# Clique or hole in claw-free graphs

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**Abstract**

Given a claw-free graph and two non-adjacent vertices $x$ and $y$ without common neighbours we prove that there exists a hole through $x$ and $y$ unless the graph contains the obvious obstruction, namely a clique separating $x$ and $y$. We derive two applications: We give a necessary and sufficient condition for the existence of an induced $x$–$z$ path through $y$, where $x, y, z$ are prescribed vertices in a claw-free graph; and we prove an induced version of Menger's theorem between four terminal vertices. Finally, we improve the running time for detecting a hole through $x$ and $y$ and for the Three-in-a-Tree problem, if the input graph is claw-free.

## 1 Introduction

Given two non-adjacent vertices $x$ and $y$ in a graph $G$, what is an obvious obstruction for the existence of a hole (an induced cycle of length $\geq 4$) through $x$ and $y$? Clearly, a clique that separates $x$ and $y$. Ideally, we would like to prove that such a clique is the only obstruction:

*there is a hole through $x$ and $y$ if and only if there does not exist any clique that separates $x$ and $y$.* (1)

If $G$ is the line graph of a graph $H$ then an easy application of Menger's theorem to $H$ shows that the statement is true. On the other hand, (1) is false in general; an example may be found in Figure 1 on the left. This is not at all surprising as Bienstock [1] (see also Corrigendum [10]) proved that the following problem is NP-complete, so that one should not expect a simple necessary and sufficient obstruction.

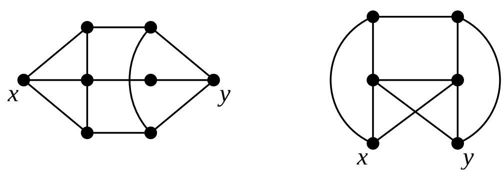


Figure 1: No clique separating $x$ from $y$ and no hole through $x$ and $y$ either

Hole-through-two-Vertices. *Given a graph $G$ and two non-adjacent vertices $x, y$, check whether there is a hole through $x$ and $y$.*

The complete bipartite graph $K_{1,3}$ is called a *claw*. The class of *claw-free* graphs, that is, the graphs not containing the claw as an induced subgraph, is a natural superclass of the class of line graphs. Many of the properties of line graphs extend to claw-free graphs. This is also the case here: Hole-through-two-Vertices becomes solvable in polynomial time as demonstrated by Lévêque, Lin, Maffray and Trotignon [7]. Thus, there is hope for (1) to extend to claw-free graphs, and indeed this is our main result:

**Theorem 1.** *Let $G$ be a claw-free graph, and let $x$ and $y$ be two non-adjacent vertices without common neighbours. Then, there exists a hole through $x$ and $y$ if and only if no clique separates $x$ and $y$.*

We remark that the exclusion of common neighbours of $x$ and $y$ is necessary, see the right graph in Figure 1. However, it is easy to modify the theorem so that common neighbours may be admitted. In fact, in order to prove Theorem 1 we will need a slightly stronger version that does allow common neighbours. We will state and prove it in the next section.

In Section 3, we will derive two applications from Theorem 1. First, we will find a similar obstruction to the existence of an induced $x$–$z$ path containing $y$, where $x, y, z$ are prescribed vertices in a claw-free graph. Second, we will investigate when there are two disjoint paths between two sets (of cardinality 2 each) so that, in addition, there are no chords between the two paths. In a way, this is an induced version of Menger's theorem for two paths.

In Section 4, we will look at algorithmic consequences. We will improve the runnning time given by Lévêque et al, and we will see that the Three-in-a-Tree problem introduced by Chudnovsky and Seymour [2] can, as one should expect, be solved considerably faster in claw-free graphs. We conclude the paper by posing two open problems in the last section.

## 2 A clique obstruction for holes

All our graphs are finite and simple. In general we follow the notation of Diestel [4]. The *centre of a claw* is the unique vertex of degree 3 of the claw.

In this section we prove a version of Theorem 1 that does allow $x$ and $y$ to have common neighbours. Moreover, for the benefit of the applications in Section 3 we will slightly relax the requirement that $G$ is claw-free. For this, let us say that a graph $G$ is *claw-free except possibly at $U$*, where $U$ is a subset of $V(G)$, if the centre of every claw is contained in $U$.

Given two vertices $x$ and $y$ we call a vertex set $S$ an *$x$–$y$ separator* (in $G$) if $S$ is disjoint from $\{x, y\}$ and if $x$ and $y$ are contained in different components of $G - S$. For two sets $X, Y \subseteq V(G)$, we allow an $X$–$Y$ separator to contain vertices of $X \cup Y$, i.e. $S \subseteq V(G)$ is an *$X$–$Y$ separator* if every $X$–$Y$ path meets $S$. This slight abuse of notation makes for cleaner statements and we hope that it does not cause much confusion. For brevity, we call a hole that contains $x$ and $y$ an *$x$–$y$ hole*. We consider the empty graph to be a clique. Thus, if two vertices $x$ and $y$ are contained in different components then there exists an $x$–$y$ clique separator, namely the empty clique.

**Theorem 2.** *Let $x$ and $y$ be two non-adjacent vertices in a graph $G$ that is claw-free except possibly at $\{x, y\}$. Then either*

(i) *there exists an $x$–$y$ hole; or*

(ii) *there exists an $x$–$y$ clique separator in $G - (N(x) \cap N(y))$, and $(N(z) \setminus \{x, y\}) \cup \{z\}$ is an $x$–$y$ separator in $G$ for every $z \in N(x) \cap N(y)$,* *but not both.*

We mention that it is quite likely that Theorem 2 can alternatively be proved with Chudnovsky and Seymour's structure theorem for claw-free graphs; see [3] for an overview. Indeed, we checked some of the cases of the structure theorem to gain confidence in the statement of Theorem 2 before formulating our proof. In the end, however, we decide against using the structure theorem. First, while it may first seem so, Theorem 2 is not a trivial consequence of the structure theorem. Second, Chudnovsky and Seymour's theorem is a very deep and complex theorem, and so it seems not warranted to use it for something that can be proved from first principles with reasonable effort. Moreover, given the (necessary) complexity of the structure theorem it is not at all clear whether using it would indeed lead to a (much) shorter proof.

We will need two lemmas for the theorem. The first of these deals with the rather special situtation when the whole graph is the disjoint union of neighbours of $x$ and $y$.

**Lemma 3.** *Let $x$ and $y$ be two non-adjacent vertices in a graph $G$ that is claw-free except possibly at $\{x, y\}$, and assume that $V(G) \setminus \{x, y\}$ is the disjoint union of $N(x)$ and $N(y)$. Then either*

(i) *there exists an $x$–$y$ hole; or*

(ii) *there exists an $x$–$y$ clique separator in $G$.*

*Proof.* We proceed by induction on $|V(G)|$. If $|V(G)| = 2$ then $G = \overline{K_2}$ and statement (ii) holds. (Note that we accept the empty set as a clique.) Now, suppose that $G$ has at least three vertices, and assume that $G$ does not possess any $x$–$y$ hole. If $y$ is an isolated vertex then clearly the empty set may serve as the desired $x$–$y$ clique separator. Thus, let $N(y) \neq \emptyset$, and pick some $p \in N(y)$. Since any $x$–$y$ hole in $G-p$ is clearly a hole in $G$ as well, it follows that induction yields a minimal $x$–$y$ clique separator $K$ in $G - p$.

