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Mathematics of Games

Exercise session 8

20.06.2013, 2pm-4pm, N24-H15

Hand-in before class starts.

1. **Theorem 1 (Abreu 1998)** *If the stage game is finite, any distribution over infinite histories that can be generated by some subgame-perfect equilibrium σ can be generated with a strategy profile σ^* that specifies that play switches to the worst equilibrium $\underline{w}(i)$ for player i if player i is the first to play an action to which σ assigns probability* 0.

   Hint: Consider $\sigma^*(h^t) = \sigma(h^t)$ as long as σ gives the history h^t positive probability. What happens if σ gives positive probability to $h^{t'}$ for all $t' < t$, and player i is the only player to play an action with probability 0 in $\sigma(h^t)$ at period t?

2. Consider a first-price, sealed-bid auction in which the bidders' valuations are independently and uniformly distributed on $[0, 1]$. Bidder i has valuation v_i for the good - that is, if bidder i gets the good and pays the price p, then i's payoff is $v_i - p$. The bids $b_i \geq 0$ are submitted simultaneously, the higher bidder wins the good and pays the bidden price, the other bidders get and pay 0. In case of a tie, the winner is determined uniformly at random. Show that if there are n bidders, then the strategy of bidding $\frac{n-1}{n}$ times one's valuation is a Bayesian Nash equilibrium of this auction.

3. Consider a Cournot duopoly operating in a market with inverse demand $P(Q) = a - Q$, where $Q = q_1 + q_2$ is the aggregate quantity on the market. Both firms have total costs $c_i(q_i) = cq_i$ with a constant c, but demand is uncertain: it is high ($a = a_H$) with probability θ and low ($a = a_L$) with probability $1 - \theta$. So the payoff depends on a and is $u_i(q_i, q_j, a) = (P(Q) - c)q_i$ for both firms. Furthermore, information is asymmetric: firm 1 knows whether demand is high or low, but firm 2 does not. All of this is common knowledge. The two firms simultaneously choose quantities. What is the pure-strategy Bayesian Nash equilibrium of this game?

4. Consider the following model of Bertrand duopoly with differentiated products. Demand for firm i is $q_i(p_i, p_j) = a - p_i - b_i \cdot p_j$. Costs are zero for both firms. The sensitivity of firm i's demand to firm j's price is either high or low. That is, b_i is either b_H or b_L, where $b_H > b_L > 0$. For each firm, $b_i = b_H$ with probability θ and $b_i = b_L$ with probability $1 - \theta$, independent of the realization of b_j. Each firm knows its own b_i but not its competitor's. All of this is common knowledge. Which (four) conditions define a pure-strategy Bayesian Nash equilibrium of this game? Solve for such an equilibrium in the case $\theta = 1$.