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Mathematics of Games

Exercise Session 2

Exercise Session 2 due on 12.05.2014, by 12:15pm, N24-H14.
Total : 20 Points
Hand-in individual!

1. **Theorem 1 (Nash, 1950)** *Every finite normal-form static game of complete information has a mixed-strategy Nash equilibrium.*

   The idea of the proof of Nash's theorem is to apply Kakutani's fixed-point theorem to the players' "reaction correspondences". Let player i's *reaction correspondence* r_i map each strategy profile σ to the set of mixed strategies that maximize player i's payoff when his opponents play σ_{-i}. We define the correspondence $r : \Sigma \to \Sigma$ to be the Cartesian product of the r_i. A *fixed point* of r is a σ such that $\sigma \in r(\sigma)$, so for every player, $\sigma_i \in r_i(\sigma)$. Thus, a fixed point of r is a Nash equilibrium. From Kakutani's theorem, the following are sufficient conditions for $r : \Sigma \to \Sigma$ to have a fixed point:

   (a) Σ is a compact, convex, nonempty subset of a finite-dimensional Euclidean space.

   (b) $r(\sigma)$ is nonempty for all σ. [1 Point]

   (c) $r(\sigma)$ is convex for all σ. [2 Points]

   (d) $r(\cdot)$ has a *closed graph*: For any two sequences $(\sigma^n), (\hat{\sigma}^n)$ with $\hat{\sigma}^n \in r(\sigma^n)$ for all n, $\sigma^n \to \sigma$ and $\hat{\sigma}^n \to \hat{\sigma}$, the following holds: $\hat{\sigma} \in r(\sigma)$. [3 Points]

   First note that each Σ_i is a simplex of dimension $(|\Sigma_i| - 1)$, which is nonempty, compact, and convex. So this also holds for the cartesian product Σ.
   Now prove that the conditions (b)-(d) are satisfied. Use the following information.

   - A set X in a linear vector space is *convex* if, for any x and y belonging to X and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y$ belongs to X.
   - In a game as in the theorem above, each player's payoff function is *multilinear*, that is,
     $$u_i(\lambda\sigma_i' + (1 - \lambda)\sigma_i'', \sigma_{-i}) = \lambda u_i(\sigma_i', \sigma_{-i}) + (1 - \lambda)u_i(\sigma_i'', \sigma_{-i})$$
     for all $\sigma_i', \sigma_i'' \in \Sigma_i$ and $\lambda \in [0, 1]$. Linear functions are continuous.
   - Continuous functions on nonempty convex and compact sets attain maxima.

   [6 Points]

2. Solve for the mixed-strategy Nash equilibria in the following normal-form game.

	L	C	R
T	2,0	1,1	4,2
M	3,4	1,2	2,3
B	1,3	0,2	3,0

[4 Points]

3. Suppose there are I farmers, each of whom has the right to graze cows on the village common. The amount of milk a cow produces depends on the number of cows, N, grazing on the green. The revenue produced by n_i cows is $n_i u(N)$ for $N < N^*$ and $u(N) = 0$ for $N \geq N^*$, where $u(0) > 0$, $u' < 0$ and $u'' < 0$. Each cow costs c, and cows are perfectly divisible. Suppose $u(0) > c$. Farmers simultaneously decide how many cows to purchase. All purchased cows will graze on the common. Write this game as a static game of complete information, give a pure-strategy Nash Equilibrium and compare it against the social optimum. How does this game relate to the Cournot oligopoly model?

[6 Points]

4. Consider the Cournot duopoly model from the lecture. Apply the first two steps of the iterated elimination of strictly dominated strategies: prove that the monopoly quantity $q_m = \frac{a-c}{2}$ strictly dominates any higher quantity and that half the monopoly quantity $q_{\frac{m}{2}} = \frac{a-c}{4}$ strictly dominates any lower quantity. How many steps are required in total?

[4 Points]