Minimal bricks have many vertices of small degree

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Abstract

We prove that every minimal brick on n vertices has at least $n/9$ vertices of degree at most 4.

1 Introduction

A key element in matching theory is the notion of a brick. We briefly and somewhat informally explain this notion and its role. For a much more detailed treatment we refer to the books of Lovász and Plummer [5] and Schrijver [8].

A *matching* (a set of independent edges) of a graph is *perfect* if every vertex is incident with a matching edge. Consider a matching covered graph, that is a connected graph with at least one edge in which every edge lies in some perfect matching. A *tight cut* of such a graph is a cut that meets every perfect matching in precisely one edge. Contracting one, or the other, side of a tight cut F we obtain two new graphs (which preserve the perfect matching structure we had in the original graph). This operation is called an 'F-contraction', or a 'split along the tight cut F'.

Clearly, we can go on splitting along tight cuts in the newly obtained graphs until arriving at graphs that contain no non-trivial tight cuts. It was shown by Lovász [4] that no matter how we choose the tight cuts we split along, we will essentially always arrive at the same set of graphs (up to multiplicity of edges). The obtained decomposition is generally called a 'tight cut decomposition' or a 'brick and brace decomposition' because the set of final graphs (without non-trivial tight cuts) is divided into those that are bipartite – called *braces* – and those that are not – the *bricks*. This decomposition allows to reduce several problems from matching theory to bricks (e.g. a graph is Pfaffian if and only if its bricks and braces are [3]).

Both bricks and braces have been characterised by Edmonds, Lovász and Pulleyblank [2] in other terms. We omit the characterisation of braces. For the one of bricks, let us first say that a graph G is *bicritical* if $G - \{u, v\}$ has a perfect matching for every choice of distinct vertices u and v. Now bricks are precisely the bicritical and 3-connected graphs [2]. For practical purposes let us consider a brick to be defined this way.

The focus of this paper lies on *minimal bricks*: Those bricks G for which $G - e$ ceases to be a brick for every edge $e \in E(G)$. Minimality often leads to sparsity in some respect. Minimal bricks are no exception: It is known [6] that any minimal brick on n vertices has average degree at most $5 - 14/n$, unless it is

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one of four special bricks (the prism or the wheel W_n for $n = 4, 6, 8$). While thus minimal bricks do have vertices of degree 3 or 4, they may conceivably be very few in number, if the average degree is very close to 5. Of particular interest are vertices that attain the smallest degree possible, which is 3 for a brick.

De Carvalho, Lucchesi and Murty [1] proved that any minimal brick contains a vertex of degree 3, which had been conjectured earlier by Lovász; see [1]. This was extended by Norine and Thomas, who showed the existence of 3 such vertices, and then went on to pose the following stronger conjecture.

Conjecture 1 (Norine and Thomas [6]). *There is an $\alpha > 0$ so that every minimal brick G contains at least $\alpha|V(G)|$ vertices of degree* 3.

Our main result yields further evidence for this conjecture.

Theorem 2. *Every minimal brick G has at least $\frac{1}{9}|V(G)|$ vertices of degree at most* 4.

We hope that the methods developed here, if substantially strengthened, will be useful for attacking Norine and Thomas' conjecture.

2 Brick generation

For practical purposes, the abstract definition of a brick as a 3-connected and bicritical graph may sometimes be less useful than knowing how to obtain a brick from another brick by a small local operation. De Carvalho, Lucchesi and Murty [1] study such operations, and prove that any brick other than the Petersen graph can be obtained by performing these operations successively, starting with either K_4 or the prism. (In particular, every graph in this sequence is a brick.) Norine and Thomas [7] show a generalisation of this result, which they obtained independently.

In particular, every brick has a generating sequence of ever larger bricks. To be useful in induction proofs about minimal bricks, however, it appears necessary that all intermediate graphs are minimal as well, which is unfortunately not guaranteed by the results above. To mend this situation, Norine and Thomas [6] introduce another family of operations, called *strict extensions*, which we shall describe below. Using strict extensions, they find that each minimal brick has a generating sequence consisting only of minimal bricks:

Theorem 3 (Norine and Thomas [6]). *Every minimal brick other than the Petersen graph can be obtained by strict extensions starting from K_4 or the prism, where all intermediate graphs are minimal bricks.*

Notice that although a strict extension of a brick is a brick, a strict extension of a minimal brick need not be a minimal brick [6].

