using static (one-stage) results derived in class, but not dynamic ones.

(12 points)

2. (a) Compute the pure and mixed strategy Nash equilibria and the corresponding payoffs in the "battle of the sexes" game seen in class as shown in the figure below.

	F	B
F	2,1	0,0
B	0,0	1,2

(b) Let $\delta = \frac{9}{10}$, and consider the infinitely repeated game based on the stage game from (a), with discount factor δ. Find a pure-strategy subgame-perfect Nash equilibrium s of the infinitely repeated game, for which the corresponding average discounted payoffs

$$(1 - \delta) \cdot \sum_{t=0}^{\infty} \delta^t g_i(s^t(h^t))$$

for both players $i = 1, 2$ are higher than double the highest symmetric payoff of the stage game, that is: the corresponding average discounted payoffs are higher than $\frac{4}{3}$.

(c) Modify this game of complete information into a game of incomplete information with a continuum of types, so that whenever the type interval tends to a point, the Bayesian Nash Equilibrium tends to a Nash Equilibrium of the complete information game. Give the transformations, as well as the respective (Bayesian) Nash Equilibria.

(4 + 4 + 4 = 12 points)

3. Consider the Copacabana restaurant game where a guest is an argentinian type with probability 0.5 or a german type with probability 0.5. The german would rather have a beer, while the argentinian would rather have a Chimarrão (sort of hot tea). The brazilian waiter loves to have a fight with argentinians, and in particular if the argentinian asks for a nice cold beer. On the other hand, the brazilian survivor and sympathy instincts tell the waiter it is better to treat very well the strong-built german, specially when germans prevented Argentinia from winning the World Cup in Maracanã. The payoffs are as in the table below. Give all pooling and separating Perfect Nash Equilibria. Which Equilibria are eliminated by the *Intuitive Criterion*, defined as seen in class, and which survive it?

(a_1, a_2)	Argentinian	German
$(Beer, not)$	2,0	3,2
$(Beer, duel)$	1,3	2,0
$(Chimarrao, not)$	3,3	2,0
$(Chimarrao, duel)$	0,1	1,1

Intuitive Criterion: Fix a vector of equilibrium payoffs u_1^* for player 1. For each strategy a_1, let $J(a_1)$ be the set of all θ such that $u_1^*(\theta) > \max u_1(a_1, a_2, \theta) \forall a_2 \in BR(T, a_1)$, where $BR(T, a_1)$ is the set of all pure-strategy best responses for player 2 to a_1 for beliefs $P(.|a_1)$ such that $P(T|a_1) = 1$. If for some a_1 there exists a $\theta' \in T$ such that $u_1^*(\theta) < \min u_1(a_1, a_2, \theta') \forall a_2 \in BR(T \setminus J(a_1), a_1)$, then the equilibrium fails this criterion.

(12 points)

4. Consider a variant of Rubinstein's infinite-horizon bargaining game where partitions (x_1, x_2), with $x_1 + x_2 = 1$, are restricted to be integer multiples of 0.01, that is, x_i can be $0, 0.01, 0.02, \ldots, 0.99$, or 1 for $i = 1, 2$. There is a common discount factor δ. Prove that, if $\delta > 0.99$, any partition can be supported as a subgame-perfect NE.

(12 points)

5. Consider a first-price, sealed-bid auction in which the bidders' valuations are independently and uniformly distributed on $[0, 1]$. Bidder i has valuation v_i for the good - that is, if bidder i gets the good and pays the price p, then i's payoff is $v_i - p$. The bids $b_i \geq 0$ are submitted simultaneously, the higher bidder wins the good and pays the bidden price, the other bidders get and pay 0. In case of a tie, the winner is determined uniformly at random. Show that if there are n bidders, then the strategy of bidding $\frac{n-1}{n}$ times one's valuation is a Bayesian Nash equilibrium of this auction.

(12 points)

6. Bonus Question (Theorem):

   (a) Give the definition of *continuous at infinity* from the lecture.

   (b) Prove the following theorem: In an infinite-horizon multi-stage game with observed actions that is continuous at infinity, profile s is subgame-perfect if and only if there is no player i and stragegy $\hat{s}_i$ that agrees with s_i except at a single t and h^t, and such that $\hat{s}_i$ is a better response to s_{-i} than s_i conditional on history h^t being reached.

(2 + 4 points)