t-perfection in P_5-free graphs

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Abstract

A graph is called t-perfect if its stable set polytope is fully described by non-negativity, edge and odd-cycle constraints. We characterise P_5-free t-perfect graphs in terms of forbidden t-minors. Moreover, we show that P_5-free t-perfect graphs can always be coloured with three colours, and that they can be recognised in polynomial time.

1 Introduction

There are three quite different views on *perfect graphs*, a view in terms of colouring, a polyhedral and a structural view. Perfect graphs can be seen as:

- the graphs for which the chromatic number $\chi(H)$ always equals the clique number $\omega(H)$, and that in any induced subgraph H;
- the graphs for which the *stable set polytope*, the convex hull of stable sets, is fully described by non-negativity and clique constraints; and
- the graphs that do not contain any *odd hole* (an induced cycle of odd length at least 5) or their complements, *odd antiholes*.

(The polyhedral characterisation is due to Fulkerson [15] and Chvátal [9], while the third item, the strong perfect graph theorem, was proved by Chudnovsky, Robertson, Seymour and Thomas [6].)

In this article, we work towards a similar threefold view on *t-perfect* graphs. These are graphs that, similar to perfect graphs, have a particularly simple stable set polytope. For a graph to be t-perfect its stable set polytope needs to be given by non-negativity, edge and odd-cycle constraints; for precise definitions we defer to the next section. The concept of t-perfection, due to Chvátal [9], thus takes its motivation from the polyhedral aspect of perfect graphs. The corresponding colouring and structural view, however, is still missing. For some graph classes, though, claw-free graphs for instance [5], the list of minimal obstructions for t-perfection is known. We extend this list to P_5-free graphs. (A graph is P_5-free if it does not contain the path on five vertices as an induced subgraph.)

Perfection is preserved under vertex deletion, and the same is true for t-perfection. There is a second simple operation that maintains t-perfection: a *t-contraction*, which is only allowed at a vertex with stable neighbourhood, contracts all the incident edges. Any graph obtained by a sequence of vertex deletions and t-contractions is a *t-minor*. The concept of t-minors makes it more convenient to characterise t-perfection in certain graph classes as it allows for more succinct lists of obstructions.

For that characterisation denote by C_n^k the kth power of the n-cycle C_n, that is, the graph obtained from C_n by adding an edge between any two vertices of distance at most k in C_n. We, moreover, write $\overline{G}$ for the complement of a graph G, and K_n for the complete graph on n vertices and W_n for the wheel with $n+1$ vertices.

Theorem 1. *Let G be a P_5-free graph. Then G is t-perfect if and only if it does not contain any of K_4, W_5, C_7^2, $\overline{C_{10}^2}$ or $\overline{C_{13}^3}$ as a t-minor.*

This answers a question of Benchetrit [2, p. 76].

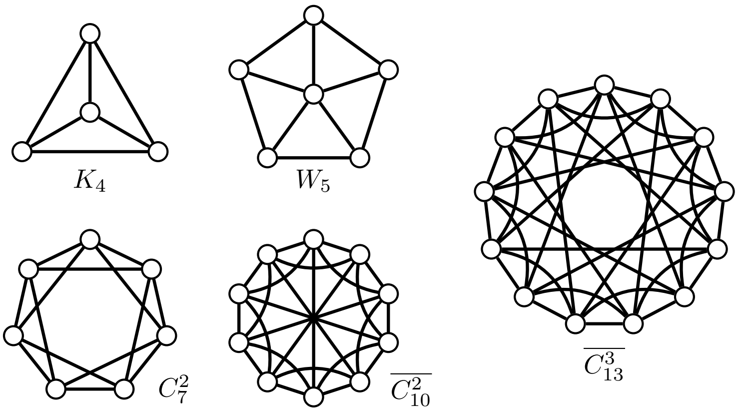


Figure 1: Forbidden t-minors in P_5-free graphs

Excluding P_5 as an induced subgraph is a serious restriction. Indeed, of the inequalities defining t-perfection only the non-negativity, edge and the 5-hole constraints remain since longer (odd) induced cycles contain an induced P_5 (see next section for the definitions). Nevertheless, the class of P_5-free t-perfect graphs is not entirely trivial: it contains, for instance, all blow-ups of triangles and 5-cycles, the graphs in which the vertices of a triangle or a 5-cycle are replaced by independent sets of vertices of arbitrary size.

While the main result is only about P_5-free graphs, we argue that it still provides a modest step towards an understanding of general t-perfect graphs. This step consists in new obstructions to t-perfection, see below, and in a decomposition method that works in any graph, not only P_5-free graphs, see Section 5.

The forbidden graphs Theorem 1 are *minimally t-imperfect*, in the sense that they are t-imperfect but any of their proper t-minors are t-perfect. Odd wheels, even Möbius ladders (see Section 3), the cycle power C_7^2 and the graph $\overline{C_{10}^2}$ are known to be minimally t-imperfect. The graph $\overline{C_{13}^3}$ appears here for the first time as a minimally t-imperfect graph. We prove this in Section 4, where we also present two more minimally t-imperfect graphs.

A starting point for Theorem 1 was the observation of Benchetrit [2, p. 75] that t-minors of P_5-free graphs are again P_5-free. Thus, any occurring minimally t-imperfect graph will be P_5-free, too. This helped to whittle down the list of prospective forbidden t-minors. We prove Theorem 1 in Sections 5 and 6.

A graph class in which t-perfection is quite well understood is the class of *near-bipartite* graphs; these are the graphs that become bipartite whenever the neighbourhood of any vertex is deleted. In the course of the proof of Theorem 1 we make use of results of Shepherd [26] and of Holm, Torres and Wagler [20]: together they yield a description of t-perfect near-bipartite graphs in terms of forbidden induced subgraphs. We discuss this in Section 3.

As a by-product of the proof of Theorem 1 we also obtain a polynomial-time algorithm to check for t-perfection in P_5-free graphs (Theorem 20).

Finally, in Section 7, we turn to the third defining aspect of perfect graphs: colouring. Shepherd and Sebő conjectured that every t-perfect graph can be coloured with four colours, which would be tight. For t-perfect P_5-graphs we show (Theorem 23) that already three colours suffice. We, furthermore, offer a conjecture that would, if true, characterise t-perfect graphs in terms of (fractional) colouring, in a way that is quite similar as for perfect graphs.

We end the introduction with a brief discussion of the literature on t-perfect graphs. A general treatment may be found in Grötschel, Lovász and Schrijver [19, Ch. 9.1] as well as in Schrijver [25, Ch. 68]. The most comprehensive source of literature references is surely the PhD thesis of Benchetrit [2]. A part of the literature is devoted to proving t-perfection for certain graph classes. For instance, Boulala and Uhry [3] established the t-perfection of series-parallel graphs. Gerards [16] extended this to graphs that do not contain an *odd-*K_4 as a subgraph (an odd-K_4 is a subdivision of K_4 in which every triangle becomes an odd circuit). Gerards and Shepherd [17] characterised the graphs with all subgraphs t-perfect, while Barahona and Mahjoub [1] described the t-imperfect subdivisions of K_4. Wagler [29] gave a complete description of the stable set polytope of antiwebs, the complements of cycle powers. These are near-bipartite graphs that also play a prominent role in the proof of Theorem 1. See also Wagler [30] for an extension to a more general class of near-bipartite graphs. The complements of near-bipartite graphs are the quasi-line graphs. Chudnovsky and Seymour [8], and Eisenbrand, Oriolo, Stauffer and Ventura [12] determined the precise structure of the stable set polytope of quasi-line graphs. Previously, this was a conjecture of Ben Rebea [24].

Algorithmic aspects of t-perfection were also studied: Grötschel, Lovász and Schrijver [18] showed that the max-weight stable set problem can be solved in polynomial-time in t-perfect graphs. Eisenbrand et al. [11] found a combinatorial algorithm for the unweighted case.

