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Generating the cycle space by induced non-separating cycles in locally finite graphs and in graphs with at most one end

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You can disappear here without knowing it.
Less Than Zero

Contents

Chapter 1

Introduction

A classical result of Tutte [7] states that in a finite 3-connected graph the cycle space (the set of mod 2 sums of cycles) is generated by induced non-separating cycles. Following Tutte we shall call those cycles *peripheral*.

Theorem 1.1 (Tutte [7]). *The cycle space of a finite* 3*-connected graph is generated by peripheral cycles.*

The purpose of this work is to extend Tutte's theorem to infinite graphs. However, for arbitrary infinite graphs the cycle space need not be generated by peripheral cycles. An obvious counterexample is the cartesian product G of a cycle C with a *double ray* (a 2-way infinite path). There, every copy of C is separating but not the sum of any set of peripheral cycles.

We pursue two ways to deal with the counterexample. In Chapter 2 we restrict ourselves to a class of (infinite) graphs that excludes graphs such as G. In fact, we show that Tutte's theorem holds for (3-connected) graphs with at most one end, thereby answering a question of Halin.

In contrast, we observe in Chapter 3 that if a generalized definition of the cycle space is used, one that admits infinite cycles and respects the end structure of the graph, then G is no longer a counterexample. Indeed, with that notion of the cycle space we see that Tutte's result generalizes neatly to (3-connected) locally finite graphs.

Chapter 4 is the odd one out. There, we look at finite graphs that are no longer required to be 3-connected. We obtain a simple extension of Tutte's result.

Each chapter is self-contained. In general our notation and terminology will be that of Diestel [2]. All our graphs will be undirected and without loops or multi-edges.

Chapter 2

Graphs with at most one end

2.1 Introduction

All known counterexamples to Tutte's result (Theorem 1.1), such as the one in Chapter 1, have at least two ends. (An *end* is the equivalence class of *rays* (1-way infinite paths), where two rays are said to be equivalent if they cannot be separated by finitely many vertices.) Halin [5], on the other hand, observed that in certain classes of (3-connected) graphs with at most one end the cycle space is indeed generated by peripheral cycles. In particular, this is true for planar one-ended graphs and for rayless graphs (graphs without rays).

Theorem 2.1 (Halin [5]). *The cycle space of a* 3*-connected rayless graph is generated by induced non-separating cycles.*

Motivated by these results he raised the following problem.

Problem 2.2 (Halin [5]). *Is the cycle space of every* 3*-connected graph with at most one end generated by induced non-separating cycles?*

Weakening Problem 2, Halin made the following conjecture.

Conjecture 2.3 (Halin [5]). *The cycle space of every* 3*-connected graph that does not contain a double ray is generated by induced non-separating cycles.*

We will give a positive answer to Problem 2.2, thereby proving Conjecture 2.3.

2.2 Generating the cycle space

We recall a standard concept that naturally arises when dealing with cycles. Here, we have taken the definition from Bondy and Murty [1].

Definition 2.4. *Let H be a subgraph of a graph G. We define an equivalence relation on $E(G)\setminus E(H)$ by $e \sim f$ if there is a path P such that*

(i) the first edge of P is e and the last is f.

(ii) P meets H at most at its ends.

A connected non-trivial subgraph B of $G - E(H)$ whose edge set is closed under this equivalence relation is called a bridge *of H. The vertices of B on H are called the* vertices of attachment *of B.*

One sees easily that a bridge is either a chord of H or a subgraph of G consisting of a component K of $G - H$ with the edges $E(K, H)$ added. Note, that a cycle is peripheral if and only if it has at most one bridge.

The key to the solution of Problem 2.2 is the following observation. Let a cycle C in a 3-connected graph G with at most one end be given. Being finite, C may not separate any two rays of G. Consequently, C has at most one bridge containing rays. Substituting that bridge through a suitable rayless one we obtain a rayless graph G' whose cycle space differs not too much from that of G. In addition, the cycle space of G' is generated by peripheral cycles (Theorem 2.1).

The following simple lemma shows how we can find an appropriate substitute bridge for the unwanted bridge.

Lemma 2.5. *Let G be a 3-connected graph and let C be a cycle in G with a bridge B that is not a chord. Denote by G' the graph obtained from G by contracting the bridge B to a vertex v_B (with any loops and multi-edges deleted). Then G' is 3-connected. Moreover, every peripheral cycle D in G' that avoids v_B is a peripheral cycle in G as well.*

Proof. First, we show that G' is 3-connected. For this, consider any vertices x and y of G'. We assume first that $x = v_B$. Let $u, v \in V(G')\setminus\{v_B, y\}$ be two vertices. Because G is 3-connected there is an u-v-path P in $G - y$. Should this path be a valid path in G' too we are done. If not, then P meets B. Denote by w_1 the first vertex of P in C and by w_2 the last vertex. With a path $Q = w_1 \dots w_2$ in $C - y$ we obtain an u-v-path $P' = uPw_1Qw_2Pv$ in $G - \{v_B, y\}$. By a similar reasoning we see that if $v_B \notin \{x, y\}$, $G' - \{x, y\}$ is connected as well.

Finally, we have to deal with the second assertion. Clearly, a cycle D in G' that avoids v_B may be viewed as a cycle in G. We claim that D has only a single bridge in G. For this, let $e \in B$ be an edge. We show that every edge $f \in E(G)\setminus D$ is equivalent to e. Should f be another edge of B this is obvious, so assume $f \notin B$. In particular, f is then an edge of G' as well. Let e' be an edge of G' incidident to v_B. D avoids v_B and has only a single bridge in G'. Consequently, there is a path $P' = f \dots e'$ in G' internally avoiding D. Denote by v the predecessor of v_B on P'. Note that v is a common vertex of C and B. Hence, there is a path $Q = v \dots e \subseteq B$ internally disjoint to C (and, thus, to D as well). Observe, that v lies not in D—otherwise P' could not be internally disjoint to D. With this, we obtain the path $P = P'vQ$ from f to e which meets D at most in its endvertices. □

We know now how to obtain a rayless graph G' from our original graph G suitable to our problem. In that graph Theorem 2.1 delivers generating cycles for an arbitrary cycle; we only have to ensure that these avoid our substitute bridge. This amounts to the following slightly strengthened version of Halin's theorem.

Lemma 2.6. *Let G be 3-connected and rayless. Let C be a cycle in G with a finite bridge B. Then C is the sum of peripheral cycles each meeting B at most in C.*

We start with the finite version of Lemma 2.6. For that, we need some further terminology. Let C be a cycle in a finite graph G. Two bridges B, B' of C are called *overlapping* if there is no path $P \subseteq C$ such that all vertices of attachment of B belong to P, but no inner vertex of P is a vertex of attachment of B'. Two bridges B, B' are called *skew* if there are vertices u, u', v, v' appearing in that circular order on C such that $u, v \in B$ and $u', v' \in B'$. It is easy to see that in a 3-connected graph two overlapping bridges are skew or have three vertices in common.

Lemma 2.7 (Tutte [7]). *Let G be a finite and 3-connected graph. Let C be a cycle in G with a bridge B. Then either C is peripheral or there is another bridge B' of C overlapping B.*

Lemma 2.8, the finite version of Lemma 2.6, is essentially Tutte's [7] original theorem. As the proof is rather short, we include it here for the convenience of the reader.

Lemma 2.8. *Let G be a finite and 3-connected graph. Let C be a cycle in G with a bridge B. Then C is the sum of peripheral cycles each meeting B at most in C.*

Proof. Let U be the set of sums of peripheral cycles that meet B at most in C. If $C \in U$ we are done, so assume otherwise. Then there is a cycle $D \notin U$ with

(i) D has a bridge $B' \supseteq B$,

(ii) no cycle $D' \notin U$ has a bridge that properly contains B'.

By Lemma 2.7 there is another bridge $\tilde{B}$ that overlaps B'.

First, let B' and $\tilde{B}$ be skew. Thus, there are vertices u, x, v, y that appear in that circular order on D, such that $u, v \in B'$ and $x, y \in \tilde{B}$. Denote by $P = x \dots y \subseteq \tilde{B}$ a path internally disjoint to D. Then $C_1 := xCyP$ and $C_2 := yCxP$ are cycles each having a bridge that properly contains B'. Thus, we have $C_1, C_2 \in U$—in contradiction to $D = C_1 + C_2$.

Finally, let B' and $\tilde{B}$ have three vertices x, y, z in common. There is a $b \in \tilde{B}$ and three paths $P_x = x \dots b$, $P_y = y \dots b$ and $P_z = z \dots b$ in $\tilde{B}$ meeting only in b and that are each internally disjoint to D. With these we obtain three cycles: $C_{xy} := xCyP_yP_x$, $C_{yz} := yCzP_zP_y$ and $C_{zx} := zCxP_xP_z$. Every one of these has a bridge properly containing B'. Thus, we have again the contradiction of $C_{xy}, C_{yz}, C_{zx} \in U$ but $D = C_{xy} + C_{yz} + C_{zx}$. □

Fortunately, Lemma 2.6 can be proven by closely following Halin's proof of Theorem 2.1. Our sole contribution is to include the bridge B into the reasoning, which is an easy task as B is assumed to be finite. Before we can start, however, we need a tool for handling rayless graphs that was developed by Schmidt [6] (see Halin [5] for an exposition in English). This tool allows transfinite inductions on rayless graphs.

Proposition 2.9 (Schmidt [6]). *On the class of rayless graphs there is a function o that assigns to each rayless graph G an ordinal such that the following is satisfied* *(i) G is finite if and only if $o(G) = 0$,*

(ii) if $o(G) > 0$ there is a finite set $F \subseteq V$ such that for every component K of $G - F$ we have $o(K) < o(G)$.

The function o will be called the order *of the graph.*

We will need the following properties below. For finitely many rayless graphs $G_1, \dots, G_n$ the order of the union is the maximum of the orders of the constituent graphs: $o(G_1 \cup \dots \cup G_n) = \max_{i=1\dots n} o(G_i)$.

It turns out, that for every infinite rayless graph G there is a unique minimal vertex set satisfying (ii). This set is called the *kernel* of G and denoted by $K(G)$.

Further, we need the notion of a *subkernel* of an infinite rayless graph G. For a vertex set $S \subseteq K(G)$ we denote by $\mathcal{C}_S$ the set of all components K of $G - K(G)$ with $N(K) = S$. Then S is called a subkernel of G if the graph $G[S \cup \bigcup \mathcal{C}_S]$ has the same order as G. Below we will make use of the observation, that for a subkernel S the set $\mathcal{C}_S$ is necessarily infinite—otherwise $G[S \cup \bigcup \mathcal{C}_S]$ would be the finite union of graphs each having a strictly smaller order than G, which is impossible.

Proof of Lemma 2.6. For the proof we make a transfinite induction on $o(G)$. For $o(G) = 0$, G is finite. Hence, Lemma 2.8 ensures the induction start.

So, assume $o(G) \geq 1$ and let the assertion be true for smaller orders. There is a set $\mathcal{S}$ of components of $G - K(G)$ with the following properties

(i) $C \cup B$ is in $G' := G[K(G) \cup \bigcup \mathcal{S}]$,

(ii) for every subkernel S of G the set of $K \in \mathcal{S}$ with $N(K) = S$ is finite but has at least $|K(G)| + 2$ elements,

(iii) $\mathcal{S}$ contains all the components K of $G - K(G)$ whose neighbourhood $N(K)$ is not a subkernel.