Let us first show that we may assume that one of $K \cap N(x)$ and $K \cap N(y)$ is empty. Suppose not, and choose $k \in K \cap N(x)$ and $\ell \in K \cap N(y)$, and consider any neighbour $v$ of $\ell$ in $N(x) \setminus K$. Since $x$ and $y$ have no common neighbours, $k$ and $y$ are non-adjacent, which means that, in order to avoid a claw on $\ell, k, v, y$ with centre $\ell$, we need to have $kv$ as an edge of $G$. This implies $N(K \cap N(y)) \cap N(x) \subseteq N(k)$ for any $k \in K \cap N(x)$. From the minimality of $K$ it follows that $k$ has a neighbour in $N(y) \setminus K$, and consequently, as no claw in $G$ may have its centre at $k$, the set $(N(k) \cap N(x)) \setminus K$ is a clique. Hence $K' := (N(K \cap N(y)) \cup K) \cap N(x)$ is an $x$–$y$ clique separator in $G - p$ that is contained in $N(x)$.

By replacing $K$ with $K'$ if necessary, and by observing that we are done if $K$ (or $K'$) separates $x$ from $y$ in $G$, we obtain in any case the following:

*for every $p \in N(y)$ there exists a minimal $x$–$y$ clique separator $K$ in $G - p$ so that $p$ has a neighbour in $N(x) \setminus K$, and either $K \subseteq N(x)$ or $K \subseteq N(y)$.* (2)

Let us deal with the case of $K \subseteq N(x)$ first. We set $K_p := K \cap N(p)$ and $\overline{K}_p := K \setminus K_p$. Observe that we may exclude that $K = K_p$, as then $K \cup \{p\}$ is an $x$–$y$ clique separator in $G$. (Possibly, though, we may have $K = \overline{K}_p$.) For every

$k \in K_p$, and every neighbour $v \neq p$ of $k$ in $N(y)$ it holds that $p$ and $v$ are adjacent as otherwise $k, p, v, x$ would be a claw. Thus, we get $N(K_p) \cap N(y) \setminus \{p\} \subseteq N(p)$. If also $N(\overline{K}_p) \cap N(y) \subseteq N(p)$ then $(N(p) \cap N(y)) \cup \{p\}$ is an $x$–$y$ separator in $G$ and a clique—the latter follows since $p$ has a neighbour in $N(x) \setminus K$ but $G$ does not contain any claws with centre at $p$. Thus, we may assume that there is a non-neighbour $w$ of $p$ in $N(\overline{K}_p) \cap N(y)$.

The set $N(p) \cap N(x)$, which is a superset of $K_p$, forms a clique as there is no claw centred at $p$. As a result, the $x$–$y$ separator $(N(p) \cap N(x)) \cup \overline{K}_p$ fails only to be a clique if there exist non-adjacent $a \in (N(p) \cap N(x)) \setminus K$ and $\ell \in \overline{K}_p$. Let $\ell' \in \overline{K}_p$ be a neighbour of $w$. If also $a$ and $\ell'$ are non-adjacent then $x\ell'wypax$ is an $x$–$y$ hole. So, let $a\ell'$ be an edge in $G$, which means that $\ell', a, \ell, w$ is a claw, unless $\ell$ is a neighbour of $w$. Then, however, $x\ell wypax$ is an $x$–$y$ hole, and we are done.

Let us now treat the case when $K \subseteq N(y)$. Since we are done if $K \cup \{p\}$ is a clique, $K$ needs to contain a vertex $\ell$ that is non-adjacent to $p$. If $|K| > 1$ then pick any $p' \in K \setminus \{\ell\}$, which is then a neighbour of $y$, and observe that (2) yields a minimal $x$–$y$ clique separator $K'$ in $G - p'$ with $K' \subseteq N(x)$ or $K' \subseteq N(y)$. Now, however, the latter case may not occur as any such $K'$ needs to contain $K \setminus \{p'\}$ and $p$, and thus contains the non-adjacent vertices $\ell$ and $p$.

Therefore, we have $K' \subseteq N(x)$, which means we have reduced to the case above. So, let $K$ consist of a single vertex $p'$, and observe that $p$ and $p'$ are non-adjacent, as otherwise $\{p, p'\}$ is an $x$–$y$ clique separator. If $(N(p) \cup N(p')) \cap N(x)$ is a clique then we have found the desired separator again. As each of $N(p) \cap N(x)$ and $N(p') \cap N(x)$ is a clique it follows therefore that there are non-adjacent $u \in N(p) \cap N(x)$ and $u' \in N(p') \cap N(x)$. Then, however, $xupyp'u'x$ is a hole. □

We will prove Theorem 2 by induction on the number of vertices. Assume that the two vertices $x$ and $y$ have common neighbours. Unless the graph has an $x$–$y$ hole we obtain from Theorem 2 that there is an $x$–$y$ clique separator once the common neighbours are deleted, and that for every common neighbour $z$ of $x$ and $y$ the set $N_G(z) \setminus \{x, y\} \cup \{z\}$ separates $x$ and $y$. However, these two pieces of information are unrelated. We do not know anything, for instance, of the position of the clique relative to all the separators given by the common neighbours. This makes it hard to use these separators in the induction proof of Theorem 2. The next lemma gives us more information to work with.

**Lemma 4.** *Let $x$ and $y$ be two non-adjacent vertices in a graph $G$ that is claw-free except possibly at $\{x, y\}$. Set $Z := N_G(x) \cap N_G(y)$. Assume that there is an $x$–$y$ clique separator in $G - Z$, and that for every $z \in Z$ the set $N_G(z) \setminus \{x, y\}$ is an $x$–$y$ separator in $G - z$. Then at least one of the following statements holds:*

(i) *there is an $x$–$y$ clique separator in $G$; or*

(ii) *for every $z \in Z$ one of $N_G(x) \cap N_G(z)$ and $N_G(y) \cap N_G(z)$ is an $x$–$y$ separator in $G - z$.*

*Proof.* Let us first note that $Z$ is a clique. Indeed, as $N_G(z) \setminus \{x, y\}$ is assumed to be an $x$–$y$ separator in $G - z$ for every $z \in Z$ we clearly have that $N_G(x) \cap N_G(y) = Z \subseteq N_G(z) \cup \{z\}$.

Since $z$ is adjacent to the two non-adjacent vertices $x, y$ all its other neighbours must be adjacent to at least one of $x$ and $y$; otherwise we would find a claw with centre at $z$. Thus, we obtain

$$N_G(z) \setminus \{x, y\} \subseteq N_G(x) \cup N_G(y) \tag{3}$$

Next, we claim that

*If $z \in Z$ and if $K$ is a minimal $x$–$y$ clique separator in $G - Z$ so that $K \nsubseteq N_G(z)$ then either $N_G(x) \cap N_G(z)$ or $N_G(y) \cap N_G(z)$ separates $x$ and $y$ in $G - z$.* (4)

To show (4) choose $S \subseteq N_G(z) \setminus Z$ to be a minimal $x$–$y$ separator in $G - Z$. Note that such a choice is possible as $N_G(z) \setminus \{x, y\}$ separates $x$ and $y$ in $G - z$. Denote by $L_x$ the component of $G - Z - K$ containing $x$, and let $L_y$ be the one containing $y$. Define in a similar way $T_x$ and $T_y$ as components of $G - Z - S$. Then both

$$X := (K \cap T_x) \cup (K \cap S) \cup (S \cap L_x) \text{ and}$$
$$Y := (K \cap T_y) \cup (K \cap S) \cup (S \cap L_y)$$

separate $x$ and $y$ in $G - Z$; see Figure 2 (a).