2 Definitions

All the graphs in this article are finite, simple and do not have parallel edges or loops. In general, we follow the notation of Diestel [10], where also any missing elementary facts about graphs may be found.

Let $G = (V, E)$ be a graph. The *stable set polytope* $\mathrm{SSP}(G) \subseteq \mathbb{R}^V$ of G is defined as the convex hull of the characteristic vectors of stable, i.e. independent, subsets of V. The characteristic vector of a subset S of the set V is the vector $\chi_S \in \{0,1\}^V$ with $\chi_S(v) = 1$ if $v \in S$ and 0 otherwise. We define a second polytope $\mathrm{TSTAB}(G) \subseteq \mathbb{R}^V$ for G, given by

$$
\begin{aligned}
& x \geq 0, \\
& x_u + x_v \leq 1 \text{ for every edge } uv \in E, \\
& \sum_{v \in V(C)} x_v \leq \left\lfloor \frac{|C|}{2} \right\rfloor \text{ for every induced odd cycle } C \text{ in } G.
\end{aligned}
$$

These inequalities are respectively known as non-negativity, edge and odd-cycle inequalities. Clearly, $\mathrm{SSP}(G) \subseteq \mathrm{TSTAB}(G)$.

Then, the graph G is called *t-perfect* if $\mathrm{SSP}(G)$ and $\mathrm{TSTAB}(G)$ coincide. Equivalently, G is t-perfect if and only if $\mathrm{TSTAB}(G)$ is an integral polytope, i.e. if all its vertices are integral vectors. It is easy to see that bipartite graphs are t-perfect. The smallest *t-imperfect* graph is K_4. Indeed, the vector $\frac{1}{3}\mathbb{1}$ lies in $\mathrm{TSTAB}(K_4)$ but not in $\mathrm{SSP}(K_4)$.

It is easy to verify that vertex deletion preserves t-perfection. Another operation that keeps t-perfection was found by Gerards and Shepherd [17]: whenever there is a vertex v, so that its neighbourhood is stable, we may contract all edges incident with v simultaneously. We will call this operation a *t-contraction at v*. Any graph that is obtained from G by a sequence of vertex deletions and t-contractions is a *t-minor of* G. Let us point out that any t-minor of a t-perfect graph is again t-perfect.

3 t-perfection in near-bipartite graphs

Part of the proof of Theorem 1 consists in a reduction to *near-bipartite* graphs. A graph is near-bipartite if it becomes bipartite whenever the neighbourhood of any of its vertices is deleted. We will need a characterisation of t-perfect near-bipartite graphs in terms of forbidden induced subgraphs. Fortunately, such a characterisation follows immediately from results of Shepherd [26] and of Holm, Torres and Wagler [20].

We need a bit of notation. Examples of near-bipartite graphs are antiwebs: an *antiweb* $\overline{C_n^k}$ is the complement of the kth power of the n-cycle C_n. The antiweb is *prime* if $n \geq 2k+2$ and $k+1$, n are relatively prime. We simplify the notation for antiwebs $\overline{C_n^k}$ slightly by writing A_n^k instead. *Even Möbius ladders*, the graphs A_{4t+4}^{2t}, are prime antiwebs; see Figure 2 for the Möbius ladder $\overline{C_8^2}$. We view K_4 alternatively as the smallest odd wheel W_3 or as the smallest even Möbius ladder $\overline{C_4^0}$. Trotter [27] found that prime antiwebs give rise to facets in the stable set polytope—we only need that prime antiwebs other than odd cycles are t-imperfect, a fact that is easier to check.

Shepherd proved:

Theorem 2 (Shepherd [26])**.** *Let G be a near-bipartite graph. Then G is t-perfect if and only if*

(i) *G contains no induced odd wheel; and*

(ii) *G contains no induced prime antiweb other than possibly an odd hole.*

Holm, Torres and Wagler [20] gave a neat characterisation of t-perfect antiwebs. For us, however, a direct implication of the proof of that characterisation

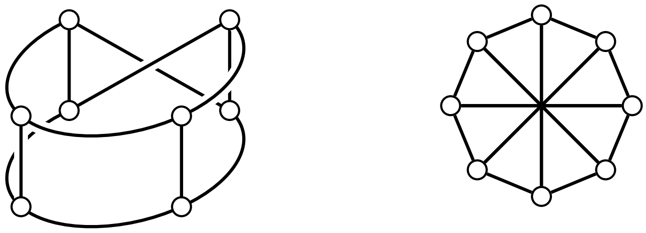


Figure 2: Two views of the Möbius ladder on 8 vertices

is more interesting: an antiweb is t-perfect if and only if it does not contain any even Möbius ladder, or any of A_7^1, A_{10}^2, A_{13}^3, A_{13}^4, A_{17}^4 and A_{19}^7 as an induced subgraph. We may omit A_{17}^4 from that list as it contains an induced A_{13}^3. Combining the theorem of Holm et al. with Theorem 2 one obtains:

Proposition 3. *A near-bipartite graph is t-perfect if and only if it does not contain any odd wheel, any even Möbius ladder, or any of A_7^1, A_{10}^2, A_{13}^3, A_{13}^4 and A_{19}^7 as an induced subgraph.*

4 Minimally t-imperfect antiwebs

Proposition 3 provides a combinatorial certificate for t-imperfection in near-bipartite graphs: any such graph either contains an odd wheel, an even Möbius ladder or one of the five graphs, C_7^2, A_{10}^2, A_{13}^3, A_{13}^4 and A_{19}^7, as an induced subgraph t-minor. These graphs are all *minimally t-imperfect*, that is, they are t-perfect but all their proper t-minors are t-perfect.

Even Möbius ladders and odd wheels, for instance, are known to be minimally t-imperfect. This follows from the result of Fonlupt and Uhry [14] that *almost bipartite* graphs are t-perfect; a graph is almost bipartite if it contains a vertex whose deletion renders it bipartite. It is easy to check that any proper t-minor of an even Möbius ladder or an odd wheel is almost bipartite.

All the other forbidden t-minors in Theorem 1 or Proposition 3 are minimally t-imperfect, too. That C_7^2 is minimally t-imperfect is proved in [5]. There, also minimality for C_{10}^2 is shown, which allows us to verify that A_{10}^2 is minimally t-imperfect as well. Indeed, for this we first observe that A_{10}^2 can be obtained from C_{10}^2 by adding diagonals of the underlying 10-cycle. The second necessary observation is that any two vertices directly opposite in the 10-cycle form a so called *odd pair*: any induced path between them has odd length. Minimality now follows from the result of Fonlupt and Hadjar [13] that adding an edge between the vertices of an odd pair preserves t-perfection.

In this section, we prove that A_{13}^3, A_{13}^4 and A_{19}^7 are minimally t-imperfect, which was not observed before. As prime antiwebs these are t-imperfect. This follows from Theorem 2 but can also be seen directly by observing that the vector $x \equiv \frac{1}{3}$ lies in TSTAB but not in SSP for any of the three graphs.

To show that the graphs are *minimally* t-imperfect, it suffices to consider the t-minors obtained from a single vertex deletion or from a single t-contraction. If these are t-perfect then the antiweb is minimally t-imperfect.

Trotter gave necessary and sufficient conditions when an antiweb contains another antiweb:

Theorem 4 (Trotter [27])**.** *$A_{n'}^{k'}$ is an induced subgraph of A_n^k if and only if*

$$n(k'+1) \geq n'(k+1) \text{ and } nk' \leq n'k.$$

We fix the vertex set of any antiweb A_n^k to be $\{0, 1, \ldots, n-1\}$, so that ij is an edge of A_n^k if and only if $|i-j| \mod n > k$.