In fact, we may choose $\mathcal{S}$ as follows. First let $\mathcal{S}_1$ be the finitely many of the components of $G - K(G)$ that meet $C \cup B$ (which is a finite set). With this we have ensured (i). Now consider a subkernel S for which the set $\mathcal{S}_1$ does not satisfy (ii). By definition of a subkernel there are infinitely many components of $G - K(G)$ whose neighbourhood is S. Thus, by adding to $\mathcal{S}_1$ for each such S the finitely many missing components we arrive at a finite set $\mathcal{S}_2$ that satisfies (i) and (ii). All that is left to deal with is (iii), which is easily satisfied by including the by (iii) required components into $\mathcal{S}_2$. Note that doing this violates neither (i) nor (ii).

Having established the existence of such a set $\mathcal{S}$ we claim that G' is 3-connected and its order $o(G')$ is strictly smaller than $o(G)$.

For the 3-connectivity consider two vertices x and y of G'. For any two other vertices u and v of $G' - \{x, y\}$ there is an u-v-path P in $G - \{x, y\}$. The only reason why P may fail to be a path in G' as well is that P meets a component K of $G - K(G)$ not contained in $\mathcal{S}$. By (iii) the neighbourhood $N(K)$ of K has to be a subkernel. As a result of (ii), there are at least three components of $G - K(G)$ in $\mathcal{S}$ with neighbourhood $N(K)$. At least one of these components is disjoint to $\{x, y\}$ and thus may be used to substitute the part of P that goes through K. By doing this for all such components K we arrive at an u-v-path P' in $G' - \{x, y\}$.

For $o(G') < o(G)$ observe that the set $\mathcal{S}'$ of components in $\mathcal{S}$ whose neighbourhood is a subkernel of G is finite. Indeed, this is because of (ii) and the fact that there are only finitely many subkernels. Hence, we have $o(G[\bigcup \mathcal{S}']) < o(G)$ as each of the components has smaller order than G. For a set $S \subseteq K(G)$ that is not a subkernel, the union of the components $K \in \mathcal{S}$ with S as neighbourhood must have smaller order. Otherwise S would be a subkernel. Since the number of subsets of $K(G)$ is finite there are only finitely many of those unions. Therefore, the order of the union of these unions is smaller than $o(G)$ too. Combined with the preceding observation for subkernels we have established $o(G') < o(G)$.

Next, we show that a peripheral cycle D in G' is still peripheral in G. It is sufficient to assume D to be separating in G as D is induced in G' which, in turn, is an induced subgraph. $G' - D$ is connected and therefore contained in a component K of $G - D$. Suppose there is a different component K' of $G - D$ and denote its set of neighbours $N(K')$ by S. K' being disjoint to G' is therefore certainly disjoint to $K(G)$ as well. But S is contained in $K(G)$ since all the neighbours of $D - K(G)$ are in G'. As a consequence, K' is a component of $G - K(G)$ and by condition (iii), S a subkernel. Condition (ii) asserts that there are $|K(G)| + 2 \geq |S| + 2$ components in $\mathcal{S}$ with neighbourhood S. As K' is separated from K by D, any other component of $G - K(G)$ with neighbourhood S that is not met by D is as well separated from K by D. But D may meet at most $|S|$ of the components in $\mathcal{S}$ with neighbourhood S and, hence, avoids at least two of those components. But then D is already separating in G'—a contradiction.

Since B is completely contained in G' it is still a bridge of C in G'. By applying the induction hypothesis to the cycle C with bridge B in G' we obtain C as the sum of peripheral cycles that each meet B at most in C. By the argument above these cycles are peripheral in G as well and the theorem is proven. □

As noted the proof of Halin's problem follows easily.

Theorem 2.10. *The cycle space of a* 3*-connected graph G with at most one end is generated by peripheral cycles.*

Proof. As every element of the cycle space is the (finite) sum of cycles it is sufficient to consider a cycle C in G. Since C is finite and G has at most one end there is a bridge B of C containing a tail for every ray in G. Hence, the graph G' obtained from G as in Lemma 2.5 is rayless and 3-connected. Lemma 2.6 yields a set of peripheral cycles in G' each avoiding v_B whose sum is C. By Lemma 2.5 these cycles are peripheral cycles in G too, as required. □

Chapter 3

Locally finite graphs

3.1 Introduction

In this chapter we show that despite obvious counterexamples Tutte's result (Theorem 1.1) generalizes to locally finite graphs. This will be achieved by admitting infinite cycles and sums as recently proposed by Diestel and Kühn [3].

Let us look at a simple example. Consider the cartesian product of a *double ray* (an infinite 2-way) with a 4-cycle (Figure 3.1). The peripheral cycles of this graph are exactly its non-separating 4-cycles. We see that the (separating) cycle C in the figure is not the sum of any such cycles, so Tutte's theorem fails for this graph.

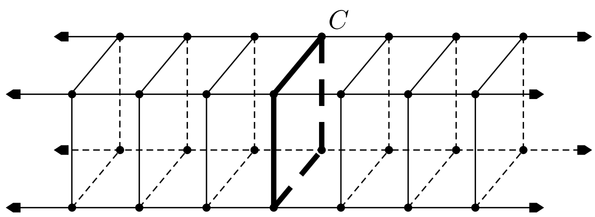

Figure 3.1: The cycle C is not the finite sum of peripheral cycles

It can be mended, however, by allowing infinite sums. Indeed, C is clearly the (infinite) sum of all the peripheral 4-cycles to the left of C (or to the right for that matter).

However, infinite sums of cycles can also produce edge sets of subgraphs such as the double ray shown in Figure 3.2, which should then also be legitimate elements of the cycle space closed under (well-defined) infinite sums. This complicates matters, but not beyond control: the subspace of the edge-space of a locally finite graph G that is generated by (possibly) infinite sums of the (finite) cycles of G has been studied by Diestel and Kühn in [3, 4] who obtained this space as an adaptation of the cycle space to *infinite cycles* involving the ends of G. (These infinite cycles are homeomorphic images of the unit circle in the standard compactification of G by its ends; for example, the double ray in Figure 3.2 forms an infinite cycle with the *left* end of the graph.) We shall

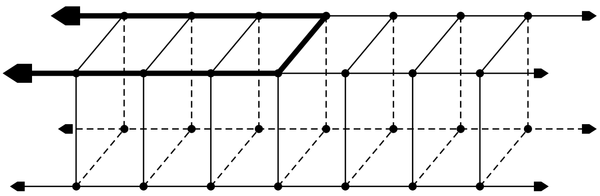

Figure 3.2: An infinite edge set that arises from an infinite sum of cycles

make use of the results in [3, 4] throughout this paper.

A short overview of the chapter follows. Section 3.2 contains the definition of the cycle space and the statement of our main result. In Section 3.3 we examine bridges and in particular the overlap graph of a cycle in a 3-connected graph. This will be seen to be connected. In Section 3.4 we prove the main lemma for our main result. In Section 3.5 we prove our main result, that every element of the (generalized) cycle space $\mathcal{C}(G)$ in a 3-connected locally finite graph is the sum of peripheral cycles. Also, we see that for some graphs G a generating set of $\mathcal{C}(G)$ consisting of peripheral cycles must contain infinite peripheral cycles. Finally, in Section 3.6 we show that following the approach presented in this chapter Tutte's result cannot be extended to arbitrarily infinite graphs.

3.2 Definitions and statement of the main result

A 1-way infinite path will be called a *ray*. A subray of a ray will be said to be a *tail* of that ray. Two rays in a graph are defined to be *equivalent* if there is no finite vertex set separating them. The resulting equivalence classes are called the *ends* of the graph. For a graph G the union of G with its ends is denoted by $\overline{G}$.

In order to define the cycle space Diestel and Kühn [3] introduced a topology on $\overline{G}$. For a finite set $S \subseteq V$ and an end ω there is exactly one component of $G - S$ that contains a tail for every ray in ω. This component will be denoted by $C_G(S, \omega)$ and we say ω *belongs* to $C_G(S, \omega)$. The union of $C_G(S, \omega)$ with all the ends belonging to that component is $\overline{C}_G(S, \omega)$. Write $E_G(S, \omega)$ for the set of all edges between S and $C_G(S, \omega)$ and let $\mathring{E}'_G(S, \omega)$ be any union of half-edges $(x, y] \subset e$, one for every $e \in E_G(S, \omega)$, with $x \in \mathring{e}$ and $y \in C_G(S, \omega)$. Then let the topology on $\overline{G}$ be generated by the open sets of G viewed as a 1-complex and the sets of the form

$$\hat{C}_G(S, \omega) := \overline{C}_G(S, \omega) \cup \mathring{E}'_G(S, \omega).$$

Throughout the paper $\overline{G}$ will be assumed to be endowed with this topology. In [3] it is shown that this topology is in a certain sense the best possible one on locally finite graphs.

A homeomorphic image of the unit interval in $\overline{G}$ is called an *arc* in $\overline{G}$. The images of 0 and 1 are the *endpoints* of the arc.

Having established a topology we may define cycles. First, we call a homeomorphic image C' of the unit circle in $\overline{G}$ a *circle*. The edge set C of C' is a *cycle*. We will, however, be a bit sloppy in so far as we shall repeatedly talk about a vertex $v \in C$ while meaning a vertex incident with one of the edges of C. An important property of the chosen topology is that for every circle C' in $\overline{G}$, the corresponding cycle $C' \cap G$ is dense in C'. Hence, given a cycle we may talk about its defining circle.

We now define infinite sums of edge sets. For this, let $(A_i)_{i \in I}$ be a family of edge sets. The family is called *thin* if every edge that is contained in one of the elements of the family lies in only finitely many of the A_i. The sum $\sum_{i \in I} A_i$ of such a thin family is defined to be the set of all edges that appear in exactly an odd number of the elements of the family. Whenever we talk about sums we will mean sums of thin families.

With this we may finally define the cycle space $\mathcal{C}(G)$ of a locally finite graph G to be the set of sums of (thin families of) cycles. One of the main results of [3, Cor. 11] is that the cycle space of a locally finite graph is closed under taking (infinite) sums. It should be noted that $\mathcal{C}(G)$ is a vector space over $\mathbb{Z}_2$.

Having defined the cycle space we may now state our main result.

Theorem 3.1. *Every element of the cycle space $\mathcal{C}(G)$ of a locally finite 3-connected graph G is a sum of peripheral cycles.*

A concept that will be essential in the proof is the following standard concept of a bridge. Here, we have taken the definition from Bondy and Murty [1].

Definition 3.2. *Let H be a subgraph of a graph G. We define an equivalence relation on $E(G) \backslash E(H)$ by $e \sim f$ if there is a path P such that*

(i) the first edge of P is e and the last is f.

(ii) P meets H at most at its ends.

A connected non-trivial subgraph B of $G - E(H)$ whose edge set is closed under this equivalence relation is called a bridge *of H. The vertices of B on H are called the* vertices of attachment *of B.*

One sees easily that a bridge is either a chord of H or a subgraph of G consisting of a component K of $G - H$ with the edges $E(K, H)$ added. Note, that a cycle is peripheral if and only if it has at most one bridge.