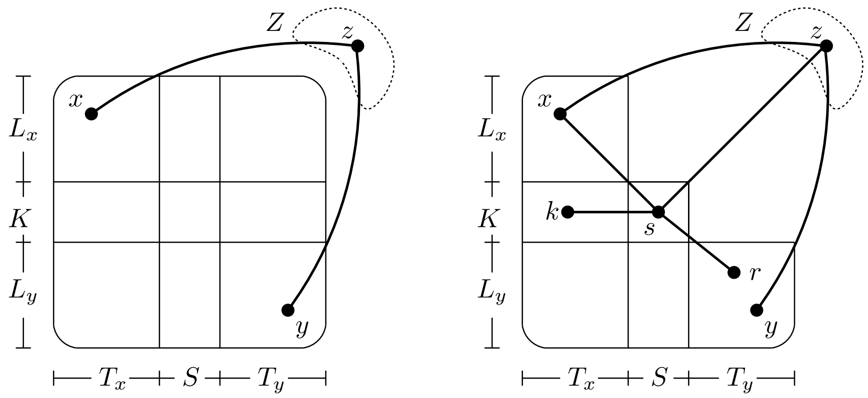


Figure 2: The separators in the proof of Lemma 4

Suppose that $S \cap L_x \neq \emptyset$ and $S \cap L_y \neq \emptyset$. Now, if $T_x \cap K = \emptyset$ then $X$ would be a proper subset of $S$, which contradicts the minimality of $S$. Hence, we obtain $T_x \cap K \neq \emptyset$ and by symmetry also $T_y \cap K \neq \emptyset$. However, since $K$ is a clique this implies that there is an edge between a vertex in $T_x$ and a vertex in $T_y$, contradicting that $S$ is a separator. Therefore, one of $S \cap L_x$ and $S \cap L_y$ must be empty. By symmetry we may assume that $S \cap L_x = \emptyset$. This, in turn, implies $X \subseteq K$, and it follows from the minimality of $K$ that $X = K$, i.e. that $K \cap T_y = \emptyset$. Now, clearly, $S \cap N_G(x) \subseteq K \cup L_x$. As $S \cap L_x = \emptyset$ this reduces to $S \cap N_G(x) \subseteq K$. We state these two facts as we will use them in the next step:

$$K \cap T_y = \emptyset \text{ and } S_x := S \cap N_G(x) \subseteq K. \tag{5}$$

Put $R := N_G(S_x) \cap T_y$, and suppose that $R \nsubseteq N_G(z)$. Pick a vertex $r \in R$ that is non-adjacent to $z$, and let $s \in S_x$ be a neighbour of $r$. Next, as $K \nsubseteq N_G(z)$ by assumption, there exists a $k \in K$ that is non-adjacent to $z$. From (5) we see that $s$ lies in $K$ too and thus is adjacent to $k$. Moreover, $k$ lies outside $T_y$, and it cannot be contained in $S \cap K$ either since $S$ is a subset of $N_G(z)$. Thus, it follows that $k \in T_x$. Now, however, we obtain a contradiction as $s, k, r, z$ induce a claw with centre $s \notin \{x, y\}$; see Figure 2 (b).

Therefore, it holds that $R \subseteq N_G(z)$. No vertex in $R$ is a neighbour of $x$ since $R \subseteq T_y$. Hence, it follows from $R \subseteq N_G(z)$ and (3) that $R \subseteq N_G(y) \cap N_G(z)$. On the other hand, from $S \cap Z = \emptyset$ and $S_x = S \cap N_G(x)$ we deduce with (3) that $S \setminus S_x \subseteq N_G(y) \cap N_G(z)$, too. Finally, it holds that $Z \subseteq N_G(z) \cup \{z\}$ as $Z$ is a clique. Thus, the set $(S \setminus S_x) \cup R \cup (Z \setminus \{z\})$ is an $x$–$y$ separator in $G - z$ that is contained in $N_G(y) \cap N_G(z)$. This establishes (4).

By the assumption of the lemma, there exists a minimal $x$–$y$ clique separator $K'$ in $G - Z$. If $K' \subseteq N_G(z)$ for every $z \in Z$ then $K' \cup Z$ is an $x$–$y$ clique separator in $G$, and the lemma follows (recall that $Z$ is necessarily a clique). Thus, assume there is a $z' \in Z$ such that $K' \nsubseteq N_G(z')$. By (4) and symmetry, we may assume that $N_G(x) \cap N_G(z')$ separates $x$ and $y$ in $G - z'$. Thus, we find in $G - Z$ a minimal $x$–$y$ separator $S'$ that is contained in $(N_G(x) \cap N_G(z')) \setminus Z$.

As $Z$ is the set of all common neighbours of $x$ and $y$ we see that $S' \subseteq N_G(z') \setminus N_G(y)$. Thus, any two $s_1, s_2 \in S'$ need to be adjacent in order to prevent $z', s_1, s_2, y$ from inducing a claw with centre at $z'$. Consequently, $S'$ is a clique.

Now, consider $z \in Z$. If $S' \subseteq N_G(z)$ then $S' \cup Z$ is an $x$–$y$ separator in $G - z$ that is contained in $N_G(z) \cap N_G(x)$. If, on the other hand, $S' \nsubseteq N_G(z)$ then (4) with $S'$ in the role of $K$ implies that one of $N_G(x) \cap N_G(z)$ and $N_G(y) \cap N_G(z)$ separates $x$ and $y$ in $G - z$. Therefore the lemma follows in either case. □

*Proof of Theorem 2.* Let us first show that (i) and (ii) cannot hold simultaneously. Clearly, the existence of a clique that separates $x$ from $y$ in $G - (N(x) \cap N(y))$ forces every hole $C$ through $x$ and $y$ to contain at least one vertex, $z$ say, in $N(x) \cap N(y)$. As $N(z) \setminus \{x, y\}$ is an $x$–$y$ separator in $G - z$ it follows that $z$ is adjacent to an interior vertex of the $x$–$y$ path $C - z$, which is impossible as $C$ is induced.

To see that at least one of (i) and (ii) always holds, we perform induction on $|V(G)|$. Assume first that $x$ and $y$ have a common neighbour. Thus, the induction hypothesis applied to $G - (N(x) \cap N(y))$ either yields a hole through $x$ and $y$ (in which case we are done) or an $x$–$y$ clique separator $K$ in $G - (N(x) \cap N(y))$. If there is a $z \in N(x) \cap N(y)$ for which $N(z) \setminus \{x, y\}$ does not separate $x$ from $y$ in $G - z$ then an $x$–$y$ path in $G - (N(z) \setminus \{x, y\}) - z$ together with $xzy$ yields a hole in $G$.