Proposition 5. *The antiweb A_{13}^3 is minimally t-imperfect.*

Proof. For A_{13}^3 to be minimally t-imperfect, every proper t-minor A_{13}^3 needs to be t-perfect. As no vertex of A_{13}^3 has a stable neighbourhood, any proper t-minor is a t-minor of a proper induced subgraph H of A_{13}^3. Thus, it suffices to show that any such H is t-perfect.

By Proposition 3, H is t-perfect unless it contains an odd wheel or one of A_7^1, A_8^2 or A_{10}^2 as an induced subgraph. Since the neighbourhood of every vertex is stable, H cannot contain any wheel. For the other graphs, we check the inequalities of Theorem 4 and see that none can be contained in H. Thus, H is t-perfect and A_{13}^3 therefore minimally t-imperfect. □

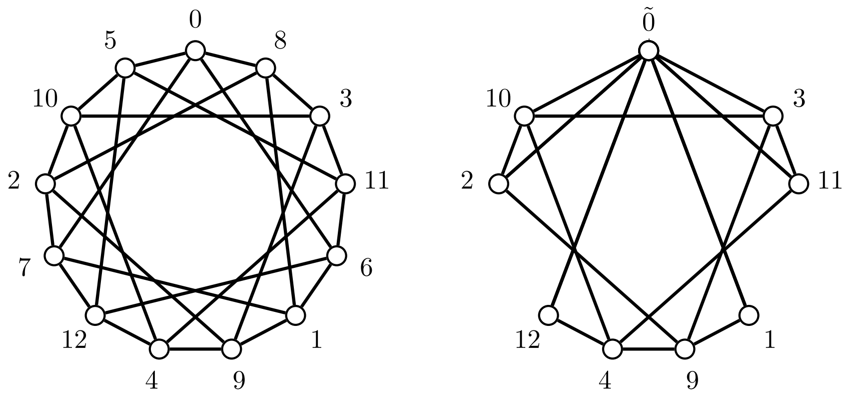


Figure 3: Antiweb A_{13}^4, and its t-minor obtained by a t-contraction at 0

Proposition 6. *The antiweb A_{13}^4 is minimally t-imperfect.*

Proof. By Proposition 3, any proper induced subgraph of A_{13}^4 that is t-imperfect contains one of A_7^1, A_8^2, or A_{10}^2 as an induced subgraph; note that A_{13}^4 does not contain odd wheels. However, routine calculation and Theorem 4 show that A_{13}^4 contains neither of these. Therefore, deleting any vertex in A_{13}^4 always results in a t-perfect graph.

It remains to consider the graphs obtained from A_{13}^4 by a single t-contraction. By symmetry, it suffices check whether the graph H obtained by t-contraction at 0 is t-perfect; see Figure 3. Denote by $\tilde{0}$ the new vertex that resulted from the contraction.

The graph H is still near-bipartite and still devoid of odd wheels. Thus, by Proposition 3, it is t-perfect unless it contains A_7^1 and A_8^2 as an induced subgraph—all the t-imperfect antiwebs of Proposition 3 are too large for the nine-vertex graph H.

Now, A_7^1 is 4-regular but H only contains five vertices of degree at least 4. Similarly, A_8^2 is 3-regular but two of the nine vertices of H, namely 1 and 12, have degree 2. We see that neither of the two antiwebs can be contained in H, so that H is t-perfect and, thus, A_{13}^4 minimally t-imperfect. □

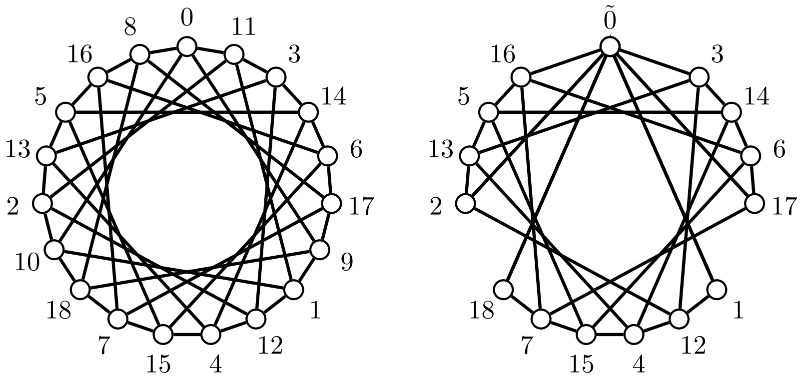


Figure 4: Antiweb A_{19}^7, and its t-minor obtained by t-contraction at 0

Proposition 7. *The antiweb A_{19}^7 is minimally t-imperfect.*

Proof. We claim that any proper induced subgraph of A_{19}^7 is t-perfect. Indeed, as A_{19}^7 does not contain any induced odd wheel, this follows from Proposition 3, unless A_{19}^7 contains one of $A_7^1, A_8^2, A_{10}^2, A_{12}^4, A_{13}^3, A_{13}^4$, or A_{16}^6 as an induced subgraph. We can easily verify with Theorem 4 that this is not the case.

It remains to check that any t-contraction in A_{19}^7 yields a t-perfect graph, too. By symmetry, we may restrict ourselves to a t-contraction at the vertex 0. Let H be the resulting graph, and let $\tilde{0}$ be the new vertex; see Figure 4.

The graph H is a near-bipartite graph on 15 vertices. It does not contain any odd wheel as an induced subgraph. Thus, by Proposition 3, H is t-perfect unless it has an induced subgraph A that is isomorphic to a graph in

$$\mathcal{A} := \{A_7^1, A_8^2, A_{10}^2, A_{12}^4, A_{13}^3, A_{13}^4\}.$$

Since this is not the case for A_{19}^7, we may assume that $\tilde{0} \in V(A)$.

Note that the graphs A_7^1, A_{10}^2, A_{13}^3 and A_{13}^4 have minimum degree at least 4. Yet, $\tilde{0}$ has only two neighbours of degree 4 or more (namely, 3 and 16). Thus, neither of these four antiwebs can occur as an induced subgraph in H.

It remains to consider the case when H contains an induced subgraph A that is isomorphic to A_8^2 or to A_{12}^4, both of which are 3-regular graphs. In particular, A is then contained in $H' = H - \{1, 18\}$ as the vertices 1 and 18 have degree 2.

As H' has only 13 vertices, A cannot be isomorphic to A_{12}^4 since deleting any single vertex of H' never yields a 3-regular graph. That leaves only $A = A_8^2$.

Since A_8^2 is 3-regular, we need to delete exactly one of the four neighbours of $\tilde{0}$ in H'. Suppose this is the vertex 3. Then, 12 has degree 2 and thus cannot be part of A. Deleting 12 as well leads to vertex 2 having degree 2, which thereby is also excluded from A. This, however, is impossible as 2 is one of the three remaining neighbours of $\tilde{0}$.

By symmetry, we may therefore assume that the neighbours of $\tilde{0}$ in A are precisely $2, 3, 16$. That 17 is not part of A entails that the vertex 7 has degree 2 and thus cannot lie in A either. Then, however, $16 \in V(A)$ has degree 2 as well, which is impossible. □

5 Harmonious cutsets

We investigate the structure of minimally t-imperfect graphs, whether they are P_5-free or not. We hope this more general setting might prove useful in subsequent research.

A structural feature that may never appear in a minimally t-imperfect graph G is a *clique separator*: any clique K of G so that $G - K$ is not connected.

Lemma 8 (Chvátal [9]; Gerards [16])**.** *No minimally t-imperfect graph contains a clique separator.*

A generalisation of clique separators was introduced by Chudnovsky et al. [7] in the context of colouring K_4-free graphs without odd holes. A tuple $(X_1, \ldots, X_s)$ of disjoint subsets of the vertex set of a graph G is *G-harmonious* if

- any induced path with one endvertex in X_i and the other in X_j has even length if and only if $i = j$; and
- if $s \geq 3$ then $X_1, \ldots, X_s$ are pairwise complete to each other.