We give a rough overview of the proof of Theorem 3.1. Let an element of the cycle space $Z \in \mathcal{C}(G)$ with a bridge B be given. We will add suitable peripheral cycles to Z such that the sum $Z' \in \mathcal{C}(G)$ has a bigger bridge $B' \supsetneq B$. Continuing in this manner, we want that eventually (after possibly countable infinitely many steps) the inflated bridge B covers all of G. As this is only possible if the resulting sum is the empty set, we have then found a generating set of peripheral cycles for Z. Finding suitable peripheral cycles for the single steps will mostly be the work of Lemma 3.10.

Directly using the definition of the cycle space it may be a bit awkward to identify a given edge set as belonging to the cycle space. Fortunately, Diestel and Kühn provided a more accessible characterization as well. For this, let $\{V_1, V_2\}$ be a partition of the vertex set of a graph G. Then the set of all edges with one endvertex in V_1 and the other in V_2 is called a *cut*.

Theorem 3.3 (Diestel and Kühn [3]). *In a locally finite graph G the following statements are equivalent for $Z \subseteq E(G)$:*

(i) $Z \in \mathcal{C}(G)$

(ii) $|Z \cap F|$ *is even for every finite cut* F *of* G.

As cycles are easier to handle than arbitrary elements of the cycle space it turns out to be convenient that we can always decompose such an element into constituent cycles.

Theorem 3.4 (Diestel and Kühn [4]). *Every element of the cycle space of a graph* G *is an edge-disjoint union of cycles in* G.

Further, when dealing with sums or unions of a family $\mathcal{F}$ we will make use of the shorthands $\sum \mathcal{F}$ respectively $\bigcup \mathcal{F}$ to express the sum $\sum_{F \in \mathcal{F}} F$ respectively the union $\bigcup_{F \in \mathcal{F}} F$.

A tree T with a distinguished vertex $r \in T$ is called a *rooted tree* with *root* r. For another vertex $t \in T$ the predecessor on the path rTt is called the *parent* of t. The vertices that have t as their parent are the *children* of t. A vertex without children is a *leaf*.

For vertices v and w of a graph G we denote by $d_G(v, w)$ the minimal length of a v-w-path. Similarily, for an edge e and a vertex v, $d_G(v, e)$ is the minimal length of a path between v and one of the endvertices of e.

3.3 Overlap graphs

This section lays the groundwork for our main lemma, Lemma 3.10. A concept that we will make extensive use of in the next section is that of a *residual arc* of a bridge in a cycle. Let C be a cycle in a finite graph G and B a bridge of C. Then the vertices of attachement subdivide C into edge-disjoint walks; these are called the residual arcs of B in C (Tutte [7]).

In an infinite graph, however, the situation is slighty more complicated due to the fact that vertices of attachment may accumulate. More precisely, let C be a cycle in an infinite (connected) graph G with a bridge B. On the defining circle C' of C may be a point x in which vertices of attachment of B accumulate. Such an x is necessarily an end of G and we will see that it shares certain properties with the vertices of attachment. Hence, we call such an x an *end of attachment* of B.

Should B have exactly one vertex of attachment then the whole circle C' is called the residual arc of B in C. Otherwise, a subarc L of C' is said to be a *residual arc* of B in C if L does not contain any vertices of attachment of B in its interior and is (inclusion-) maximal with that property.

It is easy to see that in the case of a finite graph the two definitions coincide. Furthermore, the following lemma shows that the latter definition behaves in fact in a similar way as its finite equivalent. To exclude the pathological case of C' itself being a residual arc we restrict ourselves to 2-connected graphs.

Lemma 3.5. *Let* C *be a cycle in a* 2*-connected graph* G *with a bridge* B*. Then the following statements are true.*

(i) The endpoints of a residual arc L *of* B *in* C *are vertices of attachments or ends of attachment of* B.

(ii) For a vertex $v \in C$ but $v \notin B$ there is exactly one residual arc L of B in C containing v.

(iii) For a vertex $v \in C \cap B$ there are exactly two residual arcs of B in C with v as an endpoint.

Proof. (i) Let $x \in L$ be an endpoint of L and assume that x is not a vertex of attachment of B. Take a neighbourhood U of x in C'. There must be a vertex of attachment of B in $U \backslash L$, for otherwise L could be extended into $U \backslash L$. Since this is true for every neighbourhood we have identified x as an end of attachment of B.

(ii) and (iii) Let $x \in C'$ be a point on C' that is neither a vertex of attachment of B nor an end of attachment. For any two vertices of attachment u, w denote by L_{uw} the subarc of C' with endpoints u and w that contains x (note that there is exactly one). Let $\mathcal{S}$ be the set of all pairs of two (different) vertices of attachment. Then, denote by L the component from

$$\bigcap_{\{u,w\} \in \mathcal{S}} L_{uw}$$

that contains x. Note that L is an arc. Additionally, the endpoints of L are because of construction vertices of attachment or ends of attachment, while the interior of L is devoid of them. All this identifies L as a residual arc, and as any other residual arc meeting x would have to meet L in its interior the maximality of L guarantees L to be the unique residual arc of B in C containing x.

From this, (ii) follows immediately. For (iii) take a connected neighbourhood U of v in C' that does not contain a vertex of attachment of B apart from v. $U \backslash v$ consists of two components; in each of those we choose a point and apply the construction above. In this way we obtain two residual arcs containing v—if there were more, then two of them would overlap in their interior which is impossible because of the maximality. □

The following definitions are immediate generalizations of the corresponding definitions found in Bondy and Murty [1] for finite graphs.

Definition 3.6. *Let $C \subseteq G$ be a cycle and $C' \subseteq \overline{G}$ its defining circle.*

(i) We say a bridge B of C avoids *another bridge B' of C if there is a residual arc of B that contains all vertices of attachment of B'. Otherwise, they* overlap.

(ii) Two bridges B and B' of C are called skew *if C' contains four vertices v, v', w, w' in that cyclic order such that v, w are vertices of attachment of B and v', w' vertices of attachment of B'.*

Clearly, if two bridges B, B' of a cycle C are skew they overlap. Further, for a subgraph H of a graph G we define the *overlap graph* of H in G as the graph on the bridges of H such that two bridges are adjacent if and only if they overlap.

It is easy to see that in a 3-connected graph it is impossible for a bridge of a cycle to avoid all other bridges (unless it is the only one). Thus, the overlap graph of that cycle cannot have trivial components, but, it turns out, even more is true: there is only a single component.

Lemma 3.7. *For every cycle C in a 3-connected graph G the overlap graph of C in G is connected.*

For the proof we recall the following standard concept. A linear ordering of the points of an arc A that is induced by a homeomorphism $\sigma : [0,1] \mapsto A$ is said to be an *orientation* of A. If $\vec{A}$ is an arc with an orientation and $a, b \in A$ we denote by $a\vec{A}$ the oriented subarc containing all points $c \in A$ with $c \geq a$; in a similar way we define $\vec{A}b$ and $a\vec{A}b$. Further, we write $\mathring{a}\vec{A}$ for the oriented subarc consisting of all points $c \in A$ with $c > a$. Analogously, we define $\vec{A}\mathring{b}$ and $\mathring{a}\vec{A}\mathring{b}$.

Proof. Denote by C' the defining circle of C in $\overline{G}$. Let K be a component of the overlap graph of C in G.

(i) Firstly, we show that for every vertex on C there is a bridge in K meeting that vertex. Suppose there is a vertex $v \in C$ that is not a vertex of attachment for any bridge in K. For a bridge $B \in K$ denote by M_B the residual arc of B in C containing v (Lemma 3.5). Define M to be the component of $\bigcap_{B \in K} M_B$ containing v and set $L := C' \setminus \mathring{M}$. Observe, that as M is an arc containing no vertices of attachment of bridges of K in its interior, L is an arc covering every vertex of attachment of every bridge in K. Further, because the endpoints of each of the M_B, $B \in K$, are vertices of attachment or ends of attachment (once again, Lemma 3.5) the two endpoints x, y of L are each a vertex of attachment of a bridge in K or an accumulation point of vertices of attachment of bridges of K (though not necessarily an end of attachment). Note that for x or y to be of the latter kind it has to be an end of G. (Below we will repeatedly consider a graph $H - \{x, y\}$ for a subgraph H of G. If x is in fact an end we will tacitly assume that $H - y$ is meant—the same holds for y.)

Observe that neither $C - (L \cap G)$ nor $(L \cap G) - \{x, y\}$ are empty: the former graph contains at least the vertex v, and the latter cannot be empty because L covers at least the three vertices of attachment of at least one bridge—unless K consists of only a single chord. But in that case, x and y cannot be adjacent. Now, as a consequence of the 3-connectivity of G there is a C-path P from $C - (L \cap G)$ to $(L \cap G) - \{x, y\}$ in $G - \{x, y\}$. Clearly, the bridge B of C containing P cannot be in K as L fails to cover all its vertices of attachment.

On the other hand, we claim that this bridge B is skew to a suitable bridge in K. Indeed, let $\vec{L}$ be the arc L endowed with the orientation given by $x < y$ and let z be the endvertex of P on L. Let x' be a vertex of attachment of a bridge $B_{x'} \in K$ on $x\vec{L}\mathring{z}$. Such an x' exists because if we cannot choose x for x' then x has to be an accumulation point of vertices of attachement of bridges in K. Similarly, let y' be a vertex of attachment of a bridge $B_{y'} \in K$ on $\mathring{z}\vec{L}y$. Take B' to be the first bridge on a path from $B_{x'}$ to $B_{y'}$ in K to have a vertex of attachment in $\mathring{z}\vec{L}y$. We either have $B' = B_{x'}$ or B' must be skew to the previous bridge on that path. In both cases, B' has a vertex of attachment in $x\vec{L}\mathring{z}$ and another one in $\mathring{z}\vec{L}y$ and is thus skew to B—a contradiction. Consequently, the bridges of K cover the whole cycle C with their vertices of attachment.

(ii) Finally, we show that K is the only component of the overlap graph. Consider a bridge B. Should every vertex of C be a vertex of attachment of B then every other bridge of C overlaps B (any other bridge B' has either three vertices in common with B or it is a chord; but then we have $|C| > 3$). So, let $v \in C$ be a vertex not met by B. Denote by L the residual arc of B in C

containing v. Let w be any vertex in $C'\backslash L$ (such a vertex exists as B has at least three vertices of attachment, or it is chord, in which case the two endpoints of L cannot be adjacent vertices in C). We have seen that there is a bridge $B_v \in K$ that meets v and similarily $B_w \in K$ containing w. In a similar way as in (i), the first bridge on a path in K from B_v to B_w that meets $C'\backslash L$ is skew to B. Consequently, B is contained in K and, in turn, K the only component of the overlap graph. □

Before we state and then prove the main lemma, we point out a simple fact for later use.