Let us now formally define strict extensions, following Norine and Thomas [6]. There are five types of strict extensions: Strict linear, bilinear, pseudolinear, quasiquadratic and quasiquartic extensions. The first three of these are based on an even simpler operation, the bisplitting of a vertex.

For this, consider a graph H and one of its vertices v of degree at least 4. Partition the neighbourhood of v into two sets N_1 and N_2 such that each contains at least two vertices. We now replace v by two new independent vertices,

v_1 and v_2, where v_1 is incident with the vertices in N_1 and v_2 with the ones in N_2. Finally, we add a third new vertex v_0 that is adjacent to precisely v_1 and v_2. We say that any such graph H' is obtained from H by *bisplitting* v. The vertex v_0 is the *inner vertex* of the bisplit, while v_1 and v_2 are the *outer vertices*. Any time we perform a bisplit at a vertex v we will tacitly assume v to have degree at least 4.

We will now define turn by turn the strict extensions. At the same time we will specify a small set of vertices, the *fundament* of the strict extension. One should think of the fundament as a minimal set of vertices that needs to be present, should we want to perform the extension in some other, usually smaller, graph.

Let v be a vertex of a graph G. We say that G' is a *strict linear extension of* G if G' is obtained by one of the three following operations. (See Figure 1 for an illustration.)

1. We perform a bisplit at v, denote by v_0 the inner vertex, and by v_1 and v_2 the outer vertices of the bisplit. Choose a vertex $u_0 \in V(G) - v$. Add the edge u_0v_0.
2. We perform bisplits at v and at a second vertex u, obtaining outer vertices v_1 and v_2 and inner vertex v_0 from the first bisplit and outer vertices u_1 and u_2 and inner vertex u_0 from the second. Add the edge u_0v_0.
3. We bisplit v, obtaining the inner vertex u_0, and outer vertices u_1 and u_2. We bisplit u_1, obtaining an inner vertex v_0 and outer vertices v_1 and v_2, where v_1 is adjacent to u_0. Add the edge u_0v_0.

The *fundament* of the extension depends on the subtype: For 1. the fundament is comprised of u_0, v plus any choice among the vertices of G of two neighbours of v_1 and of two neighbours of v_2; for 2. it will be u, v together with any two neighbours for each of u_1, u_2, v_1, v_2 that lie in G; and for 3. we choose v, one neighbour of v_1 and two of each of u_2 and v_2, all of them vertices of G.

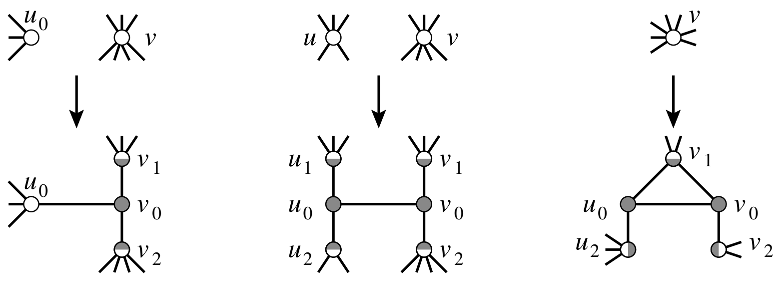

Figure 1: Strict linear extension

Next, assume u, v, w to be three vertices of G, so that w is a neighbour of u but not of v. Bisplit u, and denote by u_2 the new outer vertex that is adjacent to w, by u_1 the other outer vertex and by u_0 the new inner vertex. Subdivide the edge u_2w twice, so that it becomes a a path u_2abw, where a and b are new vertices. Let G' be the graph obtained by adding the edges bu_0 and av; see Figure 2. We say that G' is a *bilinear extension of* G. Its *fundament* consists of

u, v, w together with one neighbour of u_2, neither a nor u_0, and two neighbours of u_1, none equal to u_0.