A pair of subgraphs $\{G_1, G_2\}$ of $G = (V, E)$ is a *separation* of G if $V(G_1) \cup V(G_2) = V$ and G has no edge between $V(G_1) \setminus V(G_2)$ and $V(G_2) \setminus V(G_1)$. If both $V(G_1) \setminus V(G_2)$ and $V(G_2) \setminus V(G_1)$ are non-empty, the separation is *proper*.

A vertex set X is called a *harmonious cutset* if there is a proper separation (G_1, G_2) of G so that $X = V(G_1) \cap V(G_2)$ and if there exists a partition $X = (X_1, \ldots, X_s)$ so that $(X_1, \ldots, X_s)$ is G-harmonious.

We prove:

Lemma 9. *If a t-imperfect graph contains a harmonious cutset then it also contains a proper induced subgraph that is t-imperfect. In particular, no minimally t-imperfect graph admits a harmonious cutset.*

For the proof we need a bit of preparation.

Lemma 10. *Let $S_1 \subsetneq \ldots \subsetneq S_k$ and $T_1 \subsetneq \ldots \subsetneq T_\ell$ be nested subsets of a finite set V. Let $\sigma := \sum_{i=1}^{k} \lambda_i \chi_{S_i}$ and $\tau := \sum_{j=1}^{\ell} \mu_j \chi_{T_j}$ be two convex combinations in $\mathbb{R}^V$ with non-zero coefficients. If $\sigma = \tau$ then $k = \ell$, $\lambda_i = \mu_i$ and $S_i = T_i$ for all $i = 1, \ldots, k$.*

The lemma is not new. It appears in the context of submodular functions, where it may be seen to assert that the *Lovász extension* of a set-function is well-defined; see Lovász [21]. For the sake of completeness, we give a proof here.

Proof. By allowing λ_1 and μ_1 to be 0, we may clearly assume that $S_1 = \emptyset = T_1$. Moreover, if two elements $u, v \in V$ always appear together in the sets S_i, T_j

then we may omit one of u, v from all the sets. So, in particular, we may assume S_2 and T_2 to be singleton-sets.

Let s be the unique element of S_2. Then $\sum_{i=2}^{k} \lambda_i = \sigma_s = \tau_s \leq \sum_{j=2}^{\ell} \mu_j$. By symmetry, we also get $\sum_{i=2}^{k} \lambda_i \geq \sum_{j=2}^{\ell} \mu_j$, and thus we have equality. We deduce that $T_2 = \{s\}$, and that $\lambda_1 = \mu_1$ as $\lambda_1 = 1 - \sum_{i=2}^{k} \lambda_i = 1 - \sum_{j=2}^{\ell} \mu_j = \mu_1$. Then

$$(\lambda_1 + \lambda_2)\chi_{S_1} + \sum_{i=3}^{k} \lambda_i \chi_{S_i \setminus \{s\}} = (\mu_1 + \mu_2)\chi_{T_1} + \sum_{j=3}^{\ell} \mu_j \chi_{T_j \setminus \{s\}}$$

are two convex combinations. Induction on $|S_k|$ now finishes the proof, where we also use that $\lambda_1 = \mu_1$. □

Lemma 11. *Let G be a graph, and let (X, Y) be a G-harmonious tuple (with possibly $X = \emptyset$ or $Y = \emptyset$). If $S_1, \ldots, S_k$ are stable sets then there are stable sets $S'_1, \ldots, S'_k$ so that*

(i) $S'_1 \cap X \subseteq \ldots \subseteq S'_k \cap X$*;*

(ii) $S'_1 \cap Y \supseteq \ldots \supseteq S'_k \cap Y$*; and*

(iii) $\sum_{i=1}^{k} \chi_{S'_i} = \sum_{i=1}^{k} \chi_{S_i}$.

Proof. We start with two easy claims. First:

> *For any two stable sets S, T there are stable sets S' and T' such that $\chi_S + \chi_T = \chi_{S'} + \chi_{T'}$ and $S' \cap X \subseteq T' \cap X$.* (1)

Indeed, assume there is an $x \in (S \cap X) \setminus T$. Denote by K the component of the induced graph $G[S \cup T]$ that contains x, and consider the symmetric differences $\tilde{S} = S \triangle K$ and $\tilde{T} = T \triangle K$. Clearly, $\chi_S + \chi_T = \chi_{\tilde{S}} + \chi_{\tilde{T}}$. Moreover, K meets X only in S as otherwise K would contain an induced x–$(T \cap X)$ path, which then has necessarily odd length. This, however, is impossible as (X, Y) is G-harmonious. Therefore, $x \notin \tilde{S} \cap X \subset S \cap X$. By repeating this exchange argument for any remaining $x' \in (\tilde{S} \cap X) \setminus \tilde{T}$, we arrive at the desired stable sets S' and T'. This proves (1).

We need a second, similar assertion:

> *For any two stable sets S, T with $S \cap X \subseteq T \cap X$ there are stable sets S' and T' such that $\chi_S + \chi_T = \chi_{S'} + \chi_{T'}$, $S' \cap X = S \cap X$ and $S' \cap Y \supseteq T' \cap Y$.* (2)

To see this, assume there is a $y \in (T \cap Y) \setminus S$, and let K be the component of $G[S \cup T]$ containing y, and set $\tilde{S} = S \triangle K$ and $\tilde{T} = T \triangle K$. The component K may not meet $T \cap X$, as then it would contain an induced y–$(T \cap X)$ path. This path would have even length, contradicting the definition of a G-harmonious tuple. As above, we see, moreover, that K meets Y only in T; otherwise there would be an induced odd y–$(S \cap Y)$ path, which is impossible. Thus, $\tilde{S}$, $\tilde{T}$ satisfy the first two conditions we want to have for S', T', while $(\tilde{T} \cap Y) \setminus \tilde{S}$ is smaller than $(T \cap Y) \setminus S$. Again repeating the argument yields S', T' as desired. This proves (2).

We now apply (1) iteratively to S_1 (as S) and each of $S_2, \ldots, S_k$ (as T) in order to obtain stable sets $R_1, \ldots, R_k$ with $R_1 \cap X \subseteq R_i \cap X$ for every $i = 2, \ldots, k$

and $\sum_{i=1}^k \chi_{S_i} = \sum_{i=1}^k \chi_{R_i}$. We continue applying (1), first to R_2 and each of $R_3, \ldots, R_k$, then to the resulting R'_3 and each of $R'_4, \ldots, R'_k$ and so on, until we arrive at stable sets $T_1, \ldots, T_k$ with $\sum_{i=1}^k \chi_{S_i} = \sum_{i=1}^k \chi_{T_i}$ that are nested on X: $T_1 \cap X \subseteq \ldots \subseteq T_k \cap X$.

In a similar way, we use (2) to force the stable sets to become nested on Y as well. First, we apply (2) to T_1 (as S) and to each of $T_2, \ldots, T_k$ (as T), then to the resulting T'_3 and each of $T'_4, \ldots, T'_k$, and so on. Proceeding in this manner, we obtain the desired stable sets $S'_1, \ldots, S'_k$. □

Lemma 12. *Let (G_1, G_2) be a proper separation of a graph G so that $X = V(G_1) \cap V(G_2)$ is a harmonious cutset. Let $z \in \mathbb{Q}^{V(G)}$ be so that $z|_{G_1} \in \mathrm{SSP}(G_1)$ and $z|_{G_2} \in \mathrm{SSP}(G_2)$. Then $z \in \mathrm{SSP}(G)$.*

The lemma generalises the result by Chudnovsky et al. [7] that $G = G_1 \cup G_2$ is 4-colourable if G_1 and G_2 are 4-colourable.