Therefore, we assume from now on that

$$x \text{ and } y \text{ have no common neighbours.} \tag{6}$$

Lemma 3 takes care of the case when $V(G) = N(x) \cup N(y) \cup \{x, y\}$, so pick a vertex

$$p \notin N(x) \cup N(y). \tag{7}$$

Since any hole in $G - p$ is a hole in $G$, we may assume that induction applied to $G - p$ yields an $x$–$y$ clique separator $K$ in $G - p$, which we choose to be minimal.

Denote the component of $G - p - K$ containing $x$ by $C_x$ and denote the one containing $y$ by $C_y$. If $K$ separates $x$ from $y$ in $G$ as well, we are done. Hence, we may assume that

$$\text{\textit{$K$ is a minimal $x$–$y$ clique separator in $G-p$, and $p$ has neighbours in both $C_x$ and $C_y$.}} \tag{8}$$

As $p \notin N(x) \cup N(y)$ by (7), $C_x$ as well as $C_y$ contains more than one vertex. Thus, if $G_x$ denotes the graph obtained from $G$ by contracting all of $C_x$ to a vertex $x'$, and if $G_y$ denotes the graph obtained from contracting $C_y$ to a vertex $y'$, then both $G_x$ and $G_y$ have fewer vertices than $G$. Moreover, $G_x$ and $G_y$ are claw-free except possibly at $\{x', y\}$ and at $\{x, y'\}$, respectively.

We first observe that we may assume that

$$\text{\textit{there is no $x'$–$y$ clique separator in $G_x$, and no $x$–$y'$ clique separator in $G_y$.}} \tag{9}$$

Indeed, any such clique separator also separates $x$ from $y$ in $G$, which is one of the desired outcomes of the theorem.

The induction hypothesis applied to $G_x$ with $x'$ and $y$, and to $G_y$ with $x$ and $y'$ either yields a hole through $x'$ and $y$ (resp. through $x$ and $y'$) or an obstruction as in (ii) of the statement of the theorem. Note that $x'$ and $y$ (resp. $x$ and $y'$) may have common neighbours, and indeed if the induction yields such an obstruction as in (ii) then they will have common neighbours as otherwise we find an $x'$–$y$ clique separator in $G_x$ (resp. such a separator in $G_y$), in contradiction to (9). We distinguish two cases: either we find in $G_x$ an $x'$–$y$ hole and in $G_y$ an $x$–$y'$ hole, or at least one does not contain such a hole.

**Case I.** *Assume there is an $x'$–$y$ hole in $G_x$ and an $x$–$y'$ hole in $G_y$.*

Viewed in $G$ the holes in $G_x$ and $G_y$ yield an induced $p$–$K$ path $R = p \dots r$ through $x$, and an induced $p$–$K$ path $S = p \dots s$ through $y$. If the cycle $pRrsSp$ (note that $r = s$ or $rs \in E(G)$ as $r, s \in K$) is induced, we have found the desired hole, so assume the cycle to have a chord, which also implies that $r \neq s$. Clearly, such a chord needs to be an edge between $r$ and $\mathring{p}S\mathring{s}$ or between $s$ and $\mathring{p}R\mathring{r}$.[1] Now, if neither $r$ has a neighbour in $pS\mathring{y}$ nor $s$ a neighbour in $pR\mathring{x}$ then we may assume, by symmetry, that $r$ has a neighbour in $yS\mathring{s}$. Denoting by $v$ the first neighbour of $r$ on $ySs$, we find with $pRrvSp$ an $x$–$y$ hole, and are done.

Let $r'$ be the predecessor of $r$ on $R$, and denote by $s'$ the predecessor of $s$ on $S$. We note that

$$\text{\textit{if $r$ has a neighbour in $pS\mathring{y}$ then $r' \in N_G(s)$, and if $s$ has a neighbour in $pR\mathring{x}$ then $s' \in N_G(r)$.}} \tag{10}$$

Indeed, let $r$ have a neighbour $v$ in $pS\mathring{y}$. Since $r, r', s, v$ cannot induce a claw and since $S$ is induced it follows that $sr' \in E(G)$. We argue in a similar way for $s$ and $R$.

Suppose that both $r$ has a neighbour in $pS\mathring{y}$ and $s$ has a neighbour in $pR\mathring{x}$. Recall that $K$ is a minimal separator in $G - p$. Thus, for any $k \in K$ the sets $N_G(k) \cap C_x$ and $N_G(k) \cap C_y$ are cliques, and it follows that $r$ can have at most two neighbours, which are then consecutive, on the induced path $\mathring{p}Ss'$. Then (10)

[1]Here, and in what follows we use the notation of Diestel [4] for paths. In particular, if $P = v_1 \dots v_n$ is a path then $\mathring{v}_i P v_j$ denotes the subpath $v_{i+1} \dots v_j$.

implies that $r' = x$. In a similar way, $s$ may have at most two, necessarily consecutive, neighbours on $\mathring{p}Rr'$, and we conclude that $s' = y$. But this means that $r$ (and $s$) is a common neighbour of $x$ and $y$, which contradicts (6).

Therefore, we may assume, again by symmetry, that $r$ has a neighbour on $pS\mathring{y}$, but $s$ has no neighbour on $pR\mathring{x}$. By (10), $s$ has a neighbour on $xRr'$—pick $u$ to be the first neighbour on $xRr'$. Then $pRusSp$ is an $x$–$y$ hole, which finishes Case I.

**Case II.** *There is no $x'$–$y$ hole in $G_x$, or no $x$–$y'$ hole in $G_y$ (possibly both).*

Assume that there is no $x'$–$y$ hole in $G_x$. Set $Z := N_{G_x}(x') \cap N_{G_x}(y)$, and observe that $Z$ consists of those vertices in $K$ that are adjacent to $y$. Indeed, by (8) we have that $N_{G_x}(x') = K \cup \{p\}$ and by (7) that $p \notin N_{G_x}(y)$. Thus, $Z = K \cap N_{G_x}(y)$.

Recall that the induction hypothesis applied to $G_x$ with $x'$ and $y$ yields an $x'$–$y$ clique separator in $G_x - Z$, and that, moreover, it holds that $N_{G_x}(z) \setminus \{x', y\}$ is an $x'$–$y$ separator in $G_x - z$ for every $z \in Z$. Lemma 4 together with (9) implies that $Z \neq \emptyset$ and that for every $z \in Z$ either $N_{G_x}(z) \cap N_{G_x}(x')$ or $N_{G_x}(z) \cap N_{G_x}(y)$ is already an $x'$–$y$ separator in $G_x - z$. Denote by $Z_{x'}$ those vertices $z$ in $Z$ for which $N_{G_x}(z) \cap N_{G_x}(x')$ is an $x'$–$y$ separator in $G_x - z$, and set $Z_y := Z \setminus Z_{x'}$. Note that for every vertex $z \in Z_y$ the set $N_{G_x}(z) \cap N_{G_x}(y)$ separates $x'$ from $y$ in $G_x - z$.