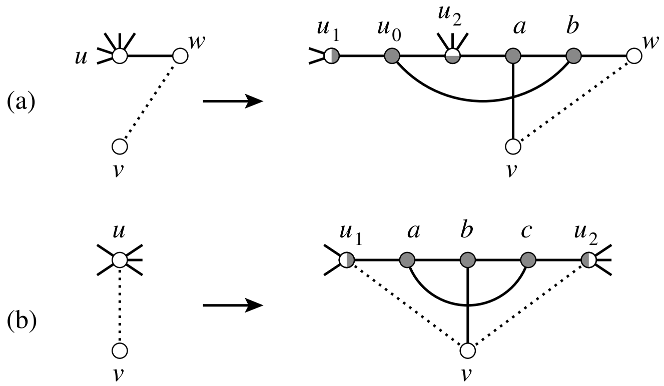

Figure 2: (a) Bilinear extension (b) Pseudolinear extension

A graph G' is called a *pseudolinear extension of* G if it may be obtained from G in the following way. Choose a vertex u of G of degree at least 4, and a non-neighbour v of u. Partition the neighbours of u into two sets N_1 and N_2 each of size at least two. Replace the vertex u by two new ones, u_1 and u_2, so that u_1 is adjacent to every vertex in N_1 and u_2 to every one in N_2. Add three new vertices a, b, c and a path u_1abcu_2, and let the graph resulting from adding the edges ac and bv be G'; see Figure 2. We define the *fundament* as u, v plus two neighbours of each of u_1 and u_2, all chosen among $V(G)$.

The penultimate extension is the *quasiquadratic extension*, shown in Figure 3. Let u and v be two distinct vertices of G, and let x and y be not necessarily distinct vertices so that $x \neq u$, $y \neq v$ and $\{u, v\} \neq \{x, y\}$. If u and v are adjacent, delete the edge between them. Add two adjacent new vertices u' and v' and join u' by an edge to u and x, and make v' adjacent to v and y. The resulting graph G' is a quasiquadratic extension of G.

Norine and Thomas distinguish those quasiquadratic extensions in which the edge uv was present in G, calling these extensions *quadratic*. As we will mostly be concerned with non-quadratic quasiquadratic extensions, let us call these extensions *conservative-quadratic*. Thus, in a conservative-quadratic extension the vertices u and v are not adjacent in G, and, in particular, G is an induced subgraph of G'. Let us remark rightaway that, as a conservative-quadratic extension is not quadratic its name is ill-chosen. To be more correct, we should call such an extension conservative-quasiquadratic. But life is far too short for such a long name.

The *fundament* of the quasiquadratic extension is simply $\{u, v, x, y\}$. For later use, let us call $\{u, v\}$ the *upper fundament* of the extension.

Finally, consider distinct vertices u, v and distinct vertices x, y so that $u \neq y$, $v \neq x$ and $\{u, v\} \neq \{x, y\}$. If present, delete the edges uv and xy. We add four new vertices u', v', x', y' and edges between them so that $u'v'y'x'u'$ is a 4-cycle. The graph obtained by adding the edges uu', vv', xx' and yy' is a *quasiquartic extension of* G. Its *fundament* consists of u, v, x, y.

Now, an extension is called *strict* if it is any of the following: quasiquadratic, quasiquartic, bilinear, pseudolinear, and strict linear. We write $G \to G'$ if G

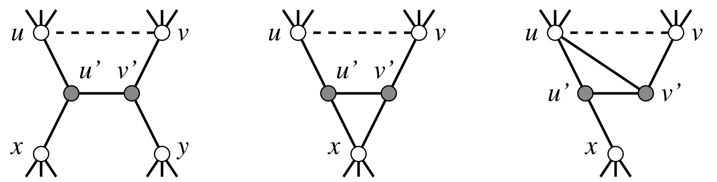

Figure 3: (Quasi-)quadratic extension with different allowed identifications

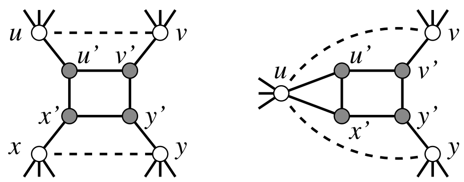

Figure 4: (Quasi-)quartic extension with different allowed identifications

is a brick and G' is obtained from G by a strict extension.