Lemma 3.8. *Let C by a cycle in a locally finite graph G. Let B be a bridge of C with an end of attachment ω. Then B contains a ray of ω.*

For the proof we need a simple lemma which can be found in [3]:

Lemma 3.9. *Let U be an infinite set of vertices in a connected locally finite graph. Then there exists a ray $R \subseteq G$ for which G contains an infinite set of disjoint U-R paths.*

Proof of Lemma 3.8. As ω is an end of attachment of B there is a ray $R \subseteq C$ in ω containing infinitely many vertices of attachments of B. Denote the set of these vertices of attachment by U and apply Lemma 3.8 to the graph B and the set U. We obtain a ray R' in B that is equivalent to R. □

3.4 Locally generating a cycle

This section is devoted to proving the following lemma, which will later be used in the induction step for the proof of our main result.

Lemma 3.10. *Let C be a cycle in the locally finite 3-connected graph G with a bridge B and let $v \in V$ be a vertex on C. Then there are peripheral cycles $D_1, \ldots, D_m$ that meet B at most in C satisfying*

$$\sum_{i=1}^{m} D_i \cap E(v) = C \cap E(v).$$

Apart from being used in the proof of Theorem 3.1, the lemma may serve as an indicator that the theorem itself is not unreasonable. Indeed, at the very least one should be able to find a peripheral cycle for any given edge—and this is in fact the case according to the lemma.

Lemma 3.10 and its proof are inspired by a result of Tutte [7, (2.2)]. For the remainder of this section G will be assumed to be 3-connected and locally finite, and whenever we talk about a cycle we will mean a cycle in G.

Consider a cycle C with a bridge B and a vertex $v \in C$. Further, let a second bridge $\tilde{B}$ overlapping B be given and let L be a residual arc of $\tilde{B}$ in C that meets v. Denote by x and y the two endpoints of L (Figure 3.3). Now, depending on the nature of x and y we distinguish different cases. Should both, x and y, be vertices of attachment of $\tilde{B}$, let P be an x-y-path through $\tilde{B}$. If one of x and y is a vertex of attachment, say x, and the other, here y, an end of attachment, let $P \subseteq \tilde{B}$ be a ray of the end y starting in x (Lemma 3.8 of

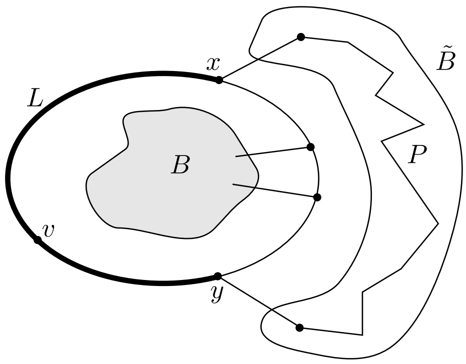

Figure 3.3: The extension step

the previous section guarantees the existence of such a P). Finally, if both x and y are ends of attachment of $\tilde{B}$, let $P \subseteq \tilde{B}$ be a double ray ending in x and in y (once again, Lemma 3.8). To check that we have covered all cases see Lemma 3.5.

It is easy to verify that in any case $C' := E(L) \cup E(P)$ is a cycle that contains v and that has a bridge $B' \supseteq B$. We say (C', B') is *gained* from (C, B) through the *extension step* $(\tilde{B}, L, v)$.

If $\tilde{B}$ contains v we know from Lemma 3.5 from the previous section that there is a second residual arc L' of $\tilde{B}$ in C meeting v. In that case we find a path or ray, as the case may be, $P' \subseteq \tilde{B}$ from v to the other endpoint of L' with the same edge incident with v as on P (because of the equivalence relation). Then $C'' := E(L') \cup E(P'')$ is as well a cycle with a bridge B'' so that (C'', B'') is gained from (C, B) by the extension step $(\tilde{B}, L', v)$. We call (C'', B'') a *twin* of (C', B') with respect to (C, B). We see that the following holds

$$C \cap E(v) = (C' + C'') \cap E(v). \qquad (*)$$

The proof of Lemma 3.10 follows roughly along similar lines as the proof of Theorem 3.1: given a cycle C_1 with a bridge B_1 we try to inflate the bridge B_1. More precisely, we will construct a sequence $(C_1, B_1), (C_2, B_2), \ldots$ of cycle-bridge pairs such that (C_i, B_i) is gained from its predecessor by an extension step. Then, we have $B_i \supseteq B_{i-1}$ and our aim is to do the extension in such a way that eventually B_i grows so big that it is the only bridge left. Clearly, the corresponding cycle is then peripheral. Unfortunately, this sequential approach may be insufficient. Rather, it is sometimes necessary to perform two alternative extension steps simultaneously. This parallel approach is captured in the concept of an extension tree we shall now introduce.

Definition 3.11. *Let C by a cycle with a bridge B and let $v \in C$ be a vertex. Let T be a finite rooted tree with root r and let there be mappings*

$$\begin{aligned} C_T : V(T) &\mapsto \{C' \in \mathcal{C}(G) \,|\, C' \text{ is a cycle}\} \\ B_T : V(T) &\mapsto \{B' \subseteq G \,|\, B' \text{ is a bridge of a cycle}\} \end{aligned}$$

satisfyfing the following

(i) for $w \in V(T)$ $B_T(w)$ is a bridge of the cycle $C_T(w)$

(ii) $C_T(r) = C$ and $B_T(r) = B$

(iii) let $p \in V(T)$ be the parent of $w \in V(T)$. Then there is a bridge $\tilde{B}$ overlapping $B_T(p)$ and a residual arc L of $\tilde{B}$ in $C_T(p)$ meeting v such that $(C_T(w), B_T(w))$ is gained from $(C_T(p), B_T(p))$ by the extension step $(\tilde{B}, L, v)$.

(iv) let $p \in V(T)$ have a child u such that $(C_T(u), B_T(u))$ has a twin with respect to $(C_T(p), B_T(p))$. Then p has exactly one other child; and that is mapped on such a twin.

If all these conditions are satisfied we call T or, more formally, (T, r, C_T, B_T) an extension tree *with parameters (C, B, v).*

Note that, firstly, (iii) and (iv) imply that a vertex in an extension tree has at most two children. Secondly, deleting all descendants of a given vertex and restricting the mappings to the remaining vertices will yield another extension tree with the same parameters. And finally, the converse operation leads to an extension tree too: let $(T_1, r, C_{T_1}, B_{T_1})$ be an extension tree with parameters (C, B, v), l a leaf of T_1 and $(T_2, l, C_{T_2}, B_{T_2})$ an extension tree with parameters $(C_{T_1}(l), B_{T_1}(l), v)$. Then the tree $T := T_1 \cup T_2$ with root r and with the from both trees induced mappings is an extension tree with parameters (C, B, v).

Lemma 3.12. *Let T be an extension with parameters (C, B, v). For every vertex $t \in V(T)$ the cycle $C(t)$ meets B at most in C.*

Proof. Induction on the depth of t—note that if p is the parent of t we have $B(p) \subseteq B(t)$. □

The next lemma is the reason why we have introduced extension trees at all, instead of employing a sequential algorithm: the cycles associated with the leaves sum to precisely the edges of C at v.

Lemma 3.13. *Let T be an extension tree with parameters (C, B, v). Then the following holds*

$$\sum_{l \text{ leaf of } T} C(l) \cap E(v) = C \cap E(v).$$

Proof. Induction on $|T|$ and the equation $(*)$. □

Consider a cycle C_1 with a bridge B_1, and let (C_2, B_2) be gained from (C_1, B_1) by the extension step $(\tilde{B}, L, v)$. In a finite graph it is trivial that the edges $C_1 \backslash E(L)$ go into the bridge B_2. Indeed, B_1 has a vertex of attachment in $C_1 \backslash E(L)$ since B_1 and $\tilde{B}$ are overlapping (Figure 3.3). Now, as $C_1 \backslash E(L)$ is clearly connected $C_1 \backslash E(L) \subseteq B_2$ follows immediately.

In an infinite graph, however, $C_1 \backslash E(L)$ does not have to be connected. Instead, it may be the disjoint union of rays and double rays. Nevertheless, $C_1 \backslash E(L) \subseteq B_2$ holds in locally finite graphs as well. In fact, to every component of $C_1 \backslash E(L)$ we find another component (provided there is more than one) such that each sends a ray into the same end. Between these two rays there are, by definition, infinitely many disjoint paths connecting them and we see in the following lemma that not all of these may meet C_2.

Lemma 3.14. *Let C_1 be a cycle with a bridge B_1 and $v \in C_1$ a vertex. Let (C_2, B_2) be gained from (C_1, B_1) by the extension step $(\tilde{B}, L, v)$. Then we have*

$$C_1 \backslash E(L) \subseteq B_2$$

and especially

$$E(C_1 \cup B_1) \subseteq E(C_2 \cup B_2).$$

Proof. Denote by C_1' the C_1 defining circle. Since B_1 and $\tilde{B}$ are overlapping B_1 has a vertex of attachment b in $C_1' \backslash L$. Let e be an edge of B_1 incident with b. For a given edge $f \in C_1 \backslash E(L)$ we shall find a path between e and f that is internally disjoint to C_2.

For this, let P be the path, ray or double ray between the two endpoints of L that was used in the extension step to construct C_2 (indeed $P = C_2 \backslash C_1$). Further, there is an arc M in $C_1' \backslash L$ with b and one of the vertices of f, denoted by w, as endpoints such that $f \cap M$ consists just of the vertex w. This implies $L \cap M = \emptyset$ (neither of b and w are in L).

Now, for every $z \in M$ we will construct an open neighbourhood of z that is disjoint to C_2 and whose intersection with G is (graph-theoretically) connected. If z is a vertex or a point in the interior of an edge such a neighbourhood can easily be found. So, let z be an end of G. First, we find a finite set of vertices S_z such that the open set $\hat{C}_G(S_z, z)$ (for the definition see Section 3.2) does not intersect with P. For this, note that z can be separated by a finite vertex set S_z' from the up to two ends in which P may send a ray (dependig on the nature of the endpoints of L). Then, only finitely many vertices of P may meet $\hat{C}_G(S_z', z)$. Adding these to S_z' we arrive at a finite set S_z such that $\hat{C}_G(S_z, z)$ is disjoint to P. Next, we find a finite vertex set T_z that accomplishes the same for L. Let U be a neighbourhood of the preimage of z in the unit circle, such that U is disjoint to the preimage of L. As $M \cap L$ is empty (and both, M and L, are closed) this can clearly be done. But then, by continuity, there is a finite vertex set T_z such that the preimage of $\hat{C}_G(T_z, z)$ is contained in U. Hence, $\hat{C}_G(T_z, z)$ is disjoint to L. To conclude, the neighbourhood $\hat{C}_G(S_z \cup T_z, z)$ of z has the desired properties: it is disjoint to $C_2 = E(L) \cup E(P)$ and its intersection with G clearly connected.

As M is compact we find finitely many of these neighbourhoods covering M. By piecing paths in these together we obtain a b-w-path Q not meeting C_2. After adding the edges e and f we have a path as required above.

For the second equation just note that $B_1 \subseteq B_2$ and $E(L) \subseteq C_2$. □

Recall that we want to inflate the bridge B so that eventually it is the only one left. To achieve this, we have to ensure that B grows in a relatively controlled way. In particular, we need to be able to perform the extension steps in such a way that after finitely many steps our favourite bridge $\tilde{B}$ can be used for the next step (if it is not already contained in the inflated bridge).