Assume that $Z_{x'} \neq \emptyset$ and consider $z \in Z_{x'}$. By (8), $N_{G_x}(x') = K \cup \{p\}$, and moreover, every vertex in $K \cup \{p\}$ has a neighbour in $C_y \subseteq G_x$. Thus, for $N_{G_x}(z) \cap N_{G_x}(x')$ to be an $x'$–$y$ separator in $G_x - z$, it is necessary that $K \cup \{p\} \subseteq N_{G_x}(z) \cup \{z\}$. Now $N_{G_x}(z) \setminus N_{G_x}(y)$, which is a superset of $(K \setminus Z) \cup \{p\}$, forms a clique since no vertex in $N_{G_x}(z) \setminus N_{G_x}(y)$ is adjacent to the neighbour $y$ of $z$; otherwise we would find a claw with centre $z$. As $K \cup \{p\} \subseteq N_{G_x}(z) \cup \{z\}$ holds for every $z \in Z_{x'}$ we get that $(K \setminus Z_y) \cup \{p\}$ is a clique. This then implies $Z \neq Z_{x'}$ as otherwise $K \cup \{p\}$ would be a clique, and thus a contradiction to (9). We have shown that

$$Z \neq Z_{x'} \textit{ and, unless } Z_{x'} = \emptyset, \ (K \setminus Z_y) \cup \{p\} \textit{ is a clique.} \tag{11}$$

As $Z \neq Z_{x'}$ there is some $z \in Z_y$. Pick a minimal $x'$–$y$ separator $S \subseteq (N_{G_x}(z) \cap N_{G_x}(y)) \setminus Z$ in $G_x - Z$ (by definition of $Z_y$ there is such an $S$). Note that $S \subseteq N_{G_x}(z')$ for every other $z' \in Z_y$, too. Furthermore, as no claw in $G_x$ has its centre at $z$ and as $x'z \in E(G_x)$, $S$ is a clique. Thus, $Z_y \cup S$ is a clique but $Z \cup S$ cannot be one, by (9) and the fact that $Z \cup S$ is an $x'$–$y$ separator in $G_x$. Hence

$$Z_y \cup S \textit{ is a clique but not } Z \cup S. \tag{12}$$

Moreover, this implies that $Z_{x'} \neq \emptyset$.

Next, we claim that

$$N_G(Z_y) \cap C_x \subseteq N_G(p). \tag{13}$$

Consider $z \in Z_y$ and a neighbour $v \in N_G(z) \cap C_x$; see Figure 3. By (12) there are non-adjacent $r \in Z_{x'}$ and $s \in S$. As otherwise $z, r, s, v$ is a claw it follows that $r$ and $v$ are adjacent. From (11) we get that $p$ is a neighbour of $r$. Since $r \in Z$, $r$ is also adjacent to $y$. So, for $r, p, v, y$ not to induce a claw, we either have $py \in E(G)$ or $pv \in E(G)$. The former, however, is impossible by (7). Thus, $p$ and $v$ are adjacent, which proves (13).

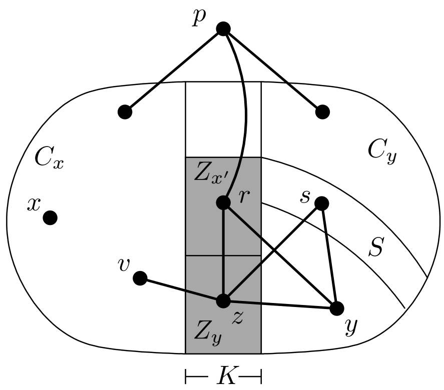


Figure 3: Illustration of (13)

An immediate consequence of (11) is that $(K \setminus N_G(y)) \cup \{p\}$ is a clique. This excludes that also in $G_y$ we do not find an $x$–$y'$ hole as then we would deduce in the same way that $(K \setminus N_G(x)) \cup \{p\}$ is a clique, too. Since $x$ and $y$ have no common neighbours by (6) it would follow that $K \cup \{p\}$ is a clique, a contradiction to (9). We conclude that

*there is an induced $p$–$K$ path $Q$ through $x$.*

Let us come to the final contradiction. Denote by $q$ the endvertex of $Q$ in $K$, and observe that as $Q$ is induced, (11) forces $q$ to lie in $Z_y$. Then, the predecessor $v$ of $q$ on $Q$ is, by (13), a neighbour of $p$, which necessitates that $Q = pxq$. Now, however, we obtain a contradiction to (7) as $p$ is a neighbour of $x$. □

# 3 Applications

We derive two applications, Theorems 5 and 6, from Theorem 2.

**Theorem 5.** *Let $x, y, z$ be three vertices in a graph $G$ that is claw-free except possibly at $\{x, y, z\}$. Then exactly one of the following two statements holds:*

(i) *There is an induced $x$–$z$ path through $y$.*

(ii) *There is a clique other than $\{y\}$ that separates $\{x, z\}$ from $\{y\}$, or $N(x) \setminus \{y\}$ separates $y$ from $z$, or $N(z) \setminus \{y\}$ separates $x$ from $y$.*

Given a graph $G$, let us call two subgraphs or vertex sets $S, T$ *non-touching* if $S$ and $T$ are disjoint and if there does not exist any edge with one endvertex in $S$ and the other in $T$.

**Theorem 6.** *Let $X, Y$ be two non-touching vertex sets of cardinality 2 in a graph $G$ that is claw-free except possibly at $X \cup Y$. Then exactly one of the following statements holds:*

(i) *There are two non-touching $X$–$Y$ paths.*

(ii) *There exists a clique separating $X$ from $Y$ in $G$; or there exists $z \in X \cup Y$ so that $X$ is separated from $Y$ by $N(z)$.*

We remark that the theorem becomes false if $X$ and $Y$ are allowed to touch. Figure 4 shows a claw-free graph with $X$ and $Y$ touching where neither (i) nor (ii) is satisfied.

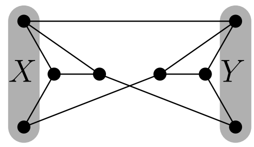


Figure 4: Theorem 6 may fail if $X$ and $Y$ touch

We will obtain Theorem 5 from Theorem 2, with some extra effort, and then deduce Theorem 6 from Theorem 5. In both these deductions we use an argument that is encapsulated in the lemma below. (So, in some sense it is used twice in the proof of Theorem 6.)

We say that two paths $P$ and $Q$ are *non-touching except at their ends* if $P$ and $Q$ meet at most in their endvertices and if for any edge $pq \notin E(P \cup Q)$ with $p \in V(P)$ and $q \in V(Q)$ it follows that $p$ is an endvertex of $P$ and $q$ one of $Q$.

**Lemma 7.** *Let $y_1, y_2$ be not necessarily distinct vertices in a graph $G$, and let $x_1, x_2$ be distinct vertices in $G - \{y_1, y_2\}$. Assume that*

(1) *$x_1, x_2, y_1, y_2$ are pairwise non-adjacent;*

(2) *$G$ is claw-free except possibly at $\{x_1, x_2, y_1, y_2\}$;*

(3) *for $j = 1$ or for $j = 2$ there exist an $x_j$–$y_1$ path $P_1$ and an $x_j$–$y_2$ path $P_2$, so that $P_1$ and $P_2$ are non-touching except at their ends (if $y_1 = y_2$ this means that there is a hole through $x_j$ and $y_1$); and*

(4) *neither $N(x_1)$ nor $N(x_2)$ separates $\{x_1, x_2\}$ from $\{y_1, y_2\}$.*

*Then $G$ has*

(i) *an $x_1$–$y_1$ path $Q_1$ and an $x_2$–$y_2$ path $Q_2$; or*

(ii) *an $x_1$–$y_2$ path $Q_1$ and an $x_2$–$y_1$ path $Q_2$,*

*so that $Q_1$ and $Q_2$ are non-touching except at their ends.*

*Proof.* Assume $P_1$ and $P_2$ and $j \in \{1, 2\}$ to be chosen to have minimal total length $|E(P_1)| + |E(P_2)|$ subject to that $P_1$ is an $x_j$–$y_1$ path, $P_2$ is an $x_j$–$y_2$ path and subject to that $P_1$ and $P_2$ are non-touching except at their ends. In particular, $P_1$ and $P_2$ are induced. By symmetry we may assume that $j = 1$.

