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Mathematics of Games

Exercise session 7

Hand-in: 17.06.2013, 12pm-2pm, N24-H15

1. Prove the following theorem.

   Theorem 1 (Friedman 1971) *Let σ^* be a static equilibrium (an equilibrium of the stage game) with payoffs e. Then for any $v \in V$ with $v_i > e_i$ for all players i, there is a $\underline{\delta}$ such that for all $\delta > \underline{\delta}$ there is a subgame-perfect equilibrium of $G(\delta)$ with payoffs v.*

   Hint: Assume initially $\exists a', u(a') = v$, and profile s where each player plays a_i' in period 0.

   Friedman's result shows that patient and identical Cournot duopolists can implicitly collude by each producing half of the monopoly output, with any deviation triggering a switch to the Cournot outcome forever after. The collusion is implicit: Each firm is deterred from breaking the agreement by the (credible) fear of provoking Cournot's competition.

2. *Cat-Dog Fight Game*: A cat and a dog are fighting for a prize whose current value at any time $t = 0, 1, \ldots$ is $v > 1$. Fighting costs 1 unit per period. If one animal stops fighting at period t, his opponent wins the prize without incurring a fighting cost that period, and the choice of the second stopping time is irrelevant. If both animals stop fighting simultaneously, both losing neither winning the prize. That is, if we consider a per-period discount factor δ, the (symmetric) payoff functions are:

   - $L(t') = -(1 + \delta + \ldots + \delta^{t'-1}) = -\frac{1-\delta^{t'}}{1-\delta}$, for the loser(s), and
   - $W(t') = L(t') + \delta^{t'} v$, for the winner (in case there is one).

   Give a symmetric subgame-perfect Nash Equilibrium for the *Cat-Dog Fight Game*.

   Hint: You may consider a mixed strategy profile.

3. This exercise aims to show how a small modification in the information sets may change completely the solution. Draw game trees for the following two game variations of Exercise 4.b from Sheet 4, where player 3 behaves slightly different, and solve for the subgame-perfect Nash Equilibria.

(a)
1. Player 1 chooses an action a_1 from the feasible set $A_1 = \{A, L, R\}$ where A ends the game with payoffs of 20 to player 1 and 1 to players 2 and 3.
2. Player 2 observes a_1. If $a_1 = L$, player 2 chooses an action a_2 from the feasible set $A_{2_L} = \{L', R'\}$ and if $a_1 = R$, player 2 chooses an action a_2 from the feasible set $A_{2_R} = \{A', L', R'\}$, where A' ends the game with payoffs 18 for player 1, 8 for player 2 and 6 for player 3.
3. Player 3 observes if $a_1 \neq A$ and if so, player 3 observes if $a_2 \neq A'$. If $a_1 \neq A$ and $a_2 \neq A'$, player 3 observes whether or not $(a_1, a_2) = (R, R')$ or $(a_1, a_2) = (R, L')$ and then chooses an action a_3 from the feasible set $A_3 = \{L'', R''\}$, which ends the game with payoffs given in the table below.

(a_1, a_2, a_3)	Player 1	Player 2	Player 3
(L, L', L'')	20	8	4
(L, L', R'')	8	0	1
(L, R', L'')	4	4	5
(L, R', R'')	2	6	1
(R, L', L'')	12	8	2
(R, L', R'')	16	4	1
(R, R', L'')	10	2	5
(R, R', R'')	20	10	6

(b)

1. Player 1 behaves the same as in (a).
2. Player 2 behaves the same as in (a).
3. Player 3 observes if $a_1 \neq A$ and if so, player 3 observes if $a_2 \neq A'$. If $a_1 \neq A$ and $a_2 \neq A'$, player 3 observes whether or not $(a_1, a_2) = (R, R')$ or $(a_1, a_2) = (L, L')$ and then chooses an action a_3 from the feasible set $A_3 = \{L'', R''\}$, which ends the game with payoffs given in the table as in (a).

4. Find all the pure-strategy Bayesian Nash equilibria in the following static Bayesian game:

1. Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.
2. Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not.
3. Player 1 chooses either T or B; player 2 simultaneously chooses either L or R.
4. Payoffs are given by the game drawn by nature.

	L	R
T	0,1	1,0
B	2,3	1,2

Game 1

	L	R
T	3,0	2,1
B	2,2	0,2

Game 2