Lemma 3.15. *Let C be a cycle with bridge B and $v \in C$ a vertex. For a bridge $\tilde{B}$ there is an extension tree T with parameters (C, B, v) such that for every leaf l of T we have*

either $\tilde{B} \subseteq B(l)$ or $\tilde{B}$ is a bridge of $C(l)$ overlapping $B(l)$.

Proof. The connectivity of the overlap graph of C in G (by Lemma 3.7) ensures the existence of a shortest B-$\tilde{B}$-path P in the overlap graph. We do an induction on the length of such a path.

So, if P is trivial we have $B = \tilde{B}$ and if $\|P\| = 1$, B and $\tilde{B}$ are overlapping. In both cases we are done. So assume P has a length of at least two. Let $P = K_1 \dots K_k$ with $K_1 = B$ and $K_k = \tilde{B}$. Note that k is at least three. We define an extension tree T' with parameters (C, B, v) as follows. For the root r we map $C_{T'}(r) := C$ and $B_{T'}(r) := B$. Now, let L be the residual arc of the bridge K_2 in C that meets v and let (C_1, B_1) be gained from (C, B) by the extension step (K_2, L, v). We assign a vertex c_1 to be a child of r and map $C_{T'}(c_1) := C_1$ and $B_{T'}(c_1) := B_1$. Should (C_1, B_1) have a twin (C_2, B_2) with respect to r we let r have a second child c_2 and define the mappings accordingly. The resulting tree T' is then an extension tree.

Consider the child c_1. The bridge K_3 overlaps K_2 and has therefore, by definition (Definition 3.6), a vertex of attachment in $C \backslash E(L)$. Together with $C \backslash E(L) \subseteq B_1$ (Lemma 3.14) this implies that the greatest index $i \leq k$ with $K_i \subseteq B_1$ is at least three. Any bridge K_j with a greater index j must necessarily have all its vertices of attachement in L and is thus still a bridge of C_1. Also, if for $j, j' > i$ K_j and $K_{j'}$ are overlapping as bridges of C then they are overlapping as bridges of C_1 as well (this is decided on L). We arrive at a shorter path $B_1 K_{i+1} \dots K_k$ in the overlap graph of C_1. If that path is trivial, we have $\tilde{B} \subseteq B_1$ and define $T_1 := \emptyset$. Otherwise, $\tilde{B} = K_k$ is a bridge of C_1 and there is a shorter path from B_1 to $\tilde{B}$ in the overlap graph of C_1. Induction yields an extension tree T_1 with parameters (C_1, B_1, v) such that the associated bridge of every leaf either contains $\tilde{B}$ or overlaps it. Now, if r has only one child the tree $T := T' \cup T_1$ with root r satisfies the assertion. Otherwise, we obtain in a similar way an extension tree T_2 with parameters (C_2, B_2, v) and $T := T' \cup T_1 \cup T_2$ is the desired tree. □

If we can find an extension tree for which the associated cycle $C(l)$ of every leaf l is peripheral we are done, as the Lemmas 3.12 and 3.13 demonstrate. We introduce a measure of how far the leaves of an extension tree are from being peripheral.

Let T be an extension tree with parameters (C, B, v) and let $b \in B$ be a fixed vertex. For a vertex t of T we define

$$d^{(b)}(t) := \sup\{N \in \mathbb{Z} \,|\, \forall e \in E(G),\, d_G(b, e) \leq N \;\Rightarrow\; e \in E(C(t) \cup B(t))\}$$

where we admit ∞. This definition ensures that every edge e with $d_G(b, e) \leq d^{(b)}(t)$ lies in $E(C(t) \cup B(t))$. If $d^{(b)}(t) = \infty$ then every edge is contained in $E(C(t) \cup B(t))$ and $C(t)$ therefore peripheral. With that we define

$$d^{(b)}(T) := \min\{d^{(b)}(l) \,|\, l \text{ is leaf of } T\}.$$

Thus, rewritting our statement above, if we can find an extension tree with $d^{(b)}(T) = \infty$ we are done. Unfortunately, this will not always be possible. Lemma 3.16, however, shows that we can achieve the next best thing, namely finding a sequence of nested extension trees with strictly increasing $d^{(b)}(T)$. First, we make the notion of nested extension trees more precise.

To keep notation as simple as possible we will just write $T \subseteq T'$ for two extension trees T and T' while tacitly assuming that both trees are extension trees with the same parameters and the same root and that the mappings C_T and B_T of T are induced by the corresponding mappings of T'.

So let $(T_n)_{n\in\mathbb{N}}$ be a family of extension trees with parameters (C, B, v) such that $T_n \subseteq T_{n+1}$ for all $n \in \mathbb{N}$. Then we call the family an *extension family* with *parameters* (C, B, v).

Lemma 3.16. *Let C be a cycle with a bridge B. Let $v \in C$ and $b \in B$ be vertices. Assume there is no extension tree T with parameters (C, B, v) and $d^{(b)}(T) = \infty$. Then there is an extension family $(T_n)_{n\in\mathbb{N}}$ with parameters (C, B, v) so that*

$$d^{(b)}(T_1) < d^{(b)}(T_2) < \dots$$

Proof. We will inductively construct nested extension trees $T_1, \dots, T_n$ with $d^{(b)}(T_1) < \dots < d^{(b)}(T_n)$. For T_1 take any extension tree with parameters (C, B, v). For $n > 1$ let $T_1, \dots, T_{n-1}$ be already constructed.

Setting $d := d^{(b)}(T_{n-1}) + 1$ (note that $d^{(b)}(T_{n-1}) < \infty$) we define

$$m(t) := |\{e \in E(G)\backslash E(C(t) \cup B(t)) \mid d_G(b, e) \le d\}|$$

for a vertex t of an extension tree T. Further, define

$$m(T) := \max_{l \text{ leaf of } T} m(l).$$

Observe, that since G is locally finite, $m(T)$ is always finite (and is, in particular, defined). We see that $m(T) = 0$ implies $d^{(b)}(T) \ge d$. Thus, our task is to find an extension tree T with $T \supseteq T_{n-1}$ and $m(T) = 0$. For this, it suffices to establish the following claim.

For each leaf p of T_{n-1} there is an extension tree T_p with root p and parameters $(C(p), B(p), v)$ such that $m(T_p) < m(T_{n-1})$. $(*)$

Indeed, the union T of all those T_p and T_{n-1} is an extension tree (with parameters (C, B, v)) with $T \supseteq T_{n-1}$ and $m(T) < m(T_{n-1})$. An induction argument then allows us to find such a T with $m(T) = 0$.

To establish the claim, consider an edge $e \notin E(C(p) \cup B(p))$ with $d_G(b, e) \le d$ and denote by $\tilde{B}$ the bridge of $C(p)$ containing e. In particular, e contributes to $m(p)$. With Lemma 3.15 we find an extension tree T' with parameters $(C(p), B(p), v)$ such that the associated bridge of every leaf of T' either contains $\tilde{B}$ or overlaps it. Let l be a leaf of T'.

Assume first that $\tilde{B} \subseteq B_{T'}(l)$. So, e is contained in $B_{T'}(l)$ as well and we have $m(l) < m(p)$.

Now, assume $\tilde{B}$ and $B_T(l)$ to be overlapping. Let L be a residual arc of $\tilde{B}$ in $C_T(l)$ containing v and let (C', B') be gained from $(C_{T'}(l), B_{T'}(l))$ by the extension step $(\tilde{B}, L, v)$. Assume further, that (C', B') has no twin with respect to l. Denote by T the extension tree obtained from T' by adding a child c to l and mapping $C_T(c) := C'$ and $B_T(c) := B'$. Observe, that e has a vertex in common with $C_T(l)$, say the vertex w (every edge with lesser distance to b is contained in $C_T(l) \cup B_T(l)$). Should w be contained in $C_T(l)\backslash E(L)$, we have by Lemma 3.14 $e \in B_T(c)$, leading to $m(c) < m(p)$. Thus, we have to deal with the case that w is one of the two endpoints of L (being a vertex of attachment, w cannot be contained in the interior of L). The w-$\tilde{B}$-edge f that is used to construct $C_T(c)$ clearly has the same distance to b as e. Consequently,

f contributes to $m_d(p)$—but not to $m(c)$. Again, this leads to $m(c) < m(p)$. Should c have a twin with respect to l we extend T' in a similar way for that twin.

Modifying T' in this way for each leaf l we arrive at an extension tree T in which for every leaf l holds: $m(l) < m(p) \leq m(T_{n-1})$. Hence, we have $m(T) < m(T_{n-1})$, thereby establishing $(*)$. □

Let $(T_n)_{n\in\mathbb{N}}$ be an extension family with parameters (C, B, v). The union $T := \bigcup_{n\geq 1} T_n$ is then an infinite rooted tree. We extend the mappings C_{T_n} and B_{T_n} of the T_n to mappings C_T and B_T of T in the natural way. T will be called an *infinite extension tree* with *parameters* (C, B, v). To distinguish clearly between these infinite extension trees and the extension trees defined earlier (in Definition 3.11) we shall speak of *finite* extension trees when the latter ones are meant.

Fix a vertex $b \in B$. Since in an extension family the sequence $d^{(b)}(T_n)$ is monotonically increasing (Lemma 3.14) the limit is well defined (if we admit ∞). We make use of that to define $d^{(b)}(T) := \lim_{n\mapsto\infty} d^{(b)}(T_n)$.

With this terminology Lemma 3.16 asserts that if we cannot find a finite extension tree of which all the (associated cycles of the) leaves are peripheral, then there is an infinite extension tree T with $d^{(b)}(T) = \infty$. Being infinite T has rays starting in the root vertex. These rays play a similar role as the leaves, and indeed we may extract a peripheral cycle from each of these rays.

Lemma 3.17. *Let C be a cycle with a bridge B. Let $v \in C$ and $b \in B$ be vertices. Let T be an infinite extension tree with parameters (C, B, v) and $d^{(b)}(T) = \infty$. Consider a ray $c_1c_2\ldots$ in T starting in the root vertex of T. Then the limit*

$$D := \lim_{n\mapsto\infty} C_T(c_n),$$

where the limit is taken edge-wise, exists and D is the union of edge-disjoint peripheral cycles that meet B at most in C.

Proof. Let (T_n) be the T defining extension family and let the family w.l.o.g. be so that $d^{(b)}(T_1) < d^{(b)}(T_2) < \ldots$. For convenience's sake we set $C_n := C_T(c_n)$ and $B_n := B_T(c_n)$ for $n \in \mathbb{N}$. Further, we set $V_i := \{w \in V \mid d_G(b, w) \leq i\}$ for $i \in \mathbb{N}_0$. The proof consists of three parts.

(i) First, we show the existence of the limit. In fact, we show that

there is a $M \in \mathbb{N}$ and a sequence $(N_m)_{m\geq M}$ of integers such that $C_n \cap G[V_m] = C_{N_m} \cap G[V_m]$, for all $m \geq M$ and $n \geq N_m$. $(*)$

Clearly, if $(*)$ holds then the limit D exists and coincides with

$$D = \bigcup_{m\geq M} C_{N_m} \cap G[V_m].$$

Choose M large enough such that $v \in V_M$. Consider an integer $m \geq M$. If $m > M$ assume $N_M, \ldots, N_{m-1}$ to be already defined. With $N \in \mathbb{N}$ such that $m \leq d^{(b)}(T_N)$ we observe that

$$C_{n+1} \cap G[V_m] \subseteq C_n \cap G[V_m] \subseteq C_N \cap G[V_m]$$

holds for any $n \geq N$. Indeed, let (C_{N+1}, B_{N+1}) be gained by the extension step $(\tilde{B}, L, v)$. Since $m \leq d(T_N)$ every edge of $G[V_m]$ lies in C_N or B_N. But $C_{N+1} \backslash C_N$ lies in $\tilde{B}$ and, hence, may not meet $G[V_m]$. Inductively, we obtain the equation above.