Let F be the fundament of the strict extension $G \to G'$. We observe two trivial properties:

Any vertex outside F has the same degree in G as in G'. (1)

We have $|F| \leq 3 \cdot (|V(G')| - |V(G)|)$. (2)

We note that the ratio 3 is attained for strict linear extensions of the first type: There the fundament consists of u, v plus four neighbours of v, while G' has only two vertices more than G.

It is easy to see that a strict extension G' of a brick G is 3-connected. Also, it is not difficult to find a perfect matching of $G' - x - y$ for any pair of vertices $x, y \in V(G')$, with exception of the pair u_0, v_0 if $G \to G'$ is a strict linear extension, and the pair $u_0 b$, or ac, if $G \to G'$ is a bi- or pseudolinear extension, respectively. These particular cases can be reduced to the exercise of finding a perfect matching in the graph obtained from G by bisplitting a vertex, deleting the new inner vertex and another vertex distinct from the new outer vertices. Using Tutte's theorem, and the fact that G is brick, this is not hard to solve.

This leads to the following lemma, which has also been observed by Norine and Thomas [6]:

Lemma 4. *Any strict extension of a brick is a brick.*

We close this section with an example. In Figure 5 we build up a triple ladder by repeatedly alternating between quasiquartic and quasiquadratic extensions, starting from a prism. As by Lemma 4, strict extensions take a brick to a brick, we deduce that the triple ladder is a brick. To see that it is a minimal brick, note that the deletion of any edge results in a graph that fails to be 3-connected.

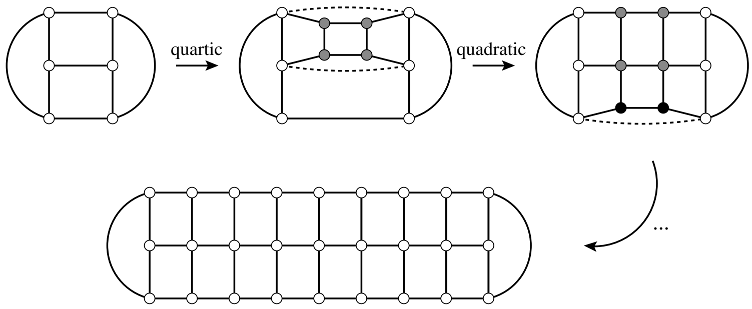

Figure 5: A minimal brick

3 Brick on brick

We will call a sequence $G_0 \to G_1 \to \ldots \to G_k$ a *brick-on-brick sequence* if all the $G_0, \ldots, G_k$ are bricks (not necessarily minimal) and if all the $G_{i-1} \to G_i$ are strict extensions. Thus, the theorem of Norine and Thomas states that every minimal brick G has such a brick-on-brick sequence that starts with K_4 or the prism and ends with G, and in which all intermediate bricks are minimal—unless G is the Petersen graph.

We formulate a simple lemma that allows us to reorder a brick-on-brick sequence.

Lemma 5. *Let $A \to B \to C$ be a brick-on-brick sequence, so that $A \to B$ is conservative-quadratic with new vertices p, q and so that p, q do not lie in the fundament of $B \to C$. Then there exists a brick B' so that $A \to B' \to C$ is a brick-on-brick sequence and $B' \to C$ is conservative-quadratic with new vertices p, q.*

Proof. Since $A \to B$ is conservative-quadratic, we have that $B - \{p, q\} = A$. It is easy to verify that thus $A \to C - \{p, q\}$ is a strict extension (of the same type as $B \to C$). For this, it is important to note that by assumption, p and q are not in the fundament of $B \to C$. In particular, any bisplittings of $B \to C$ can also be performed in A at vertices of degree ≥ 4. Using Lemma 4, we see that $B' := C - \{p, q\}$ is a brick.

It remains to show that $B' \to C$ is a conservative-quadratic extension. This is easy to check if none of the vertices of the fundament F of $A \to B$ has suffered a bisplit during the operation $A \to B'$. So assume there is a vertex $s \in F$ which is bisplit in $A \to B'$, and say s is adjacent to p in B. Then, however, s is also bisplit in $B \to C$, and in C, one of the new outer vertices, say s_1, is adjacent to p. So $B' \to C$ is a quasiquadratic extension.