Proof of Lemma 12. Let $(X_1, \ldots, X_s)$ be a G-harmonious partition of X. As $z|_{G_j} \in \mathrm{SSP}(G_j)$, for $j = 1, 2$, we can express $z|_{G_1}$ as a convex combination of stable sets $S_1, \ldots, S_m$ of G_1, and $z|_{G_2}$ as a convex combination of stable sets $T_1, \ldots, T_{m'}$ of G_2. Since z is a rational vector, we may even assume that

$$z|_{G_1} = \frac{1}{m} \sum_{i=1}^{m} \chi_{S_i} \text{ and } z|_{G_2} = \frac{1}{m} \sum_{i=1}^{m} \chi_{T_i}.$$

Indeed, this can be achieved by repeating stable sets.

We first treat the case when $s \le 2$. If $s = 1$, then set $X_2 = \emptyset$, so that whenever $s \le 2$, we have $X = X_1 \cup X_2$.

Using Lemma 11, we find stable sets $S'_1, \ldots, S'_m$ of G_1 so that $z|_{G_1} = \frac{1}{m} \sum_{i=1}^m \chi_{S'_i}$ and

$$S'_1 \cap X_1 \subseteq \ldots \subseteq S'_m \cap X_1, \text{ and } S'_1 \cap X_2 \supseteq \ldots \supseteq S'_m \cap X_2$$

holds. Analogously, we obtain a convex combination $z|_{G_2} = \frac{1}{m} \sum_{i=1}^m \chi_{T'_i}$ of stable sets $T'_1, \ldots, T'_m$ of G_2 that are increasingly nested on X_1 and decreasingly nested on X_2.

Define $\overline{S}_1 \subsetneq \ldots \subsetneq \overline{S}_k$ to be the distinct restrictions of the sets S'_i to X_1. More formally, let $1 = i_1 < \ldots < i_k < i_{k+1} = m + 1$ be so that

$$\overline{S}_t = S'_i \cap X_1 \text{ for all } i_t \le i < i_{t+1}$$

We set, moreover, $\lambda_t = \frac{1}{m}(i_{t+1} - i_t)$. Equivalently, $m\lambda_t$ is the number of S'_i with $\overline{S}_t = S'_i \cap X_1$. Then $z|_{X_1} = \sum_{t=1}^k \lambda_t \chi_{\overline{S}_t}$ is a convex combination.

We do exactly the same in G_2 in order to obtain $z|_{X_1} = \sum_{t=1}^k \mu_t \chi_{\overline{T}_t}$, where the sets $\overline{T}_t$ are the distinct restrictions of the T'_i to X_1. With Lemma 10, we deduce first that $\overline{S}_t = \overline{T}_t$ and $\lambda_t = \mu_t$ for all t, from which we get that

$$S'_i \cap X_1 = T'_i \cap X_1 \text{ for all } i = 1, \ldots, m.$$

The same argument, only applied to the restrictions of S'_i and of T'_i to X_2, yields that also

$$S'_i \cap X_2 = T'_i \cap X_2 \text{ for all } i = 1, \ldots, m.$$

Thus, $R_i = S'_i \cup T'_i$ is, for every $i = 1, \ldots, m$, a stable set of G. Consequently, $z = \frac{1}{m}\sum_{i=1}^m \chi_{R_i}$ is a convex combination of stable sets and thus a point of SSP(G).

It remains to treat the case when the harmonious cutset has at least three parts, that is, when $s \geq 3$. We claim that there are sets $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_s$ of stable sets of G_1 so that

(a) $z|_{G_1} = \frac{1}{m}\sum_{j=0}^s \sum_{S\in\mathcal{S}_j} \chi_S$ and $\sum_{j=0}^s |\mathcal{S}_j| = m$;

(b) for $j = 1, \ldots, s$ if $S \in \mathcal{S}_j$ then $X_j \cap S$ is non-empty; and

(c) for $j = 0, \ldots, s$ if $S, T \in \mathcal{S}_j$ then $X_j \cap S \subseteq X_j \cap T$ or $X_j \cap S \supseteq X_j \cap T$.

Moreover, there are analogous sets $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_s$ for G_2.

To prove the claim note first that each S_i meets at most one of the sets X_j as each two induce a complete bipartite graph. Therefore, we can partition $\{S_1, \ldots, S_m\}$ into sets $\mathcal{S}'_0, \ldots, \mathcal{S}'_s$ so that (a) and (b) are satisfied. Next, we apply Lemma 11 to each $\mathcal{S}'_j$ and $(X_j, \emptyset)$ in order to obtain sets $\mathcal{S}''_j$ that satisfy (a) and (c) but not necessarily (b); property (a) still holds as Lemma 11 guarantees $\sum_{S\in\mathcal{S}'_j} \chi_S = \sum_{S\in\mathcal{S}''_j} \chi_S$ for each j. If (b) is violated, then only because for some $j \neq 0$ there is $S \in \mathcal{S}''_j$ that is not only disjoint from X_j but also from all other $X_{j'}$. In order to repair (b) we remove all stable sets S in $\bigcup_{j=1}^s \mathcal{S}''_j$ that are disjoint from $\bigcup_{j=1}^s X_j$ from their respective sets and add them to $\mathcal{S}''_0$. The resulting sets $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_s$ then satisfy (a)–(c). The proof for the $\mathcal{T}_j$ is the same.

As a consequence of (a) and (b) it follows for $j = 0, 1, \ldots, s$ that

$$\sum_{S\in\mathcal{S}_j} \chi_{S\cap X_j} = m \cdot z|_{X_j} = \sum_{T\in\mathcal{T}_j} \chi_{T\cap X_j} \tag{3}$$

Now, consider $j \neq 0$. Then, by (b) and (c), there is a vertex $v \in X_j$ that lies in every $S \in \mathcal{S}_j$. Thus, we have $\sum_{S\in\mathcal{S}_j} \chi_S(v) = |\mathcal{S}_j|$.

Evaluating (3) at $v \in X_j$, we obtain

$$|\mathcal{S}_j| = m \cdot z(v) = \sum_{T\in\mathcal{T}_j} \chi_T(v) \leq |\mathcal{T}_j|.$$

Reversing the roles of $\mathcal{S}_j$ and $\mathcal{T}_j$, we also get $|\mathcal{T}_j| \leq |\mathcal{S}_j|$, and thus that $|\mathcal{T}_j| = |\mathcal{S}_j|$, as long as $j \neq 0$. That this also holds for $j = 0$ follows from $m = \sum_{j=0}^s |\mathcal{S}_j| = \sum_{j=0}^s |\mathcal{T}_j|$, so that we get $m_j := |\mathcal{S}_j| = |\mathcal{T}_j|$ for every $j = 0, 1, \ldots, s$.

Together with (3) this implies, in particular, that

$$\frac{1}{m_j}\sum_{S\in\mathcal{S}_j} \chi_{S\cap X_j} = \frac{1}{m_j}\sum_{T\in\mathcal{T}_j} \chi_{T\cap X_j}$$

We may, therefore, define a vector y^j on $V(G)$ by setting

$$y^j|_{G_1} := \frac{1}{m_j}\sum_{S\in\mathcal{S}_j} \chi_S \textit{ and } y^j|_{G_2} := \frac{1}{m_j}\sum_{T\in\mathcal{T}_j} \chi_T \tag{4}$$

For any $j = 0, \ldots, s$, define $G^j = G - \bigcup_{r \neq j} X_r$, and observe that X_j is a harmonious cutset of G^j consisting of only one part. (That is, X_j is G^j-harmonious.) Moreover, as (4) shows, the restriction of y^j to $G_1 \cap G^j$ lies in $\text{SSP}(G_1 \cap G^j)$, while the restriction to $G_2 \cap G^j$ lies in $\text{SSP}(G_2 \cap G^j)$. Thus, we can apply the first part of this proof, when $s \leq 2$, in order to deduce that $y^j \in \text{SSP}(G^j) \subseteq \text{SSP}(G)$.