Now, since $N(x_1)$ does not separate $\{x_1, x_2\}$ from $\{y_1, y_2\}$ there is an induced $x_2$–$\{y_1, y_2\}$ path $R$ that is disjoint from $N(x_1)$, and then also from $x_1$. Let $v$ be the first vertex of $R$ that has a neighbour on $P_1 \cup P_2$. Suppose that $v$ is adjacent to an interior vertex $p_1$ of $P_1$ and to an interior vertex $p_2$ of $P_2$. If $v \neq x_2$, i.e. if $v$ has a predecessor $u$ on $R$, then, since there is no claw with centre $v$, one of the two non-adjacent vertices $p_1$ and $p_2$ is adjacent to $u$, which contradicts the choice of $v$.

So suppose that $v = x_2$. Then if $p_1$ is chosen as the last neighbour of $x_2$ on $P_1$, and if $p_2$ is chosen to be the last neighbour of $x_2$ on $P_2$ then $P_1' := x_2p_1P_1y_1$ and $P_2' := x_2p_2P_2y_2$ have shorter total length than $P_1$ and $P_2$ and are thus a contradiction to the choice of $P_1$ and $P_2$ unless $x_1$ is adjacent to both $p_1$ and $p_2$. Then, however, $p_1, x_1, x_2$ together with the successor of $p_1$ on $P_1$ induce a claw, a contradiction. (Observe that $p_1 \notin \{y_1, y_2\}$ as $\{x_1, x_2, y_1, y_2\}$ are assumed to be pairwise non-adjacent.)

Hence we may assume that $v$ has no neighbours in the interior of $P_1$. If $v$ has no neighbour at all on $P_2$ then $v$ has to be adjacent to $y_1$, and $Q_1 := P_2$ and $Q_2 := x_2Rvy_1$ are as desired. So, let $w$ be the last vertex on $P_2$ from $x_1$ to $y_2$ that is adjacent to $v$, and put $Q_1 := P_1$ and $Q_2 := x_2RvwP_2y_2$. Suppose that there is an edge $e \notin E(Q_1 \cup Q_2)$ with one endvertex in $Q_1$ and the other in $Q_2$. By definition of $Q_1$ and $Q_2$ this is only possible if $e = vy_1$. (Recall that $R$ is disjoint from $N(x_1)$, which implies $vx_1 \notin E(G)$.) Furthermore, $y_1$ and $y_2$ need to be distinct vertices as otherwise we would have chosen $w = y_2 = y_1$, which implies $e \in E(Q_2)$. As $x_2$ is not adjacent to $y_1$ it follows that $v \neq x_2$. Now, however, we obtain a contradiction as $v, y_1, w$ together with the predecessor of $v$ on $R$ induce a claw with centre $v$ (note that $y_1$ and $y_2$ are required to be non-adjacent, which takes care of the case when $w = y_2$). Therefore, $Q_1$ and $Q_2$ are non-touching except at their ends, as desired. □

*Proof of Theorem 5.* Assume (ii) holds and let us see that then (i) cannot be true. Let $P$ be any $x$–$z$ path through $y$. If there is a clique not equal to $\{y\}$ that separates $\{x, z\}$ from $\{y\}$ then $P$ needs to go twice through the clique and therefore cannot be induced. If, on the other hand, $N_G(x) \setminus \{y\}$ separates $y$ from $z$ then $yPz$ meets $N_G(x) \setminus \{y\}$, and again $P$ is not induced; we argue in a similar way in the remaining case.

Now, assume that (ii) does not hold. We will show that this implies the existence of an induced $x$–$z$ path through $y$. Clearly, $x$ and $z$ cannot be adjacent as otherwise $xz$ is a clique separating $\{x, z\}$ from $\{y\}$. Let us now deal with the case when $y$ is adjacent to $x$ or $z$. If $xy \in E(G)$ then there is an induced $y$–$z$ path in $G - (N_G(x) \setminus \{y\})$ as $N_G(x) \setminus \{y\}$ does not separate $y$ from $z$. This path together with $xy$ yields an induced $x$–$z$ path through $y$.

Thus, we assume from now on that $x, y, z$ are pairwise non-adjacent. Denote by $\tilde{G}$ the graph obtained from $G$ by identifying $x$ and $z$ to a vertex $\tilde{x}$, and observe that $\tilde{G}$ is claw-free except possibly at $\{\tilde{x}, y\}$.

Assume that there exists an $\tilde{x}$–$y$ hole in $\tilde{G}$. Viewed in $G$, such a hole either yields the desired induced $x$–$z$ path through $y$, or it yields a hole through $x$ and $y$, or through $z$ and $y$. The last two cases are symmetric, so assume that there is a hole through $x$ and $y$ in $G$. Then, the theorem follows from Lemma 7 with $x, z, y, y$ in the roles of $x_1, x_2, y_1, y_2$.

Therefore, it remains to deal with the case when

$$\tilde{G} \textit{ does not contain any } \tilde{x}\text{–}y \textit{ hole.} \tag{14}$$

We will show that (14) contradicts our assumption that (ii) does not hold, which then concludes the proof of the theorem. In order to do so, we apply Theorem 2 and Lemma 4 to $\tilde{G}, \tilde{x}, y$. As a clique that separates $\tilde{x}$ from $y$ in $\tilde{G}$ separates $\{x, z\}$ from $\{y\}$ in $G$, it follows that $Q := N_{\tilde{G}}(\tilde{x}) \cap N_{\tilde{G}}(y) \neq \emptyset$ and that for every $q \in Q$ one of $N_{\tilde{G}}(\tilde{x}) \cap N_{\tilde{G}}(q) \cup \{q\}$ and $N_{\tilde{G}}(y) \cap N_{\tilde{G}}(q) \cup \{q\}$ separates $\tilde{x}$ from $y$ in $\tilde{G}$.

Denote by $Q_x$ those $q \in Q$ that are adjacent to $x$, and let $Q_z \subseteq Q$ be those $q$ adjacent to $z$. Because no claw may have its centre in $Q$, we deduce that $Q_x$ and $Q_z$ are disjoint, and consequently, $Q$ is the disjoint union of $Q_x$ and $Q_z$. Moreover, we observe that $Q$ is a clique. Indeed, otherwise we easily find an $\tilde{x}$–$y$ hole in $\tilde{G}$.