$C_n \cap G[V_m]$, $n \geq N$ is finite, thus there is an n that minimizes that intersection. This is the desired N_m. We have established $(*)$.

(ii) Next, we show that D is the edge-disjoint union of cycles each meeting B at most in C. For this, let F be a finite cut of G. Choose m large enough for $F \subseteq G[V_m]$. We obtain

$$F \cap D = (F \cap G[V_m]) \cap D = F \cap (D \cap G[V_m]) = F \cap C_{N_m}$$

where the last equality is because of $(*)$. As the cycle C_{N_m} clearly is an element of the cycle space Theorem 3.3 asserts that the cardinality of $F \cap C_{N_m}$ is even. This, in turn, results in $|F \cap D|$ being even for every finite cut F. Employing Theorem 3.3 once again (only now the other direction of the equivalence) we see that D is an element of the cycle space. As an element of the cycle space D is the edge-disjoint union of a set $\mathcal{F}$ of cycles (Theorem 3.4).

Since every C_{N_m} meets B at most in C (see Lemma 3.12) this holds for the limit D as well. In particular, this is still true for the cycles in $\mathcal{F}$ (being subsets of D).

(iii) Finally, we see that the cycles of $\mathcal{F}$ are indeed peripheral. For this, we begin by claiming that D has exactly one bridge and this bridge meets every vertex of G. Indeed, let e and f be edges not contained in D. Choose m large enough so that at the same time $e, f \notin C_{N_m}$ holds and the distance of e and f to b is at most $d(T_{N_m})$. Clearly, this choice of m implies $e, f \in B_{N_m}$. Thus, there is a path P in B_{N_m} internally disjoint to C_{N_m} with first edge e and last edge f. For every $n \geq m$ B_{N_n} is a superset of B_{N_m}. Thus P is still internally disjoint to C_{N_n}, and hence to the limit D. We see that e and f are contained in the same bridge of D—which is then the only bridge of D as e and f were arbitrarily chosen. Now observe, that G has a minimal degree of at least three. Consequently, every vertex is incident with an edge not contained in D (every vertex of D has degree two in D) and therefore lies in the bridge of D, which then meets every vertex.

To finish the proof consider a cycle D' of $\mathcal{F}$. Now, as a subset of D, D' has a bridge covering every vertex of G—so this is the only bridge of D' which may have vertices not meeting D'. But D' cannot have chords either as these would be chords of D as well, so D' has exactly one bridge. $\square$

We have seen, that the desired peripheral cycles may be obtained from the leaves and rays (starting in the root vertex) of a suitable extension tree. Now, Lemma 3.10 requires the number of the peripheral cycles to be finite. Fortunately, even an infinite extension tree has only finitely many leaves and rays.

Lemma 3.18. *Let T_∞ be an infinite extension tree. Then T_∞ has only finitely many leaves and only finitely many rays starting in its root vertex.*

Proof. Let T_∞ have the parameters (C, B, v). To establish the assertion it suffices to show that there is a $N \in \mathbb{N}$ such that for any finite extension tree T with parameters (C, B, v) the number of leaves is bounded by N. For a vertex t

of T let $k(t)$ be the number of edges of G incident with v that are not contained in $C(t) \cup B(t)$. We prove by induction on $|T|$ that

$$\sum_{l \text{ leaf of } T} 2^{k(l)} \leq 2^{k(r)},$$

where r denotes the root vertex of T. Note that every leaf is counted in the sum, as $2^{k(l)} \geq 1$, and that the righthand side is the same for all extension trees with parameters (C, B, v).

Clearly, the inequality holds for $|T| = 1$. So assume $|T| > 1$. Then, we find a vertex p of T that has children which are all leaves. Deleting all these children leads to an extension tree T' with fewer vertices than T. If p has only a single child c, we have $k(c) \leq k(p)$ by Lemma 3.14. Now, assume that p has two children c and d. Let $(C(c), B(c))$ be gained from its parent by the extension step $(\tilde{B}, L, v)$. Consider the edge $f \in C(c) \backslash C(p)$ incident with v (such an edge exists as $\tilde{B}$ meets v by Definition 3.11, (iv)). We see that $f \in E(C(c) \cup B(c))$ but $f \notin E(C(p) \cup B(p))$, leading to $k(c) < k(p)$ (note Lemma 3.14). By similar reasoning this holds for d as well, yielding $2^{k(c)} + 2^{k(d)} \leq 2^{k(p)}$. Summing over all leaves of T we obtain

$$\begin{aligned}
\sum_{l \text{ leaf of } T} 2^{k(l)} &\leq \sum_{\substack{l \text{ leaf of } T, \\ \text{no child of } p}} 2^{k(l)} + 2^{k(p)} \\
&\leq \sum_{l' \text{ leaf of } T'} 2^{k(l')} \\
&\leq 2^{k(r)}
\end{aligned}$$

where the last inequality is because of the induction hypothesis. □

We can now put the pieces together.

Proof of Lemma 3.10. Fix a vertex $b \in B$. If there is a finite extension tree T with parameters (C, B, v) such that $d^{(b)}(T) = \infty$ then the associated cycles of the leaves of T are by definition peripheral and by Lemma 3.13 add up to precisely the two edges of C at v. In this case we are done.

So assume otherwise. By Lemma 3.16 there is an extension family (T_n) with parameters (C, B, v) such that $d^{(b)}(T_1) < d^{(b)}(T_2) < \ldots$. Hence, we obtain an infinite extension tree $T := \cup_{n \geq 1} T_n$ with parameters (C, B, v) and with $d^{(b)}(T) = \infty$. The number of leaves of T is because of Lemma 3.18 finite. We denote those leaves by $l_1, \ldots, l_k$ and set $D_1 := C(l_1), \ldots, D_k := C(l_k)$. These D_i are by definition peripheral and meet C at most in B. Lemma 3.18 also ensures that there are only finitely many rays starting in the root vertex. Let those rays be $R_{k+1}, \ldots, R_m$. Applying Lemma 3.17 to such a ray R_i yields a peripheral cycle D_i meeting B at most in C with $v \in D_i$ (for $i = k+1, \ldots, m$). In addition, as these D_i are obtained from the limit of the ray there is vertex t_i on the ray R_i satisfying $C(t_i) \cap E(v) = D_i \cap E(v)$. For an $N \in \mathbb{N}$ big enough the tree T_N contains each of $l_1, \ldots, l_k$ and of $t_{k+1}, \ldots, t_m$. Thus, the tree T' obtained from T_N by deleting every descendant of t_i (for $i = k+1, \ldots, m$) is itself a finite extension tree. The set of leaves of T' is

$$\{l_1, \ldots, l_k, t_{k+1}, \ldots, t_m\}.$$

As a consequence of Lemma 3.13 we obtain

$$\begin{aligned} C \cap E(v) &= \sum_{l \text{ leaf of } T'} C(l) \cap E(v) \\ &= \sum_{i=1}^{k} C(l_i) \cap E(v) + \sum_{i=k+1}^{m} C(t_i) \cap E(v) \\ &= \sum_{i=1}^{m} D_i \cap E(v), \end{aligned}$$

where the last equality is by the reasoning above. Note that each of the D_i is peripheral and meets B at most in C. □

3.5 Generating the cycle space

As mentioned Lemma 3.10 is to be used in the induction step for the proof of Theorem 3.1. However, Lemma 3.10 is only applicable to cycles, but we are dealing with arbitrary elements of the cycle space. In order to overcome this, we strengthen Lemma 3.10 in the following lemma.

Lemma 3.19. *Let G be locally finite and 3-connected. Let $Z \in \mathcal{C}(G)$ be an element of the cycle space with a bridge B and let $v \in B$ be a vertex. Then there are peripheral cycles $D_1, \ldots, D_m$ that meet B at most in Z and satisfy*

(i) $Z + \sum_{i=1}^{m} D_i$ *leaves a bridge* $B' \supseteq B$

(ii) $E(v) \subseteq B'$.

Proof. First note, that (i) holds for any cycles $D_1, \ldots, D_m$ that meet B at most in Z. In fact, consider egdes $e, f \in E(B)$. By definition there is a path $P \subseteq B$ internally disjoint to Z whose first edge is e and whose last is f. P is also internally disjoint to each of the D_i. Hence, e and f are therefore equivalent with respect to $Z \cup \bigcup_{i=1}^{m} D_i$ as well. Thus, $Z + \sum_{i=1}^{m} D_i$ being a subset of $Z \cup \bigcup_{i=1}^{m} D_i$ has a bridge $B' \supseteq B$.

By Theorem 3.4 Z is the edge-disjoint union of cycles; denote by $C_1, \ldots, C_n$ those of these cycles that contain v. As a subset of Z each of these C_i leaves a bridge B_i containing B. Therefore, applying Lemma 3.10 to C_i with bridge B_i yields peripheral cycles $D_{i1}, \ldots, D_{im_i}$ meeting $B_i \supseteq B$ at most in $C_i \subseteq Z$ with

$$\sum_{j=1}^{m_i} D_{ij} \cap E(v) = C_i \cap E(v)$$

(for $i = 1, \ldots, n$). Summing up, we arrive at

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} D_{ij} \cap E(v) = \sum_{i=1}^{n} C_i \cap E(v) = Z \cap E(v).$$

By the argument above, $Z' := Z + \sum_{i=1}^{n} \sum_{j=1}^{m_i} D_{ij}$ has a bridge $B' \supseteq B$. Also, the preceding equation implies $v \notin Z'$. Together with $v \in B$ this leads to $E(v) \subseteq B'$ as required. □

We are now ready to prove our main result. For technical reasons we shall first assume that $Z \in \mathcal{C}(G)$ has a bridge. This bridge then is extended in a controlled way, so that it eventually covers every edge of the graph. The extension is done in steps, each using Lemma 3.19.

Theorem 3.1. *Every element of the cycle space $\mathcal{C}(G)$ of a locally finite 3-connected graph G is a sum of peripheral cycles.*

Proof. We begin by proving the following statement.

> *Let $Z \in \mathcal{C}(G)$ be an element of the cycle space with a bridge B. Then Z is the sum of a thin family $\mathcal{D}$ of peripheral cycles each meeting B at most in Z.* $(*)$

Fix a vertex $b \in B$ and let $\{e_1, e_2, \ldots\}$ be an enumeration of the edge set of G such that $d_G(b, e_i) < d_G(b, e_j)$ implies $i < j$. We will obtain $\mathcal{D}$ as the union of inductively constructed finite sets of peripheral cycles. More formally, for all $n \in \mathbb{N}$ we inductively show the existence of finite sets $\mathcal{D}_n$ of peripheral cycles and of bridges B_n of $Z_n := Z + \sum \mathcal{D}_n$ satisfying

(i) $B_{n-1} \subseteq B_n$

(ii) $\mathcal{D}_{n-1} \subseteq \mathcal{D}_n$

(iii) every $D \in \mathcal{D}_n \backslash \mathcal{D}_{n-1}$ meets B_{n-1} at most in Z_{n-1}

(iv) $\{e_1, \ldots, e_n\} \subseteq E(B_n)$

where $\mathcal{D}_0 := \emptyset$, $B_0 := B$ and $Z_0 := Z$.