To finish the proof we observe, with (a) and (4), that

$$z = \sum_{j=0}^{s} \frac{m_j}{m} y^j$$

As, by (a), $\sum_{j=0}^{s} m_j = m$, this means that z is a convex combination of points in $\text{SSP}(G)$, and thus itself an element of $\text{SSP}(G)$. □

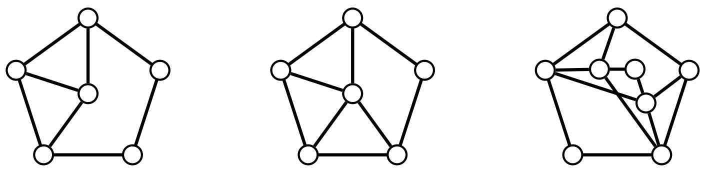
Corollary 13. *Let (G_1, G_2) be a proper separation of G so that $X = V(G_1) \cap V(G_2)$ is a harmonious cutset. Then G is t-perfect if and only if G_1 and G_2 are t-perfect.*

Proof. Assume that G_1 and G_2 are t-perfect, and consider a rational point $z \in \text{TSTAB}(G)$. Then $z|G_1 \in \text{SSP}(G_1)$ and $z|G_2 \in \text{SSP}(G_2)$, which means that Lemma 12 yields $z \in \text{SSP}(G)$. Since this is true for all rational z it extends to real z as well. □

The corollary directly implies Lemma 9.

6 P_5-free graphs

Let $\mathcal{F}$ be the set of graphs consisting of P_5, K_4, W_5, C_7^2, A_{10}^2 and A_{13}^3 together with the three graphs in Figure 5. Note that the latter three graphs all contain K_4 as a t-minor: for (a) and (b) K_4 is obtained by a t-contraction at any vertex of degree 2, while for (c) both vertices of degree 2 need to be t-contracted. In particular, every graph in $\mathcal{F}$ besides P_5 is t-imperfect. We say that a graph is *$\mathcal{F}$-free* if it contains none of the graphs in $\mathcal{F}$ as an induced subgraph.



(a) (b) (c)

Figure 5: Three graphs that t-contract to K_4

We prove a lemma that implies directly Theorem 1:

Lemma 14. *Any $\mathcal{F}$-free graph is t-perfect.*

We first examine how a vertex may position itself relative to a 5-cycle in an $\mathcal{F}$-free graph.

Lemma 15. *Let G be an $\mathcal{F}$-free graph. If v is a neighbour of a 5-hole C in G then v has either exactly two neighbours in C, and these are non-consecutive in C; or v has exactly three neighbours in C, and these are not all consecutive.*

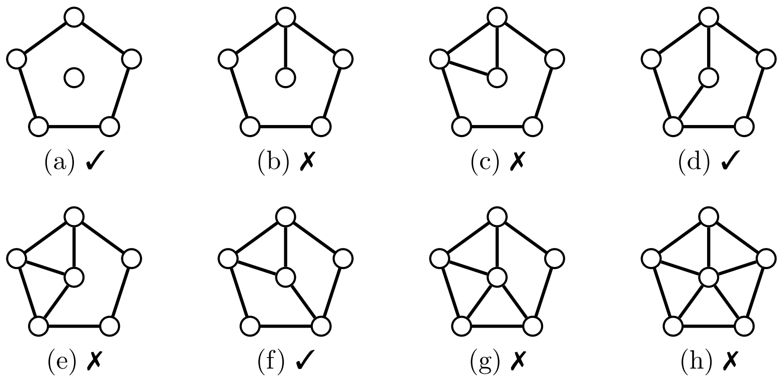


Figure 6: The types of neighbours of a 5-hole

Proof. See Figure 6 for the possible types of neighbours (up to isomorphy). Of these, (b) and (c) contain an induced P_5; (e) and (g) are the same as (a) and (b) in Figure 5 and thus in $\mathcal{F}$; (h) is W_5. Only (d) and (f) remain. □

Lemma 16. *Let G be an $\mathcal{F}$-free graph, and let u and v be two non-adjacent vertices such that both of them have precisely three neighbours in a 5-hole C. Then u and v have either all three or exactly two non-consecutive neighbours in C in common.*

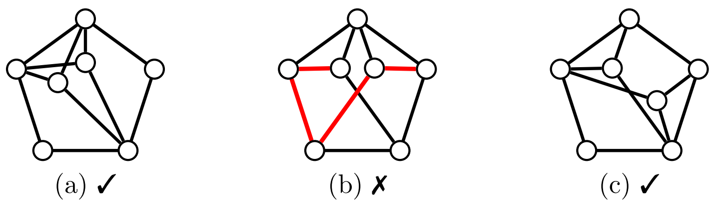


Figure 7: The possible configurations of Lemma 16

Proof. By Lemma 15, both of u and v have to be as in (f) of Figure 6. Figure 7 shows the possible configurations of u and v (up to isomorphy). Of these, (b) is impossible as there is an induced P_5—the other two configurations (a) and (c) may occur. □

A subgraph H of a graph G is *dominating* if every vertex in $G - H$ has a neighbour in H.

Lemma 17. *Let G be an $\mathcal{F}$-free graph. Then, either any 5-hole of G is dominating or G contains a harmonious cutset.*

Proof. Assume that there is a 5-hole $C = c_1 \dots c_5 c_1$ that fails to dominate G. Our task consists in finding a harmonious cutset. We first observe:

$$\text{Let } u \in N(C) \text{ be a neighbour of some } x \notin N(C). \text{ Then } u \text{ has exactly three neighbours in } C, \text{ not all of which are consecutive.} \tag{5}$$

So, such a u is as in (f) of Figure 6. Indeed, by Lemma 15, only (d) or (f) in Figure 6 are possible. In the former case, we may assume that the neighbours of u in C are c_1 and c_3. Then, however, $xuc_1c_4c_5$ is an induced P_5. This proves (5).

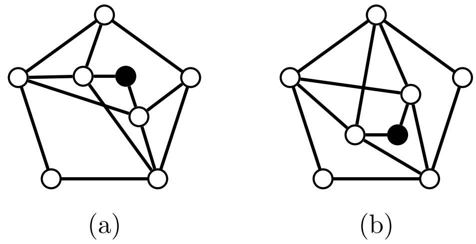


Figure 8: x in solid black.

Consider two adjacent vertices $y, z \notin N(C)$, and assume that there is a $u \in N(y) \cap N(C)$ that is not adjacent to z. We may assume that $N(u) \cap C = \{c_1, c_2, c_4\}$ by (5). Then, $zyuc_2c_3$ is an induced P_5, which is impossible. Thus:

$$N(y) \cap N(C) = N(z) \cap N(C) \text{ for any adjacent } y, z \notin N(C). \tag{6}$$

Next, fix some vertex x that is not dominated by C (and, by assumption, there is such a vertex). As a consequence of (6), $N(x) \cap N(C)$ separates x from C. In particular,

$$X := N(x) \cap N(C) \text{ is a separator.} \tag{7}$$

Consider two vertices $u, v \in X$. Then, by (5), each of u and v have exactly three neighbours in C, not all of which are consecutive. We may assume that $N(u) \cap V(C) = \{c_1, c_2, c_4\}$.

First, assume that $uv \in E(G)$, and suppose that the neighbourhoods of u and v in C are the same. This, however, is impossible as then u, v, c_1, c_2 form a K_4. Therefore, $uv \in E(G)$ implies $N(u) \cap V(C) \neq N(v) \cap V(C)$.

Now assume $uv \notin E(G)$. By Lemma 16, there are only two possible configurations (up to isomorphy) for the neighbours of v in C; these are (a) and (c) in Figure 7. The first of these, (a) in Figure 8, is impossible, as this is a graph of $\mathcal{F}$; see Figure 5 (c). Thus, we see that u, v are as in (b) of Figure 8, that is, that u and v have the same neighbours in C.