Next, we claim that

$$N_{\tilde{G}}(y) \cap N_{\tilde{G}}(q) \cup \{q\} \textit{ is an } \tilde{x}\textit{–}y \textit{ separator in } \tilde{G} \textit{ for every } q \in Q. \tag{15}$$

Suppose the contrary, and without loss of generality let us assume that some $q_x \in Q_x$ violates (15). As $x$ and $y$ are two non-adjacent neighbours of $q_x$ and as there is no claw with centre $q_x$ it follows that $N_G(q_x) \setminus \{x, y\} \subseteq N_G(x) \cup N_G(y)$. In particular, we get that

$$\begin{aligned} N_{\tilde{G}}(\tilde{x}) \cap N_{\tilde{G}}(q_x) &= (N_G(x) \cap N_G(q_x)) \cup (N_G(z) \cap N_G(q_x)) \\ &= (N_G(x) \cap N_G(q_x)) \cup (N_G(y) \cap N_G(z) \cap N_G(q_x)) \\ &\subseteq (N_G(x) \cap N_G(q_x)) \cup Q_z. \end{aligned} \tag{16}$$

(In fact, we have equality in the last line since $Q$ is a clique.)

Now, if $Q_z = \emptyset$ then $N_G(x) \cap N_G(q_x) = N_{\tilde{G}}(\tilde{x}) \cap N_{\tilde{G}}(q_x)$. However, the latter set together with $q_x$ is an $\tilde{x}$–$y$ separator, which means that $N_G(x) \cap N_G(q_x) \cup \{q_x\} \subseteq N_G(x)$ separates $\{y\}$ from $\{x, z\}$ in $G$. Hence, we have a contradiction to our assumption that (ii) does not hold. Thus,

$$Q_z \neq \emptyset. \tag{17}$$

By assumption, $(N_G(q_x) \cap N_G(y)) \cup \{q_x\} \supseteq Q$ does not separate $\{x, z\}$ from $\{y\}$. As a result, there is an induced $\{x, z\}$–$y$ path $P$ in $G - Q$ that avoids $N_G(q_x) \cap N_G(y)$. On the other hand, $N_{\tilde{G}}(\tilde{x}) \cap N_{\tilde{G}}(q_x) \cup \{q_x\}$ does separate $\{x, z\}$ from $y$ in $G$, so $N_G(x) \cap N_G(q_x)$ is an $\{x, z\}$–$y$ separator in $G - Q$. Therefore, $P$ meets $N_G(x)$, and we may thus assume that $P$ starts in $x$ (rather than in $z$). Moreover, as $N_G(x)$ separates $z$ from $y$ in $G - Q$, the only neigbour on $P$ that $z$ could possibly have is the vertex of $P$ in $N_G(x)$, which we denote by $x^+$. However, since $x^+, x, z$ together with the successor of $x^+$ on $P$ cannot induce a claw we deduce $z$ cannot be adjacent to $x^+$ either, which means that

$$z \textit{ has no neighbour on } P. \tag{18}$$

Since $Q_z \neq \emptyset$ by (17), we may pick $q_z \in Q_z$, and consider the $x$–$z$ path $R := P \cup y q_z z$, which contains $y$. Clearly, we are done if $R$ is induced (which, in fact, constitutes a contradiction to (14)). So, suppose that $R$ has a chord $e$, and observe that because of $N_G(q_z) \subseteq N_G(y) \cap N_G(z) \cup \{y, z\}$ and since $z$ has no neighbour on $P$, it follows that $e = q_z y^-$, where $y^-$ is the predecessor of $y$ on $P$. Let us check that $q_z y^- \notin E(G)$.

As $P$ is chosen to be disjoint from $N_G(q_x) \cap N_G(y)$ we deduce that $q_x$ is not adjacent to $y^-$. Moreover, $z$ cannot be a neighbour of $y^-$ either by (18), and $z$ and $q_x$ are non-adjacent as $q_x \in Q_x = Q \setminus Q_z$. Consequently, as $q_z, z, q_x, y^-$ cannot induce a claw, $q_z$ and $y^-$ cannot be adjacent. This concludes the proof of Claim (15).

Claim (15) asserts that for all $q \in Q$ the set $N_{\tilde{G}}(y) \cap N_{\tilde{G}}(q) \cup \{q\}$ separates $\tilde{x}$ from $y$ in $\tilde{G}$. For some $q' \in Q$ choose a minimal $\tilde{x}$–$y$ separator $S$ in $\tilde{G} - Q$ that is contained in $N_{\tilde{G}}(y) \cap N_{\tilde{G}}(q')$, and observe that we must have $S \subseteq N_{\tilde{G}}(q)$ for all $q \in Q$. Now, recall that $Q$ is a clique, and note that $S$ is a clique, too, since there are no claws with centre $q'$. Thus, the fact that $S \subseteq N_{\tilde{G}}(q)$ for all $q \in Q$ implies that $S \cup Q$ is a clique. As $S \cup Q$, furthermore, separates $\tilde{x}$ from $y$ in $\tilde{G}$, and then also $\{x, z\}$ from $y$ in $G$, we obtain a contradiction to our assumption that (ii) does not hold. □

*Proof of Theorem 6.* Assume that (ii) does not hold. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$, and observe that we may assume that $\{x_1, x_2, y_1, y_2\}$ are pairwise non-adjacent. Indeed, since $X$ and $Y$ are non-touching, we may only have $x_1x_2$ or $y_1y_2$ as edges in $G$. Both edges, however, constitute a clique as in (ii).

Denote by $\tilde{G}$ the graph obtained by identifying $x_1$ and $x_2$ to a vertex $\tilde{x}$. Application of Theorem 5 to $\tilde{G}$ and $y_1, \tilde{x}, y_2$ in the roles of $x, y, z$ yields an induced $y_1$–$y_2$ path $\tilde{P}$ through $\tilde{x}$ in $\tilde{G}$. Viewed in $G$, $\tilde{P}$ either splits into two disjoint induced $X$–$Y$ paths $R_1$ and $R_2$, or we obtain an induced path $y_1$–$y_2$ path $P$ through $x_1$ or through $x_2$; let us say through $x_1$. In the former case, as $\tilde{P}$ is induced $R_1$ and $R_2$ are non-touching, except when the second vertex $v$ of $R_1$ or of $R_2$ is adjacent to both of $x_1$ and $x_2$. Then, however, $v, x_1, x_2$ and the successor of $v$ induce a claw.

So, let us consider the case when we obtain an induced path $y_1$–$y_2$ path $P$ through $x_1$. Then application of Lemma 7 to the $x_1$–$y_1$ and $x_1$–$y_2$ subpaths of $P$ finishes the proof. □

# 4 Algorithmic consequences

Lévêque, Lin, Maffray and Trotignon's algorithm [7] for detecting a hole through two given vertices $x$ and $y$ in a claw-free graph $G$ (the Hole-through-two-Vertices problem) has a running time of $O(|V(G)|^4)$—provided $x$ and $y$ have degree 2. By performing the algorithm once for each pair of a neighbour of $x$ and of $y$ while deleting all others, we trivially can always reduce to the degree 2 case, at the cost of incurring an even higher running time. (We should point out, though, that the algorithm given in [7] covers more general inputs than claw-free graphs.)

In contrast, the structure result in Theorem 2 allows us to check for a hole in $O(|E(G)| \cdot |V(G)|)$-time without any requirements on the degree of $x$ and $y$. As a tool we use that clique decompositions as introduced by Wagner [12] can be computed efficiently, for instance with the algorithm of Tarjan [11] or of Whitesides [13].