For $n \in \mathbb{N}$ let $\mathcal{D}_{n-1}$ and B_{n-1} be already constructed. We claim that e_n is incident with a vertex $v \in B_{n-1}$. If $d_G(b, e_n) = 0$ this is obvious, so let there be an edge e_i adjacent to e_n with strictly lesser distance to b. By the choice of the enumeration we obtain $i < n$ and in turn $e_i \subseteq B_{n-1}$. Hence, the vertex v incident with both edges, e_i and e_n, is contained in B_{n-1} as well.

By applying Lemma 3.19 to Z_{n-1} with bridge B_{n-1} and vertex v we obtain peripheral cycles $D_1, \ldots, D_m$ meeting B_{n-1} at most in Z_{n-1}. Further, $Z_{n-1} + \sum_{i=1}^{m} D_m$ leaves a bridge $B' \supseteq B_{n-1}$ such that $E(v) \subseteq B'$. Since $e_n \in E(v)$ the conditions (i)-(iv) are clearly satified by setting

$$\mathcal{D}_n := \mathcal{D}_{n-1} \cup \bigcup_{i=1}^{m} D_i \text{ and } B_n := B'.$$

We claim that $\mathcal{D} := \cup_{n \geq 1} \mathcal{D}_n$ satisfies the assertion of the claim $(*)$. To see this, consider an edge e_n of G. First note that because of the conditions (iii,iv) the edge e_n may only be used by the finitely many cycles in $\mathcal{D}_n$ and by none other in $\mathcal{D}$—proving $\mathcal{D}$ to be a thin family. Further from the condition (iv) above we see that

$$e_n \notin Z + \sum \mathcal{D}_n \text{ and hence } e_n \notin Z + \sum \mathcal{D}$$

by the preceding argument. Thus, $(*)$ is established.

Now, if an element Z of the cycle space leaves a bridge, $(*)$ guarantees that Z is the sum of peripheral cycles. If, on the other hand, Z does not leave a bridge let C be any cycle in G. Then, both C and $Z+C$ have at least one bridge and the statement $(*)$ may be applied to each of them. Clearly, the union of the two generating sets of peripheral cycles is a generating set of Z. □

In the introductory example in Section 3.1 we were able to generate our given cycle using only *finite* peripheral cycles. One might wonder whether in general any cycle, or at least every finite one, is the sum of finite peripheral cycles. This, however, is not true as figure 3.4 demonstrates. The edge e there is not

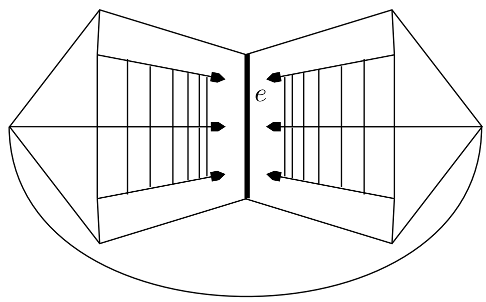

Figure 3.4: There is no finite peripheral cycle containing the edge e

contained in any finite peripheral cycle. Consequently, any cycle containing e may not be generated by finite peripheral cycles. On the other hand, it is easy to see that e lies on exactly two infinite peripheral cycles. Note that the graph shown is indeed 3-connected.

3.6 Two counterexamples for arbitrarily infinite graphs

In this section we present two examples of graphs whose cycle space is not generated by peripheral cycles. Clearly, these graphs cannot be locally finite. Therefore, before we come to the examples, we have to adjust our definition of the cycle space to accomodate the possibly non-locally finite situation. Once again, we are following Diestel and Kühn who have studied the cycle space of arbitrarily infinite graphs in [4].

The topology on arbitrarily infinite graphs will be the same as for locally finite graphs and, naturally, a cycle will still be the homeomorphic image of the unit circle. However, we have to change our definition of a thin family slightly. Namely, a family $(A_i)_{i \in I}$ of edge sets is called *thin* if no vertex is incident to infinitely many members of the family. Clearly, this definition is stronger but coincides in the case of locally finite graphs with the one given previously. With this, the cycle space is defined as before, as the set of sums of thin families of cycles.

Now, one of the reasons why the cycle space may fail to be generated by peripheral cycles is that we find an edge that is not contained in any peripheral cycle. That this is clearly possible demonstrates our first example.

For this, let D be a double ray $\dots r_{-1} r_0 r_1 \dots$ and s and t two vertices not contained in D. Denote by G_1 the graph that we obtain by joining every vertex of D to s and to t and by joining s to t (see figure 3.5). Note that G_1 is 3-connected.

Observe, that G_1 has no infinite cycles. Indeed, assume C to be an infinite cycle and C' its defining circle. For C to be infinite it has to contain a ray, say

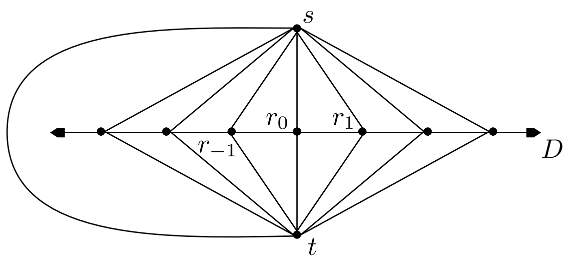

Figure 3.5: The edge st in the graph G_1 is not contained in any peripheral cycle

$r_N r_{N+1} \ldots$. Now, if ω is the end containing that ray and $S := \{r_N, s, t\}$ then $\hat{C}_{G_1}(S,\omega)$ is an open neighbourhood of ω. Because $r_N \notin \hat{C}_{G_1}(S,\omega)$ but $r_N \in C$ we see that $(C' \cap \hat{C}_{G_1}(S,\omega)) \backslash \{\omega\}$ consists of at least two components of C'. So with the observation

$$(C' \cap \hat{C}_{G_1}(S,\omega)) \backslash \{\omega\} = C \cap \hat{C}_{G_1}(S,\omega)$$

we should expect at least two (graph-theoretical) components of C in that intersection. Only, all vertices of $\hat{C}_{G_1}(S,\omega)$ lie already on the ray $r_{N+1} r_{N+2} \ldots$, which is clearly a connected subgraph of C—a contradiction.

Now suppose there is a peripheral cycle C that meets the edge st. Lacking infinite cycles C has to be finite. Since C contains more vertices than s and t, C meets the double ray D. On the other hand, there is an $N \in \mathbb{N}$ such that $C \cap D \subseteq r_{-N} \ldots r_N$ as C is finite. But then r_{N+1} is separated from $r_{-(N+1)}$ by C.

We conclude that no peripheral cycle meets the edge st. Therefore, any cycle containing st (for instance the cycle str_0s) cannot be the sum of peripheral cycles.

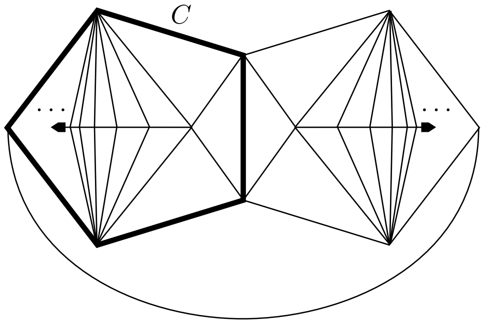

Figure 3.6: In the graph G_2 the cycle C cannot be generated by peripheral cycles

In the second example, the graph G_2 shown in figure 3.6, every edge is contained in a peripheral cycle. Nevertheless, the cycle C shown there cannot be the sum of peripheral cycles. Indeed, such a sum would have to include all the peripheral triangles incident to at least one of the infinite stars of G_2—which is clearly not allowed. It should be noted that this counterexamples depends on the fact that there are edges that lie on only one peripheral cycle.

Chapter 4

Finite graphs of arbitrary connectivity

4.1 Introduction

Before we start we should mention that the results of this chapter are plain obvious—if one knows Tutte's original paper [7]. See the notes at the end of Section 4.3.

In this chapter we generalize Tutte's result to finite graphs of arbitrary connectivity. In particular, all the graphs in this chapter are finite.

In a planar (2-connected) graph the cycle space is easily seen to be generated by the face boundaries. For 3-connected planar graphs Tutte showed that these can be characterized independently of the concrete drawing: the face boundaries are precisely the peripheral cycles, which generate the cycle space. We take this relationship between face boundaries and generators as a motivation and look for an abstract characterization of the face boundaries. For a 3-connected graph this can be accomplished only because every two drawings in a 3-connected graph are equivalent, that is the sets of their face boundaries coincide. In a 2-connected graph, however, this is not necessarily the case and consequently a characterization of the face boundaries is not possible. Instead, we find an abstract condition that identifies those cycles in an arbitrary planar graph G that occur as a face boundary in some drawing of G, and prove that the cycles satisfying this condition generate the cycle space of G, even when G is not planar.

4.2 Free cycles in planar graphs

Our first aim is to find an abstract condition such that for every cycle in a planar graph satisfying this condition there is a suitable drawing in which that cycle occurs as a face boundary. To state this condition we recall a standard concept that naturally arises when dealing with cycles. Here, we have taken the definition from Bondy and Murty [1].

Definition 4.1. *Let H be a subgraph of a graph G. We define an equivalence relation on $E(G)\backslash E(H)$ by $e \sim f$ if there is a path P such that* *(i) the first edge of P is e and the last is f.*

(ii) P meets H at most at its ends.

A connected non-trivial subgraph B of $G - E(H)$ whose edge set is closed under this equivalence relation is called a bridge *of H. The vertices of B on H are called the* vertices of attachment *of B.*

One sees easily that a bridge is either a chord of H or a subgraph of G consisting of a component K of $G - H$ with the edges $E(K, H)$ added. Note, that a cycle is peripheral if and only if it has at most one bridge.

Further we say that two bridges B and B' *avoid one another* if there is a subpath P of C such that all vertices of attachment of B belong to P, but no inner vertex of P is a vertex of attachment of B'. Otherwise they *overlap*.

Two bridges B, B' are said to be *skew* if there are two (non-adjacent) vertices of attachment u, v of B and two vertices of attachment u', v' of B' such that u' and v' are in different components of $C - \{u, v\}$. In particular two overlapping bridges are either skew or have three vertices of attachment in common.

With this we can state our condition.

Definition 4.2. *An induced cycle C of G is called* free *if it has no two overlapping bridges.*

In a plane graph we can distinguish between *inner* and *outer* bridges, that is, between bridges that are contained in the interior of the cycle and those that are not.

The following result from Bondy and Murty [1] allow us to quickly verify our claim that any free cycle occurs in a suitable drawing.