To sum up, we have proved that:

$$uv \in E(G) \Leftrightarrow N(u) \cap V(C) \neq N(v) \cap V(C) \text{ for any two } u, v \in X \tag{8}$$

An immediate consequence is that the neighbourhoods in C partition X into stable sets $X_1, \dots, X_k$ such that X_i is complete to X_j whenever $i \neq j$. As X cannot contain any triangle—together with x this would result in a K_4—it follows that $k \leq 2$. If $k = 1$, we put $X_2 = \emptyset$ so that always $X = X_1 \cup X_2$.

We claim that X is a harmonious cutset. As X is a separator, by (7), we only need to prove that (X_1, X_2) is G-harmonious. For this, we have to check the parities of induced X_1-paths and of X_2-paths; since X_1 is complete to X_2 any induced X_1–X_2 path is a single edge and has therefore odd length.

Suppose there is an odd induced X_1-path or X_2-path. Clearly, we may assume there is such a path P that starts in $u \in X_1$ and ends in $v \in X_1$. As X_1 is stable, and as G is P_5-free, it follows that P has length 3. So, let $P = upqv$.

Let us consider the position of p and q relative to C. We observe that neither p nor q can be in C. Indeed, if, for instance, p was in C then p would also be a neighbour of v since $N(u) \cap V(C) = N(v) \cap V(C)$, by (8). This, however, is impossible as P is induced.

Next, assume that $p, q \notin N(C)$ holds. Since p and q are adjacent, we can apply (6) to p and q, which results in $N(p) \cap N(C) = N(q) \cap N(C)$. However, as u lies in $N(p) \cap N(C)$ it then also is a neighbour of q, which contradicts that $upqv$ is induced.

It remains to consider the case when one of p and q, p say, lies in $N(C)$. As p is adjacent to u but not to v, both of which lie in X_1 and are therefore non-neighbours, it follows from (8) that $p \notin X$. In particular, p is not a neighbour of x, which means that $puxv$ is an induced path.

Suppose there is a neighbour $c \in V(C)$ of p that is not adjacent to u. By (8), c is not adjacent to v either, so that $cpuxv$ forms an induced P_5, a contradiction. Thus, $N(p) \cap V(C) \subseteq N(u)$ has to hold. By (5), we may assume that the neighbours of u in C are precisely c_1, c_2, c_4. As u and p are adjacent, p cannot be neighbours with both of c_1 and c_2, as this would result in a K_4. Thus, we may assume that $N(p) \cap V(C) = \{c_2, c_4\}$. (Note, that p has at least two neighbours in C, by Lemma 15.)

To conclude, we observe that $pc_4c_5c_1c_2p$ forms a 5-hole, in which u has four neighbours, namely c_1, c_2, c_4, p. This, however, is in direct contradiction to Lemma 15, which means that our assumption is false, and there is no odd induced X_1-path, and no such X_2-path either. Consequently, (X_1, X_2) is G-harmonious, and $X = X_1 \cup X_2$ therefore a harmonious cutset. □

Proposition 18. *Let G be a t-imperfect graph. Then either G contains an odd hole or it contains K_4 or C_7^2 as an induced subgraph.*

Proof. Assume that G does not contain any odd hole and neither K_4 nor C_7^2 as an induced subgraph. Observe that any odd antihole of length ≥ 9 contains K_4. Since the complement of a 5-hole is a 5-hole, and since C_7^2 is the odd antihole of length 7, it follows that G cannot contain any odd antihole at all.

Now, by the strong perfect graph theorem it follows that G is perfect. (Note that we do not need the full theorem but only the far easier version for K_4-free graphs; see Tucker [28].) Since G does not contain any K_4 it is therefore t-perfect as well. □

Lemma 19. *Let G be an $\mathcal{F}$-free graph. If G contains a 5-hole, and if every 5-hole is dominating then G is near-bipartite.*

Proof. Let G contain a 5-hole, and assume every 5-hole to be dominating. Suppose that the lemma is false, i.e. that G fails to be near-bipartite. In particular, there is a vertex v such that $G - N(v)$ is not bipartite, and therefore contains an induced odd cycle T. As any 5-hole is dominating and any k-hole with $k > 5$ contains an induced P_5, T has to be a triangle. Let $T = xyz$. We distinguish two cases, both of which will lead to a contradiction.

Case: v lies in a 5-hole C.
Let $C = c_1 \dots c_5 c_1$, and $v = c_1$. Then T could meet C in $0, 1$ or 2 vertices. If T has two vertices with C in common, these have to be c_3 and c_4 as the others are neighbours of v. Then, the third vertex of T has two consecutive neighbours in C, which means that by Lemma 15 its third neighbour in C has to be $c_1 = v$, which is impossible.

Next, suppose that T meets C in one vertex, $c_3 = z$, say. By Lemma 15, each of x, y has to have a neighbour opposite of c_3 in C, that is, either c_1 or c_5. As $c_1 = v$, both of x, y are adjacent with c_5. The vertices x, y could have a third neighbour in C; this would necessarily be c_2. However, not both can be adjacent to c_2 as then x, y, c_2, c_3 would induce a K_4. Thus, assume x to have exactly c_3 and c_5 as neighbours in C. This means that $C' = c_3 x c_5 c_1 c_2 c_3$ is a 5-hole in which y has at least three consecutive neighbours, c_3, x, c_5, which is impossible (again, by Lemma 15).

Finally, suppose that T is disjoint from C. Each of x, y, z has at least two neighbours among $c_2, \dots, c_5$, and no two have c_3 or c_4 as neighbour; otherwise we would have found a triangle in $G - N(v)$ meeting C in exactly one vertex, and could reduce to the previous subcase. Thus, we may assume that x is adjacent to c_2 and c_5. Moreover, since no vertex of x, y, z can be adjacent to both c_3 and c_4 (as then it would also be adjacent to c_1, by Lemma 15) and no $c_i \in C$ can be adjacent to all vertices of T (because otherwise c_i, x, y, z would form a K_4), it follows that we may assume that y is adjacent to c_2 but not to c_5, while z is adjacent to c_5 but not to c_2. Then, $c_1 c_2 y z c_5 c_1$ is a 5-hole in which x has four neighbours, in obvious contradiction to Lemma 15. Therefore, this case is impossible.

Case: v does not lie in any 5-hole.
Let $C = c_1 \dots c_5 c_1$ be a 5-hole. Since every 5-hole is dominating, v has a neighbour in C, and thus is, by Lemma 15, either as in (f) of Figure 6 or as in (d). The latter, however, is impossible since then v would be contained in a 5-hole. Therefore, we may assume that the neighbours of v in C are precisely $\{c_1, c_2, c_4\}$. As a consequence, T can meet C in at most c_3 and c_5, both not in both as C is induced.

Suppose $T = xyz$ meets C in $x = c_3$. If y is not adjacent to either of c_1 and c_4, then $c_1 v c_4 x y$ forms an induced P_5. If, on the other hand, y is adjacent to c_4 then, by Lemma 15, also to c_1. Thus, y is either adjacent to c_1 or to both c_1 and c_4. The same holds for z. Since y and z are adjacent, they cannot both have three neighbours in C (otherwise G would contain a K_4). Suppose $N(y) \cap C = \{x, c_1\}$. But then $x c_4 c_5 c_1 y x$ forms an induced 5-cycle in which z has at least three consecutive neighbours; a contradiction to Lemma 15.