Note that the number of edges gained in $A \to B'$ and in $B \to C$ is the same (i.e. $|E(B')| - |E(A)| = |E(C)| - |E(B)|$), and so, also the number of edges gained in $A \to B$ and in $B' \to C$ is the same. Thus, as both extensions $A \to B$ and $B' \to C$ are quasi-quadratic, with $A \to B$, also $B' \to C$ is conservative-quadratic. □

Let us now examine how the edge density changes in a brick-on-brick sequence. Suppose $G = (V, E)$ is a minimal brick other than the Petersen graph, and let $G_0 \to \ldots \to G_k$ be a brick-on-brick sequence for G as given by Theorem 3, that is, $G = G_k$ and G_0 is either the K_4 or the prism. For a set of indices $I \subseteq \{1, \ldots, k\}$ we define $\nu(I)$ to be the total number of vertices added in extensions corresponding to I:

$$\nu(I) := \sum_{i \in I} (|V(G_i)| - |V(G_{i-1})|).$$

Similarly, we define

$$\epsilon(I) := \sum_{i \in I} (|E(G_i)| - |E(G_{i-1})|).$$

Now, let I_1 be the set of indices $i \in \{1, \ldots, k\}$ for which $G_{i-1} \to G_i$ is a strict linear, bilinear or pseudolinear extension, and set $\nu_1 = \nu(I_1)$ and $\epsilon_1 = \epsilon(I_1)$. We define analogously I_2, ν_2 and ϵ_2 (resp. I_2^c, ν_2^c and ϵ_2^c) for quasiquadratic (resp. conservative-quadratic) extensions and I_3, ν_3 and ϵ_3 for quasiquartic extensions.

Finally, let $\nu_0 := |V(G_0)|$ and $\epsilon_0 := |E(G_0)|$. As G_0 is either K_4 or the prism it follows that $(\nu_0, \epsilon_0) \in \{(4, 6), (6, 9)\}$. Moreover, we clearly have that

$$|V(G)| = \nu_0 + \nu_1 + \nu_2 + \nu_3 \text{ and } |E(G)| = \epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3. \tag{3}$$

It is easy to calculate that

$$\epsilon_0 = \frac{3}{2}\nu_0,\ \epsilon_1 \leq \frac{3}{2}\nu_1,\ (\epsilon_2 - \epsilon_2^c) = \frac{4}{2}(\nu_2 - \nu_2^c),\ \epsilon_2^c = \frac{5}{2}\nu_2^c \text{ and } \epsilon_3 \leq \frac{8}{4}\nu_3. \tag{4}$$

From (4), we see that the 'edge density gain' is largest when performing conservative-quadratic extensions. In fact, the greater the average degree of a minimal brick, the more conservative-quadratic extensions must have been used in any of its brick-on-brick sequences:

Lemma 6. *Let $\delta > 0$, and let G be a minimal brick with average degree $d(G) \geq 4 + \delta$. For any brick-on-brick sequence $G_0 \to \ldots \to G_k$ with $G = G_k$ and $G_0 \in \{K_4, Prism\}$ it holds that $\nu_2^c \geq \delta |V(G)|$.*

Proof. Let $G = (V, E)$. Using (3) and (4), we find that

$$\begin{aligned}
\frac{4+\delta}{2} &\leq \frac{|E|}{|V|} \\
&= \frac{1}{|V|}\left(\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3\right) \\
&\leq \frac{1}{|V|}\left(\frac{3}{2}\nu_0 + \frac{3}{2}\nu_1 + 2(\nu_2 - \nu_2^c) + \frac{5}{2}\nu_2^c + 2\nu_3\right) \\
&\leq \frac{1}{|V|}\left(2|V| + \frac{1}{2}\nu_2^c\right),
\end{aligned}$$

and consequently, $\nu_2^c \geq \delta |V|$. □

On the other hand, we can show that two conservative-quadratic extensions cannot happen directly 'on top of each other':

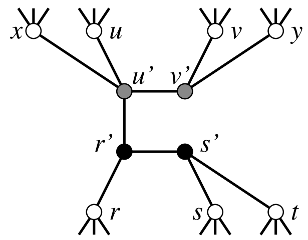

Figure 6: Applying two conservative-quadratic extensions on top of each other, as in Lemma 7.