Lemma 4.3 (Bondy and Murty [1]). *Let B be an inner bridge in a plane graph G avoiding all outer bridges. Then there is a drawing of G such that B is an outer bridge but the position of all other bridges remains unchanged.*

As a result, an easy induction argument immediately shows that a drawing can be modified so that any given free cycle is indeed a face boundary. On the other hand it is straightforward to see that in every (2-connected) plane graph the face boundaries are also free cycles. Since the face boundaries of a planar graph generate the cycle space we have the following observation which is the planar case—and the motivation—of the main result of this chapter.

Proposition 4.4. *The free cycles of a planar graph generate its cycle space.*

4.3 Free cycles in non-planar graphs

Now we extend this result to non-planar graphs. The first step will be to show that in a 3-connected graph the induced free cycles are precisely the peripheral cycles. Then, Tutte's theorem will allow us to restrict our attention to graphs that are separable by two vertices. Additionally, we see that in the case of a 3-connected graph our generators of the cycle space coincide with Tutte's.

Lemma 4.5. *Let G be 3-connected. Then the induced free cycles are exactly the peripheral cycles of G.*

Proof. Consider an induced cycle $C \subseteq G$. Now, if C is peripheral it has only a single bridge and is therefore free.

Conversely, let C be free. First note that the vertices of attachment of every bridge B cover the whole cycle. Suppose not. Let $Q \subseteq C$ be a path with minimal length covering all vertices of attachment of B; then Q fails to cover all of C. Thus, the two end vertices a and b of Q separate C into the components $\mathring{a}Q\mathring{b}$ and $C - Q$. Both these sets are non-empty: the former because B has at least three vertices of attachment and the latter because of our assumption. Because G is 3-connected there is a path P between $\mathring{a}Q\mathring{b}$ and $C - Q$. Observe that P does not lie in B as it meets $C - Q$. By the minimality of Q, a and b are vertices of attachment of B, hence P is contained in a bridge that is skew to B. Contradiction.

Finally, suppose there is a second bridge B' of C. Both B and B' are attached to the whole cycle and have therefore at least three neighbours in common. Consequently, the two bridges overlap—contradiction. So C is peripheral. □

The proof of the main theorem will rest on the following two lemmas as they provide the necessary means for an induction. For this induction to work we will split G into two subgraphs $G_1 \cup G_2 = G$ such that $G_1 \cap G_2$ consists of exactly two vertices x, y. Preferably we would like those two vertices to be adjacent, because then every induced cycle of G lies either in G_1 or in G_2. Lemma 4.6 ensures that in that case every free cycle in the smaller graphs G_1 or G_2 is one in G too.

However, x and y cannot be expected to be neighbours (as the example of a simple cycle shows) and indeed there may be an induced cycle in G which lies neither in G_1 nor in G_2. In particular the intersection of this cycle and G_i yields an x-y-path, not a cycle, and therefore is not even a member of the cycle space of G_i $(i = 1, 2)$. To mend this we simply add the edge xy to those paths and get proper cycles in $G_1 + xy$ and $G_2 + xy$. Now how do the free cycles of those modified graphs relate to the ones in G? Lemma 4.6 asserts that the only reason why a free cycle C_1, in $G_1 + xy$ say, may fail to be one in G, is that it is not even a proper cycle of G, i.e. C_1 contains the edge xy. Fortunately, that can be repaired provided we find a free cycle C_2 in $G_2 + xy$ that shares the same deficiency, i.e. $xy \in C_2$. In that case Lemma 4.7 shows that at least the sum $C_1 + C_2$ is free in G.

Lemma 4.6. *Let the* 2*-connected graph* G *be the union of two graphs* G_1 *and* G_2 *such that* $G_1 \cap G_2$ *consists of two (adjacent or non-adjacent) vertices* x, y *and* $|G_i| \geq 3$ *for* $i = 1, 2$*. Define* $H_i := G_i + xy$ *for* $i = 1, 2$ *and let* C *be a free cycle in* H_1 *that is still a cycle in* G*. Then* C *is also free in* G*.*

Proof. We begin by taking a look at how the bridges of C in G correspond to those in H_1. Consider a bridge B of C in G.

First assume, $B \subseteq G_1$. Now, if B meets x then we must have $x \in C$, as otherwise $G_2 \subseteq B$ (note that G is 2-connected). The same holds for y. Hence, B is also a bridge in H_1.

Now assume, $B \subseteq G_2$. If xy is an edge of C, B cannot prevent C from being free and we may disregard this bridge. If, on the other hand, $xy \notin C$ the chord xy is a bridge in H_1 with the same vertices of attachment as B.

Finally assume, that neither $B \subseteq G_1$ nor $B \subseteq G_2$. We claim that $B' := B \cap H_1 + xy$ is a bridge in H_1. Let e be an edge of $B \cap G_1$. First, let $f \neq xy$ be an edge in $B \cap G_2$. Then there is a path from e to f internally avoiding C. This path meets the separator $\{x, y\}$, and immediately we get $e \sim xy$ in H_1. Now, let f be any edge of $B \cap H_1$. From $e \sim f$ in G we obtain a path $P \subseteq G$ from e to f that meets C at most at its ends. This path either lies in H_1 or can be shortened to a path in H_1 by replacing its subpath xPy by xy. Thus, we have $e \sim f$ in H_1 as well. To show that B' is closed under $\sim$, let f be any edge of $E(H_1)\backslash E(C)$ with $e \sim f$. Hence, there is a path in H_1 from e to f internally avoiding C. This path is already a path in G or it makes use of the edge xy. In the latter case we obtain a path in G by replacing xy by an x-y-path in G_2 (note that $G_2 \subseteq B$ as G is 2-connected). Consequently, f is already an edge of B and therefore of B'. Thus, B' is a bridge in H_1.

In all those cases (except in the one we may disregard) there is a corresponding bridge with a set of vertices of attachment that contains the original set. Now suppose there are two overlapping bridges in G. But then the corresponding bridges in H_1 exhibit the same property contradicting that C is free in H_1. □

Lemma 4.7. *Let G be the union of two graphs G_1 and G_2 such that $G_1 \cap G_2$ consists of two non-adjacent vertices x, y. Define $H_i := G_i + xy$ and let C_i be a free cycle of H_i containing xy for $i = 1, 2$. Then $C := C_1 + C_2$ is a free cycle in G.*

Proof. Let B be a bridge of C in G. As $x, y \in C$, B must lie completely in either G_1 or G_2, say in G_1. Thus, B is clearly a bridge of C_1 in H_1 (note that $x, y \in C_1$).

Now suppose that C is not free in G. So C has two overlapping bridges B and B'. In particular the separator $\{x, y\}$ forces B and B' to be both contained in the same G_i, say G_1. Consequently, B and B' are each a bridge of C_1, contradicting that C_1 is free in H_1. □

We now prove the main result of this chapter: in any graph (planar or not), the free cycles generate the cycle space. It is even enough to take only induced free cycles.

Theorem 4.8. *The induced free cycles of a graph generate its cycle space.*

Proof. We prove the assertion by induction on $|G|$. For $|G| = 1, \ldots, 4$ it is obviously true.

First note that we can assume that G is 2-connected since the cycle space of G is the direct sum of the cycle spaces of the blocks of G. If G is even 3-connected then, by Tutte's theorem, the cycle space is generated by the peripheral cycles, which are, in turn, induced free cycles (Lemma 4.5). So we may assume that there are two vertices x, y separating G into two smaller graphs G_1 and G_2, that is $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{x, y\}$. It suffices to show that the free induced cycles generate every induced cycle as these generate the cycle space. So consider an induced cycle C. If $xy \in G$ then C lies either in G_1 or in G_2 and is by induction the sum of induced free cycles of G_1 or of G_2. Those free cycles are also free in G (Lemma 4.6) and we are done.

So assume that $xy \notin G$. Define $H_i := G_i + xy$ for $i = 1, 2$. First let C be completely contained in G_1 or G_2, say in G_1. By induction there are induced free cycles $D_1, \ldots, D_m$ in H_1 whose sum is C. Let w.l.o.g. $D_1, \ldots, D_k$ be those of the cycles that contain the edge xy. Note that since $xy \notin C$, k is even. The other cycles $D_{k+1}, \dots, D_m$ do not contain xy and are therefore already free in G (Lemma 4.6). Now let E be an induced free cycle in H_2 with $xy \in E$. Such an E exists as there is a cycle in H_2 that contains the edge xy, and hence, by induction, there is a generating set of induced free cycles. One of those must contain xy. Lemma 4.7 ensures that the induced cycles $\tilde{D}_j := D_j + E$ ($j = 1, \dots, k$) are free. Because k is even E vanishes in the sum

$$\tilde{D}_1 + \dots + \tilde{D}_k = D_1 + \dots + D_k$$

All in all we obtain the sum of induced free cycles

$$\tilde{D}_1 + \dots + \tilde{D}_k + D_{k+1} + \dots + D_m = C.$$

Finally, let C be the union of x-y-paths $P_1 \subseteq G_1$ and $P_2 \subseteq G_2$. For $i = 1, 2$ define $C_i := P_i + xy$. Those C_i are cycles in H_i and therefore by induction generated by induced free cycles of the H_i: C_1 by $D_1, \dots, D_m$ and C_2 by $E_1, \dots, E_n$, say. Let w.l.o.g. $D_1, \dots, D_k$ and $E_1, \dots, E_l$ be those of the cycles that contain xy where $l \leq k$.

The other cycles $D_{k+1}, \dots, D_m$ and $E_{l+1}, \dots, E_n$ are already free in G, as above. Further, from Lemma 4.7 it follows that the sums $F_1 := D_1 + E_1, \dots, F_l := D_l + E_l$ are also free cycles in G.

So it remains to deal with the cycles $D_{l+1}, \dots, D_k$. For this, first note that E_1 does indeed contain xy (i.e. that $l \geq 1$) as $xy \in C_2$. From, once again, Lemma 4.7 it follows that the induced cycles $\tilde{D}_j := D_j + E_1$ for $j = l+1, \dots, k$ are free in G. Now observe that k and l have to be odd in order to ensure that the edge xy lies in the sums

$$\begin{aligned} C_1 &= D_1 + \dots + D_k + D_{k+1} + \dots + D_m \text{ and} \\ C_2 &= E_1 + \dots + E_l + E_{l+1} + \dots + E_n \end{aligned}$$

Hence the number $k - l$ of the $\tilde{D}_j$ is even and as a result we obtain C as the sum

$$C = F_1 + \dots + F_l + \tilde{D}_{l+1} + \dots + \tilde{D}_k + D_{k+1} + \dots + D_m + E_{l+1} + \dots + E_n$$

where all the summands are induced free cycles in G. So the induction is proved. □

The results of this chapter are almost immediate consequences of Tutte's paper. In fact, Tutte uses a lemma similar to Lemma 4.5 to show that a non-peripheral cycle has two overlapping bridges. Then, it is shown that cycles with two overlapping bridges are not needed to generate the cycle space.

Unfortunately, at the time of the writing of this chapter I knew only the proof of Thomassen for Theorem 1.1 (see [2]) which pursues a totally different approach. In particluar, Thomassen manages to avoid introducing bridges at all. So this chapter may be seen as an idle exercise to prove Theorem 4.8 with only Thomassen's proof available.

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Diplomarbeit ohne fremde Hilfe selbständig verfasst und nur die angegebenen Quellen und Hilfsmittel benutzt habe.

Henning Bruhn
Hamburg, 24. September 2001