Given a graph $G$, if there is a clique separator $K$ we can decompose $G$ into two parts $G_1, G_2$, where $G_1, G_2$ are induced subgraphs of $G$ each properly containing $K$ so that $G = G_1 \cup G_2$ and $K = G_1 \cap G_2$. By repeating this process as long as possible we arrive at a set of induced subgraphs of $G$ that do not contain any clique separator anymore; these are the *atoms* of the concrete clique decomposition. We note the following:

*for two vertices $x$ and $y$ there exists an atom containing them both if and only if there does not exist any $x$–$y$ clique separator in $G$.* (19)

Let $G$ be a claw-free graph on $n$ vertices and $m$ edges, and let $x$ and $y$ be two non-adjacent vertices of $G$. We now describe an algorithm that decides whether there is a hole containing $x$ and $y$.

(1) Check whether $x$ and $y$ are in the same component $C$ of $G$. If they are not, output NO; otherwise replace $G$ by $C$.

(2) Set $Z := N(x) \cap N(y)$ and for each $z \in Z$ check whether $x$ and $y$ are in different components of $G - N(z) - z$. If this is not the case for some $z$ output YES.

(3) Use Tarjan's algorithm [11] in order to compute a clique decomposition in at most $n - 1$ atoms $G_1, \ldots, G_k$ of $G - Z$.

(4) Check whether there is a $G_i$ that contains both $x$ and $y$—if that is the case output YES, else output NO.

Theorem 2 in conjunction with (19) asserts that the algorithm is correct. Let us turn to the running time. The only purpose of Step (1) is to take care of the exceptional case when we have far fewer edges than vertices as replacing $G$ by $C \cup \{y\}$ allows us to assume that $m \geq n - 2$. Checking whether two vertices are in the same component can be done in $O(m+n)$-time, so that Step (1) takes $O(m+n)$ time and Step (2) at most $O(mn)$ time. The running time of Tarjan's algorithm is $O(mn)$, and every of the $\leq n-1$ $G_i$ of Step (3) contains at most $n$ vertices, which means we need at most $O(n^2)$-time for this step. In conclusion, we have proved the following:

**Theorem 8.** *Let a claw-free graph $G$ and two non-adjacent vertices $x$ and $y$ be given. If $G$ has $n$ vertices and $m$ edges then it can be checked in $O(mn)$-time whether there is a hole containing $x$ and $y$.*

The algorithm by Lévêque et al rests on Chudnovsky and Seymour's algorithm [2] for the THREE-IN-A-TREE problem:

THREE-IN-A-TREE. *Given a graph $G$ and three vertices $x, y, z$ decide whether there exists an induced subtree of $G$ containing $x, y, z$.*

Chudnovsky and Seymour show that THREE-IN-A-TREE can be solved in $O(|V(G)|^4)$-time. In a claw-free graph every induced tree is a path, so application of Theorem 5 permits to reduce the running time for claw-free graphs considerably.

**Theorem 9.** THREE-IN-A-TREE *can be solved in $O(mn)$-time in claw-free graphs, where $m$ is the number of edges and $n$ the number of vertices.*

*Proof.* As in Theorem 8 Tarjan's algorithm can be used to check whether Theorem 5 (ii) holds or not. □

# 5 Open Questions

We conclude the paper with several open questions. The first one concerns a natural generalisation of Theorem 1.

**Question 10.** *Given a claw-free graph $G$, and three pairwise non-adjacent vertices $x, y, z$, when is there a hole through $x, y$ and $z$?*

Which conditions would be necessary or at least sufficient to force the existence of such a hole? For a hole through just two predetermined vertices, the absence of common neighbours made for a simpler statement. So, it appears prudent to first focus on the case when there are no vertices adjacent to two of $x, y, z$. Next, if $G$ does contain a hole through $x$, $y$ and $z$ then it also contains an induced $u$–$v$ path through $w$ for any permutation $(u, v, w)$ of $(x, y, z)$. Thus, the obstructions described in Theorem 5 become relevant here. These are: a clique that separates two of $\{x, y, z\}$ from the third, and a permutation $(u, v, w)$ of $(x, y, z)$ so that $N(u)$ separates $v$ and $w$.

Excluding these two obstructions is not enough to guarantee the desired hole, not even when $G$ is the line graph of some graph $H$. Then the problem reduces to the question whether there is a (not necessarily induced) cycle through three independent edges in $H$. Therefore, the case of $k = 3$ of the Lovász-Woodall conjecture might suggest additional conditions:

**Conjecture 11** (Lovász [8]; Woodall [14])**.** *Let $F$ be a set of $k$ independent edges in a $k$-connected graph. If $k$ is even or $G - F$ is connected, then $G$ admits a cycle containing every edge of $F$.*

Several partial results are known. In particular, the case $k = 3$ can be found in Lovász [9, Ex. 6.67]. Recently, Kawarabayashi announced a full proof of the conjecture, the first part of which appeared in [6].

Returning to Question 10, let us translate the assumptions of the conjecture on $H$ to its line graph $G$. The condition that $x, y, z$ should not form an edge cut in $H$, turns into the requirement that $\{x, y, z\}$ should not be a cut-set of $G$, a requirement that can easily be seen to be necessary for arbitrary claw-free graphs $G$. Assuming that $H$ is 3-connected means that $G = L(H)$ is 3-*clique-connected*, that is, there are no two cliques $K, L$ so that $G - K - L$ has two non-singleton components.

To sum up, is it true that any claw-free graph $G$ contains a hole through any non-adjacent vertices $x, y, z$, provided that no vertex is adjacent to two of $\{x, y, z\}$, that $N(u)$ does not separate $v$ from $w$ for any permutation $(u, v, w)$ of $(x, y, z)$, that $G - \{x, y, z\}$ is connected and that $G$ is 3-clique-connected?

There is another quite obvious direction in which Theorem 1 could possibly be extended. Our initial motivation stemmed from the polynomial time algorithm of Lévêque et al [7] for the Hole-through-two-vertices problem in claw-free graphs. In fact, Lévêque et al prove the existence of such an algorithm not only for claw-free graphs but for $H$-free graphs, where $H$ is any subdivision of a claw. Does Theorem 1 likewise generalise to $H$-free graphs? Naturally, the obstructions to the existence of a hole would become more general.

Secondly, how (if at all) does Theorem 6 generalise to larger path systems?

**Question 12.** *Let $X$ and $Y$ be two non-touching sets of cardinality $k$ in a claw-free graph $G$. When does $G$ admit $k$ pairwise non-touching $X$–$Y$ paths in $G$?*

Considering Theorem 6, we can immediately think of two extremal types of obstructions:

- an $X$–$Y$ separator that consists of at most $k - 1$ cliques, and
- a subset $Z$ of $X \cup Y$ of cardinality $< k$ such that $N(Z)$ separates $X$ from $Y$.

However, there exists a number of obstructions which fall between the above two extremes. An example would be a set $Z \subseteq X \cup Y$ of cardinality $r$ together with $s < k - r$ cliques $K_1 \ldots, K_s$ so that $N(Z) \cup \bigcup_{i=1}^{s} K_i$ separates $X$ from $Y$.

We have discussed possible extensions of two of the main results. What about the third, Theorem 5? Under what circumstances does there exist an induced path through $k$ given vertices in a claw-free graph? While Fiala, Kamiński, Lidický and Paulusma [5] prove that there is a polynomial-time algorithm to decide whether such a path exists if $k$ is fixed, it appears doubtful that a nice and simple structural result can be obtained.

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