Lemma 7. *Let G be a brick, and let G'' be a conservative-quadratic extension of a conservative-quadratic extension G' of G. Let u' and v' be the new vertices of G'. If one of u', v' is used for the fundament of $G' \to G''$ then G'' is not a minimal brick.*

Proof. We shall use the notation from Figure 6, that is, $\{x, u, v, y\}$ is the fundament of the conservative-quadratic extension $G \to G'$, and $\{u', r, s, t\}$ is the fundament of the conservative-quadratic extension $G' \to G''$, with new vertices r' and s', where r' is adjacent to u' and r, and s' is adjacent to s and t. Several of these vertices may be identified, some of them are by definition distinct:

$$u \neq x,\ v \neq y,\ u' \neq r,\ s \neq t,\ \text{ and } u', v', r', s' \text{ are pairwise distinct.}$$

Assume for contradiction that G'' is a minimal brick. We start by proving that

$$\{s, t\} \cup \{x, u, v'\} = \emptyset. \tag{5}$$

Indeed, suppose otherwise, i.e. there is a vertex $w \in \{s, t\} \cup \{x, u, v'\}$. Then, as $G'' - u'w$ is a quadratic extension of G', the graph $G'' - wu'$ is a brick. Thus G'' is not minimal, against our assumption.

Now, we know that at least one of x and u is not in $\{v, y\}$, say $x \notin \{v, y\}$. Also, as s and t are distinct, at most one of them is equal to u', say $s \neq u'$. Together with (5), this implies that $\tilde{G} := G' - uu' \cup u's$ is a conservative-quadratic extension of G.

As G'' is a conservative-quadratic extension of G', we know that $u' \neq r$, and $\{u', r\} = \{s, t\}$. Thus, the graph $G'' - uu'$ is a quadratic extension of $\tilde{G}$. Thus G'' is not a minimal brick, a contradiction, as desired. □

We now combine the previous lemmas to find many vertices of degree 3 in the case that our minimal brick G has a rather high average degree.

Lemma 8. *Every minimal brick G of average degree $d(G) \geq 4 + \delta$ with $\delta > 0$ has at least $(4\delta - 3)|V(G)|$ vertices of degree 3.*

Proof. By Theorem 3, there is a brick-on-brick sequence $\mathcal{B} := G_0 \to \ldots \to G_k$ for G, where all intermediate graphs are minimal bricks. With Lemma 6 we find that

$$\nu_2^c \geq \delta|V(G)|. \tag{6}$$

This means that there is a set Q of at least $\delta|V(G)|$ vertices that arise as new vertices in some conservative-quadratic extension of $\mathcal{B}$. Denote by Q_1 the set of those vertices in Q that are used in the fundament of any later extension of $\mathcal{B}$, and let $Q_2 := Q \setminus Q_1$. Then $Q_2 \subseteq V(G)$ and the vertices of Q_2 have degree 3 in G by (1).

Hence if $|Q_2| \geq (4\delta - 3)|V(G)|$, then we are done. So assume otherwise. Then

$$|Q_1| = |Q| - |Q_2| > \delta|V(G)| - (4\delta - 3)|V(G)| = 3(1 - \delta)|V(G)|. \tag{7}$$

Let I be the set of indices of extensions of $\mathcal{B}$ that use some vertex of Q_1 in their fundament which has not been used in the fundament of earlier extensions of $\mathcal{B}$. Then (2) together with (7) implies that $\nu(I) > (1 - \delta)|V(G)|$.

This means that by (3) and by (6), there is an index $j \in I$ that corresponds to a conservative-quadratic extension $G_{j-1} \to G_j$ of $\mathcal{B}$. Let $q \in Q_1$ lie in the fundament of this extension.

We apply Lemma 5 repeatedly in order to finally obtain a brick G'_{j-2} so that

$$G'_{j-2} \to G_{j-1} \to G_j$$

is a brick-on-brick sequence, with q being one of the new vertices in the conservative-quadratic extension $G'_{j-2} \to G_{j-1}$. This contradicts Lemma 7. □

We are now ready to prove our main theorem.

Proof of Theorem 2. Given a minimal brick G we distinguish two cases. If the average degree of G is at least $4 + \frac{7}{9}$, then we apply Lemma 8 to see that at $\frac{1}{9}|V(G)|$ of the vertices have degree 3.