Consequently, T is disjoint from every 5-hole. By Lemma 15, each of x, y, z has neighbours in C as in (d) or (f) of Figure 6. However, if any of x, y, z has only two neighbours in C as in (d) then that vertex together with four vertices of C forms a 5-hole that meets T—this is precisely the situation of the previous subcase. Thus, we may assume that all vertices of T have three neighbours in C as in (f) of Figure 6. If we consider the possible configurations of two non-adjacent vertices which have three neighbours in C (namely v and a vertex of T) as we have done in Lemma 7, we see that only (a) and (c) in Figure 7 are possible. But then each vertex of T has to be adjacent to c_4, which means that T together with c_4 induces a K_4, which is impossible. □

Proof of Lemma 14. Suppose that G is a t-imperfect and but $\mathcal{F}$-free. By deleting suitable vertices we may assume that every proper induced subgraph of G is t-perfect. In particular, by Lemma 9, G does not admit a harmonious cutset. Since G is t-imperfect it contains an odd hole, by Proposition 18, and since G is P_5-free, the odd hole is of length 5. From Lemma 17 we deduce that any 5-hole is dominating. Lemma 19 implies that G is near-bipartite.

Noting that both A^4_{13} and A^7_{19}, as well as any Möbius ladder or any odd wheel larger than W_5, contain an induced P_5, we see with Proposition 3 that G is t-perfect after all. □

By Lemma 14, a P_5-free graph is either t-perfect or contains one of eight t-imperfect graphs as an induced subgraph. Obviously, checking for these forbidden induced subgraphs can be done in polynomial time, so that we get as immediate algorithmic consequence:

Theorem 20. *P_5-free t-perfect graphs can be recognised in polynomial time.*

We suspect, but cannot currently prove, that t-perfection can be recognised as well in polynomial time in near-bipartite graphs.

7 Colouring

Can t-perfect graphs always be coloured with few colours? This is one of the main open questions about t-perfect graphs. A conjecture by Shepherd and Sebő asserts that four colours are always enough:

Conjecture 21 (Shepherd; Sebő [23])**.** *Every t-perfect graph is 4-colourable.*

The conjecture is known to hold in a number of graph classes, for instance in claw-free graphs, where even three colours are already sufficient; see [5]. It is straightforward to verify the conjecture for near-bipartite graphs:

Proposition 22. *Every near-bipartite t-perfect graph is 4-colourable.*

Proof. Pick any vertex v of a near-bipartite and t-perfect graph G. Then $G - N(v)$ is bipartite and may be coloured with colours $1, 2$. On the other hand, as G is t-perfect the neighbourhood $N(v)$ necessarily induces a bipartite graph as well; otherwise v together with a shortest odd cycle in $N(v)$ would form an odd wheel. Thus we can colour the vertices in $N(v)$ with the colours $3, 4$. □

Near-bipartite t-perfect graphs can, in general, not be coloured with fewer colours. Indeed, this is even true if we restrict ourselves further to complements of line graphs, which is a subclass of near-bipartite graphs. Two t-perfect graphs in this class that need four colours are: $\overline{L(\Pi)}$, the complement of the line graph of the prism, and $\overline{L(W_5)}$. The former was found by Laurent and Seymour (see [25, p. 1207]), while the latter was discovered by Benchetrit [2]. Moreover, Benchetrit showed that any 4-chromatic t-perfect complement of a line graph contains one of $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ as an induced subgraph.

How about P_5-free t-perfect graphs? Applying insights of Sebő and of Sumner, Benchetrit [2] proved that P_5-free t-perfect graphs are 4-colourable. This is not tight:

Theorem 23. *Every P_5-free t-perfect graph G is 3-colourable.*

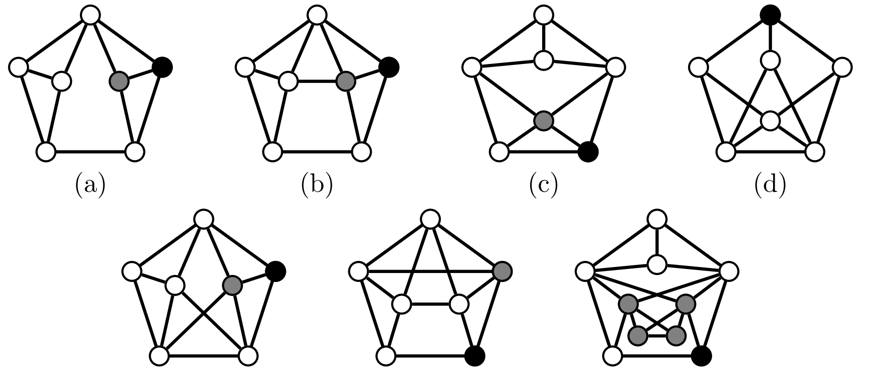


Figure 9: The remaining 4-critical P_5-free graphs of Theorem 24; in Maffray and Morel [22] these are called F_3–F_8 and F_{10}. In each graph, deleting the grey vertices and then t-contracting at the black vertex results in K_4.

For the proof we use that there is a finite number of obstructions for 3-colourability in P_5-free graphs:

Theorem 24 (Maffray and Morel [22])**.** *A P_5-free graph is 3-colourable if and only if it does not contain K_4, W_5, C_7^2, A_{10}^2, A_{13}^3 or any of the seven graphs in Figure 9 as an induced subgraph.*

(Maffray and Morel call these graphs F_1–F_{12}. The graphs K_4, W_5, C_7^2, A_{10}^2, A_{13}^3 are respectively F_1, F_2, F_9, F_{11} and F_{12}.) A similar result was obtained by Bruce, Hoàng and Sawada [4], who gave a list of five forbidden (not necessarily induced) subgraphs.

Proof of Theorem 23. Any P_5-free graph G that cannot be coloured with three colours contains one of the twelve induced subgraphs of Theorem 24. Of these twelve graphs, we already know that K_4, W_5, C_7^2, A_{10}^2, A_{13}^3 are t-imperfect, and thus cannot be induced subgraphs of a t-perfect graph. It remains to consider the seven graphs in Figure 9. These graphs are t-imperfect, too: each can be turned into K_4 by first deleting the grey vertices and then performing a t-contraction at the respective black vertex. □

We mention that Benchetrit [2] also showed that P_6-free t-perfect graphs are 4-colourable. This is tight: both $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ (and indeed all complements of line graphs) are P_6-free. We do not know whether P_7-free t-perfect graphs are 4-colourable.

We turn now to fractional colourings. A motivation for Conjecture 21 was certainly the fact that the *fractional chromatic number* $\chi_f(G)$ of a t-perfect graph G is always bounded by 3. More precisely, if $og(G)$ denotes the *odd girth* of G, that is, the length of the shortest odd cycle, then $\chi_f(G) = 2\frac{og(G)}{og(G)-1}$ as long as G is t-perfect (and non-bipartite). This follows from linear programming duality; see for instance Schrijver [25, p. 1206].

Recall that a graph G is perfect if and only if $\chi(H) = \omega(H)$ for every induced subgraph H of G. As odd cycles seem to play a somewhat similar role for t-

perfection as cliques play for perfection, one might conjecture that t-perfection is characterised in an analogous way:

Conjecture 25. *A graph G is t-perfect if and only if $\chi_f(H) = 2\frac{og(H)}{og(H)-1}$ for every non-bipartite t-minor H of G.*

Note that the conjecture becomes false if, instead of t-minors, only induced subgraphs H are considered. Indeed, in the t-imperfect graph obtained from K_4 by subdividing some edge twice, all induced subgraphs satisfy the condition (but not the t-minor K_4).

An alternative but equivalent formulation of the conjecture is: $\chi_f(G) > 2\frac{og(G)}{og(G)-1}$ holds for every minimally t-imperfect graph G. It is straightforward to check that all minimally t-imperfect graphs that are known to date satisfy this. In particular, it follows that the conjecture is true for P_5-free graphs, for near-bipartite graphs, as well as for claw-free graphs; see [5] for the minimally t-imperfect graphs that are claw-free.

Acknowledgment

We thank Oliver Schaudt for pointing out [22]. We also thank one of the referees for bringing the work of Holm et al. [20] to our attention, as well as identifying inaccuracies that led to a (hopefully!) clearer presentation of Theorems 20 and 23.

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Version January 26, 2017

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