So, we may assume that G has average degree at most $5 - \frac{2}{9}$. Denote by $V_{\leq 4}$ the set of all vertices of degree at most 4, and by $V_{\geq 5}$ the set of all vertices of degree at least 5. Then

$$\left(5 - \frac{2}{9}\right)|V(G)| \geq \sum_{v \in V(G)} d(v) \geq 3|V_{\leq 4}| + 5|V_{\geq 5}| = 5|V(G)| - 2|V_{\leq 4}|,$$

which leads to $|V_{\leq 4}| \geq \frac{1}{9}|V(G)|$.

In either case we find that at least a ninth of the vertices of G have degree at most 4. □

4 Discussion

In this work, we proved that in a minimal brick the number of vertices of degree ≤ 4 is a positive fraction of the total number of vertices. On the other hand, if we look for large degree vertices in a minimal brick, it is not difficult to find examples with a few vertices of arbitrary large degree (for instance even wheels). It seems less evident that one can also construct minimal bricks with many vertices of degree ≥ 5. We provide an example in Figure 7, where about a seventh of the vertices have degree 6. This graph is a indeed a brick, since it can be built from the triple ladder of Figure 5 by performing two quadratic extensions at triples like r, s, t. It is a minimal brick as clearly every edge is necessary for 3-connectivity.

Vertices of degree ≤ 4 and even cubic vertices seem to be abundant in all examples. In the example with fewest proportion of degree 3 vertices we know,

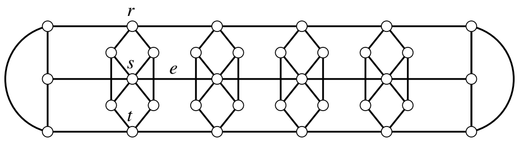

Figure 7: A minimal brick

the triple ladder in Figure 5, they still make up two thirds of the vertices. In that respect, our result with a fraction of $\geq \frac{1}{9}$ of the vertices seems quite low.

The main aim of this paper was to develop ideas and techniques that ultimately should serve to settle the Norine-Thomas conjecture. While we believe to have done a substantial step in that direction, there are still serious obstacles lying on that route. Let us briefly outline some of them.

Clearly, an average degree of at most $4 - \gamma$ (for some small constant $\gamma > 0$) yields a positive fraction of degree 3 vertices. We may therefore assume that our minimal bricks have average degree of about 4 and higher. While an average degree of about 5 and higher leads to a brick-on-brick sequence with many conservative-quadratic extensions (cf. Lemma 6), the now lower bound on the average degree will give us less information on the kind of extensions our brick-on-brick sequence is composed of. In particular, quadratic and conservative-quartic (those that do not involve edge deletions) might appear, as they push the average degree towards 4. Even worse, because conservative-quadratic extensions yield a relatively large edge-density increase, we may also have lots of strict linear, bilinear or pseudolinear extensions.

To handle this, we would seem to need a much stronger version of Lemma 7, that also forbids two chained quadratic extensions, say, that increase the degree of a fundament vertex. Unfortunately, two such extension might actually occur while still yielding a minimal brick: This is exactly what happened to produce the degree 6 vertices in Figure 7.

5 Acknowledgment

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References

[1] M.H. de Carvalho, C.L. Lucchesi, and U.S.R. Murty, *How to build a brick*, Disc. Math. **306** (2006), 2386–2410.

[2] J. Edmonds, L. Lovász, and W.R.Pulleyblank, *Brick decomposition and the matching rank of graphs*, Combinatorica **2** (1982), 247–274.

[3] C.H.C Little and F. Rendl, *Operations preserving the Pfaffian property of a graph*, J. Austral. Math. Soc (Ser. A) **50** (1991), 248–275.

[4] L. Lovász, *Matching structure and the matching lattice*, J. Combin. Theory (Series B) **43** (1987), 187–222.

[5] L. Lovász and M.D. Plummer, *Matching theory*, Ak. Kiadó, North Holland, 1986.

[6] S. Norine and R. Thomas, *Minimal bricks*, J. Combin. Theory (Series B) **96** (2006), 505–513.

[7] ______, *Generating bricks*, J. Combin. Theory (Series B) **97** (2007), 769–817.

[8] A. Schrijver, *Combinatorial Optimization. Polyhedra and efficiency*, Springer, 2003.

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